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NEW CLASSES OF FUZZY OPERATOR IDEALS

MANAF ADNAN SALEH SALEH[1], ZAINAB RIYADH SHAKER AL-YASIRI[2,*]

[1]Department of Biology Science, College of Sciences, Al-Nahrain University, Baghdad, Jadriya, Iraq

[2]Department of Mathematics and Computer Applications, College of Sciences, Al-Nahrain University, Baghdad, Jadriya, Iraq

[*]Corresponding author: manaf.adnan@nahrainuniv.edu.iq

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Abstract. In this paper, we are building new classes of fuzzy operator ideals theory so-called interpolative simple fuzzy bounded linear operators and absolutely simple fuzzy (p, θ)-summing operators, $0 \leq \theta < 1$ and $1 < p < \infty$, between arbitrary complete fuzzy normed spaces. We show that the resulting classes of fuzzy bounded linear operators are fuzzy operator ideals and establish several properties and inclusion resuls of these classes. We define fuzzy norms of the aforementioned terminologie and prove its fuzzy norm is fuzzy real number. We then prove a powerful characterization of absolutely simple fuzzy (p, θ)-summing operators by establishing a fuzzy version of Pietsch Domination Theorems. Finally, we present conclusions and raise open questions which we think are interesting.

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1. Notations and Preliminaries

The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ represent the sets of all real numbers, all positive real numbers, and all positive integers, respectively. The ordered pairs $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ will represent Banach spaces. A Banach space E's dual space is E^*, and its closed unit ball is represented by B_E. The space of all bounded linear operators from E into F is denoted by $\mathfrak{L}(E, F)$.

Definition 1. [1] *A function $\eta : \mathbb{R} \longrightarrow [0, 1]$ is said to be a fuzzy real number, and its α-level set is represented by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$ if it fulfills two conditions:*

(1) *There is $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*
(2) *For each $\alpha \in (0, 1] : [\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha], -\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$.*

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Definition 2. [1] *Let η and δ belong to $\mathscr{F}$. Define a partial ordering by $\eta \preceq \delta$ if and only if $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ for all α in the interval $(0,1]$.*

Definition 3. [1] *Let η belong to $\mathscr{F}$.When $\eta(t) = 0$ and $t < 0$, η is referred to be "a positive fuzzy real number". The symbol $\mathscr{F}^+$ stands for "the set of all positive fuzzy real numbers".*

Lemma 4. [1] *Assume η belongs to $\mathscr{F}$. Then $\eta \in \mathscr{F}^+$ if and only if $\eta_\alpha^- \geq 0$ for each $\alpha \in (0,1]$.*

Lemma 5. [2] *Assume that $\eta \in \mathscr{F}$ and $\delta \in \mathscr{F}$. Let $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then*

$$[\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+],$$
$$[\eta \ominus \delta]_\alpha = [\eta_\alpha^- - \delta_\alpha^+, \eta_\alpha^+ - \delta_\alpha^-],$$
$$[\eta \odot \delta]_\alpha = [\eta_\alpha^- \cdot \delta_\alpha^-, \eta_\alpha^+ \cdot \delta_\alpha^+], \text{ for } \eta, \delta \in \mathscr{F}^+.$$

Definition 6. [3] *Let E be a real vector space and let $\widehat{\|\cdot\|}$ be a function from E into $\mathscr{F}^+$. The order pair $(E, \widehat{\|\cdot\|})$ is said to be a fuzzy normed space if the following conditions are satisfied:*

($\mathbf{FN_0}$) *If $x \neq 0$, then $\inf_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$.*
($\mathbf{FN_1}$) *$\widehat{\|x\|} = \hat{0}$ if and only if $x = 0$.*
($\mathbf{FN_2}$) *$\widehat{\|r \cdot x\|} = \widehat{|r|} \odot \widehat{\|x\|}$, for all $r \in \mathbb{R}$ and x in E.*
($\mathbf{FN_3}$) *$\widehat{\|x+y\|} \preceq \widehat{\|x\|} \oplus \widehat{\|y\|}$, for all x and y in E, where a function $\widehat{|\cdot|}$ from $\mathbb{R}$ into $\mathscr{F}^+$ defined by the rule*

$$\widehat{|r|}(t) = \begin{cases} 1 & \text{if } t = |r| \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7. [3] *Let $\eta \in \mathscr{F}$. Then $\eta \in \mathscr{F}^+$ if and only if $\hat{0} \preceq \eta$.*

Definition 8. [4] *Let $\eta \in \mathscr{F}^+$ and $p \neq 0$ be a real number. We define η^p as*

$$\eta^p(t) = \begin{cases} \eta(t^{1/p}), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Remark 9. [4] *η^p is a fuzzy real number. Also it is clear that if $p > 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^-)^p, (\eta_\alpha^+)^p]$ for each $\alpha \in (0,1]$.*

Theorem 10. [4] *Let $\eta, \delta \in \mathscr{F}^+$ and $p, q > 0$. Then*

(1) $\eta^p \odot \eta^q = \eta^{p+q}$.
(2) $(\eta^p)^q = \eta^{p \cdot q}$.
(3) $\eta^p \odot \delta^p = (\eta \odot \delta)^p$.

Theorem 11. [5] *If $\eta_j, \gamma_j \in \mathscr{F}^+$ $(j = 1, \ldots, m)$ and $p > 1$, then*

$$\left[\sum_{j=1}^{m}(\eta_j \oplus \gamma_j)^p\right]^{\frac{1}{p}} \preceq \left[\sum_{j=1}^{m}\eta_j^p\right]^{\frac{1}{p}} \oplus \left[\sum_{j=1}^{m}\gamma_j^p\right]^{\frac{1}{p}}.$$

Definition 12. [3] *Let $(E, \widehat{\|\cdot\|})$ and $(F, \widehat{\|\cdot\|})$ be fuzzy normed spaces. If a fuzzy number η belongs to the set $\mathscr{F}^+$, then the linear operator T, which maps from $(E, \widehat{\|\cdot\|})$ to $(F, \widehat{\|\cdot\|})$, is considered fuzzy bounded, such that*

$$\widehat{\|T(x)\|} \preceq \eta \odot \widehat{\|x\|}, \forall x \in E. \tag{1}$$

The class of all fuzzy bounded linear operators from $(E, \widehat{\|\cdot\|})$ into $(F, \widehat{\|\cdot\|})$ is denoted by $\boldsymbol{FB}(E, F)$.

Definition 13. [3] *Suppose that $(E, \widehat{\|\cdot\|})$ and $(F, \widehat{\|\cdot\|})$ be fuzzy normed spaces and let T from $(E, \widehat{\|\cdot\|})$ into $(F, \widehat{\|\cdot\|})$ be fuzzy bounded linear operator. The fuzzy norm $\widehat{\|T\|}$ defined as follows.*

$$[\widehat{\|T\|}]_\alpha = [\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, \inf\left\{\eta_\alpha^+ : \widehat{\|T(x)\|} \preceq \eta \odot \widehat{\|x\|}\right\}], \forall \alpha \in (0, 1].$$

Theorem 14. [3] *If $T \in \boldsymbol{FB}(E, F)$, then $\widehat{\|T\|}$ belongs to $\mathscr{F}^+$.*

Lemma 15. [3] *If $T \in \boldsymbol{FB}(E, F)$, then $\widehat{\|T(x)\|} \preceq \widehat{\|T\|} \odot \widehat{\|x\|}, \forall x \in E$.*

Definition 16. [6] *Suppose that $(E, \widehat{\|\cdot\|})$ be fuzzy normed space. A linear operator f from $(E, \widehat{\|\cdot\|})$ into $(\mathbb{R}, \widehat{|\cdot|})$ is called fuzzy bounded linear functional on $(E, \widehat{\|\cdot\|})$if η is a fuzzy number that belongs to $\mathscr{F}^+$ such that*

$$\widehat{|f(x)|} \preceq \eta \odot \widehat{\|x\|},$$

for all $x \in E$. The closed unit ball of a fuzzy normed space $(F, \widehat{\|\cdot\|})$ define by $\mathfrak{B}_F = \left\{e \in F : \widehat{\|e\|} \preceq \hat{1}\right\}$. The class of all fuzzy bounded linear functional from $(E, \widehat{\|\cdot\|})$ into $(\mathbb{R}, \widehat{|\cdot|})$ is denoted by $\boldsymbol{F}E^*$. The complete fuzzy normed space $\boldsymbol{F}E^*$ is called fuzzy dual of E. We consider the closed unit ball $\mathfrak{B}_{\boldsymbol{F}E^*} = \left\{f : \widehat{\|f\|} \preceq \hat{1}\right\}$ is compact space with respect to the $\sigma(\boldsymbol{F}E^*, E)$-topology. The space of all continuous real-valued functions on $\mathfrak{B}_{\boldsymbol{F}E^*}$ is denoted by $C(\mathfrak{B}_{\boldsymbol{F}E^*})$.

Remark 17. [6] *Let $0 \neq f \in \boldsymbol{F}E^*$ and $0 \neq e \in F$. It can be easily checked the following rule:*

$$f \boxdot e : x \longmapsto f(x) \cdot e$$

be a fuzzy, bounded linear operator. Moreover $\widehat{\|f \boxdot e\|} \preceq \widehat{\|f\|} \odot \widehat{\|e\|}$.

2. Introduction

First, we define the operator ideal notion in [7] as follows. Assume a subset $\mathfrak{A}(E, F)$ of $\mathfrak{L}(E, F)$ exists for each pair of Banach spaces E and F. The class

$$\mathfrak{A} := \bigcup_{E,F} \mathfrak{A}(E, F)$$

is said to be an operator ideal, if the following requirements are satisfied:

($\mathbf{OI_0}$) $a^* \otimes e \in \mathfrak{A}(E,F)$, for a^* belongs to E^* and $e \in F$.
($\mathbf{OI_1}$) $S+T$ belongs to $\mathfrak{A}(E,F)$, for $S,\ T$ are belong to $\mathfrak{A}(E,F)$.
($\mathbf{OI_2}$) $BTA \in \mathfrak{A}(E_0,F_0)$ for $A \in \mathfrak{L}(E_0,E)$, $B \in \mathfrak{L}(F,F_0)$ and $T \in \mathfrak{A}(E,F)$. Condition ($\mathbf{OI_0}$) implies that $\mathfrak{A}$ contains nonzero operators.

Urs Matter defined in his seminal paper [8] a new class of interpolative ideal procedure as follows. Suppose that $0 \le \theta < 1$ and let $[\mathfrak{A}, \mathbf{A}]$ is a normed operator ideal. A bounded operator T from E into F belongs to $\mathfrak{A}_\theta(E,F)$ if there exist a Banach space G and a bounded operator $S \in \mathfrak{A}(E,G)$ such that

$$\|Tx\| \le \|Sx\|^{1-\theta} \cdot \|x\|^{\theta}, \tag{2}$$

for all $x \in E$. Setting $\mathbf{A}_\theta(T) := \inf \mathbf{A}(S)^{1-\theta}$, for each $T \in \mathfrak{A}_\theta(E,F)$ involves taking the infimum over all bounded operators S accepted in (2). Also, he proved the following fundamental characterization of absolutely (p,θ)-summing operators.

Theorem 18. [8] *For every bounded linear operator T from the space E to itself, the following statements are identical:*

(i) $T \in \mathfrak{P}_p^\theta(E,F)$.
(ii) *There is a probability measurement μ defined on B_{E^*} and a constant $C \ge 0$ such that:*

$$\|Tx\| \le C \cdot \left(\int_{B_{E^*}} \left(|\langle a,x\rangle|^{1-\theta} \cdot \|x\|^\theta \right)^{\frac{p}{1-\theta}} d\mu(a) \right)^{\frac{1-\theta}{p}}, \ \forall x \in E.$$

(iii) *There exists a constant $C \ge 0$, so that for any finite sequence $x_1, \ldots, x_m$ in E, the inequality*

$$\left(\sum_{j=1}^m \|Tx_j\|^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}} \le C \cdot \sup_{a \in B_{E^*}} \left(\sum_{j=1}^m |\langle x_j, a\rangle|^{1-\theta} \|x_j\|^\theta \right)^{\frac{1-\theta}{p}},$$

holds. $\mathsf{P}_p^\theta(T)$ *in the present case equals the infimum of the constants C in (ii) or (iii).*

The following are important specific instances of operator ideals as described by M. A. S. Saleh and L. K. Shaakir [9] .

Definition 19. *Suppose $0 < p < \infty$.If there is a constant $\zeta \ge 0$, then an operator $S \in \mathcal{L}(E,F)$ is said to be "absolutely simple p-summing", such that*

$$\left[\sum_{j=1}^m \varkappa_j \left\| S\left(\sum_{i=1}^N \phi(\alpha_i^j)\, e_i \right) \right\|^p \right]^{\frac{1}{p}} \leqq \zeta \cdot \sup_{x^* \in U_{E^*}} \left[\sum_{j=1}^m \varkappa_j \left| \left\langle x^*, \sum_{i=1}^N \phi(\alpha_i^j)\, e_i \right\rangle \right|^p \right]^{\frac{1}{p}} \tag{3}$$

holds for all $(\varkappa_j)_{j=1}^m$ in $\mathbb{R}^+$, $((\alpha_i^j)_{i=1}^N)_{j=1}^m$ in $\mathbb{R}$, and $m, N \in \mathbb{N}$.

For more information on the (crisp-fuzzy) operator ideals can be found in the monographs [10–12]. Now we illustrate the contents of the manuscript. In Section 3, we will prove some elementary results related to fuzzy analysis, which are later used. Then we recall a fuzzy operator ideal terminology and introduce some basic examples. One particularly classes of fuzzy operator ideals are interpolative simple fuzzy bounded linear operators and absolutely simple fuzzy (p,θ)-summing operators, $0 \le \theta < 1$ and $1 < p < \infty$, between arbitrary complete fuzzy normed spaces. We also define the absolutely fuzzy (p,θ)-summing norm $\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}$ and show that it is a fuzzy real number. We prove some properties and inclusion results and establish the resulting classes of fuzzy bounded operators are fuzzy operator ideals. Afterwards we then prove a powerful characterization of absolutely simple fuzzy (p,θ)-summing operators by establishing a fuzzy version of Pietsch Domination Theorems. In Section 4, we introduce conclusions and in Section 5 aim to raise open questions that we find interesting.

3. Fuzzy Operator Ideals

We will prove some elementary results related to fuzzy analysis, which are later used.

Proposition 20. *Let $1 < p < \infty$. If $(E_1, \widehat{\|\cdot\|})$ and $(E_2, \widehat{\|\cdot\|})$ be complete fuzzy normed spaces, then the linear space $E_1 \boxplus_p E_2$, the algebraic fuzzy direct sum of E_1 and E_2, equipped with the fuzzy real number-valued function $\widehat{\|\cdot\|}$ defined on $E_1 \boxplus E_2$ form a complete fuzzy normed space, where*

$$\widehat{\|x_1 \boxplus x_2\|_p} = \begin{cases} \left(\widehat{\|x_1\|}^p \oplus \widehat{\|x_2\|}^p\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty \\ \max\left\{\widehat{\|x_1\|}, \widehat{\|x_2\|}\right\}, & \text{if } p = \infty. \end{cases} \tag{4}$$

Proof. If $p = 1$ the statement is obvious. When $1 < p < \infty$. First to prove Condition $(\mathbf{FN_0})$. If $x_1 \boxplus x_2 \neq 0$, then either $x_1 \neq 0$ or $x_2 \neq 0$. Say $x_1 \neq 0$ we obtain $\inf\limits_{0<\alpha\le 1} \|x_1\|^-_\alpha > 0$. From (4) we have $\|x_1 + x_2\|^-_\alpha = (\|x_1\|^{p,-}_\alpha + \|x_2\|^{p,-}_\alpha)^{\frac{1}{p}}$. Since $\|x_2\|^{p,-}_\alpha \ge 0, \forall\ \alpha \in (0,1]$ we get $\|x_1\|^{p,-}_\alpha \le \|x_1\|^{p,-}_\alpha + \|x_2\|^{p,-}_\alpha$. Thus $\|x_1\|^-_\alpha = (\|x_1\|^{p,-}_\alpha)^{\frac{1}{p}} \le (\|x_1\|^{p,-}_\alpha + \|x_2\|^{p,-}_\alpha)^{\frac{1}{p}}, \forall\ \alpha \in (0,1]$. Therefore $0 < \inf\limits_{0<\alpha\le 1} \|x_1\|^-_\alpha \le \inf\limits_{0<\alpha\le 1} (\|x_1\|^{p,-}_\alpha + \|x_2\|^{p,-}_\alpha)^{\frac{1}{p}}$. It can be easily checked Conditions $(\mathbf{FN_1})$ and $(\mathbf{FN_2})$. Now we prove that Condition $(\mathbf{FN_3})$, let $x_1 \boxplus x_2$ and $y_1 \boxplus y_2$ in $E_1 \boxplus_p E_2$ we have

$$\begin{aligned}
\widehat{\|(x_1 \boxplus x_2) + (y_1 \boxplus y_2)\|_p} &= (\widehat{\|x_1 + y_1\|}^p \oplus \widehat{\|x_2 + y_2\|}^p)^{\frac{1}{p}} \\
&\preceq \left((\widehat{\|x_1\|} \oplus \widehat{\|y_1\|})^p \oplus (\widehat{\|x_2\|} \oplus \widehat{\|y_2\|})^p\right)^{\frac{1}{p}} \text{ (By Condition } (\mathbf{FN_3})) \\
&\preceq \left((\widehat{\|x_1\|}^p \oplus \widehat{\|x_2\|}^p)\right)^{\frac{1}{p}} \oplus \left((\widehat{\|y_1\|}^p \oplus \widehat{\|y_2\|}^p)\right)^{\frac{1}{p}} \text{ (By Theorem 11)} \\
&= \widehat{\|x_1 \boxplus x_2\|_p} \oplus \widehat{\|y_1 \boxplus y_2\|_p}.
\end{aligned}$$

To show the completeness, let $(x_{1n} \boxplus x_{2n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $E_1 \boxplus_p E_2$. Then $\|x_{1m} - x_{1n}\|^+_\alpha \le \|(x_{1m} \boxplus x_{2m}) - (x_{1n} \boxplus x_{2n})\|^+_\alpha \longrightarrow 0$ as $m,n \to \infty$, so $(x_{1n})_{n\in\mathbb{N}}$ is a Cauchy sequence in E_1, and similarly for $(x_{2n} \boxplus x_{2n})_{n\in\mathbb{N}}$. Therefore, these sequences have limits $r_1 \in E_1$, $r_2 \in E_2$. We have $\|(x_{1n} \boxplus x_{2n}) - (r_1 \boxplus r_2)\|_\alpha^+ = (\|x_{1n} - r_1\|_\alpha^{p,+} + \|x_{2n} - r_2\|_\alpha^{p,+})^{\frac{1}{p}} \longrightarrow 0$ as $n \to \infty$, showing that the original sequence has limit $r_1 \boxplus r_2$ in $E_1 \boxplus_p E_2$. The case $p = \infty$ is similar. □

Recall the fuzzy operator ideal definition from [6]:

Definition 21. *Let us assume that for every pair of the complete fuzzy normed spaces $(E, \widehat{\|\cdot\|})$ and $(F, \widehat{\|\cdot\|})$, a subset $\boldsymbol{F}\mathfrak{A}(E, F)$ is provided within $\boldsymbol{FB}(E, F)$. The class*

$$\boldsymbol{F}\mathfrak{A} := \bigcup_{E,F} \boldsymbol{F}\mathfrak{A}(E, F)$$

is defined as "a fuzzy operator ideal" if it satisfies the following conditions:

- $(\mathbf{I_0})$ *$f \boxdot e \in \boldsymbol{F}\mathfrak{A}(E, F)$, where $f \in \mathfrak{B}_{\boldsymbol{F}E^*}$ and $e \in F$.*
- $(\mathbf{I_1})$ *$S + T \in \boldsymbol{F}\mathfrak{A}(E, F)$, wherer S and $T \in \boldsymbol{F}\mathfrak{A}(E, F)$.*
- $(\mathbf{I_2})$ *$BT \in \boldsymbol{F}\mathfrak{A}(E, G)$, where $T \in \boldsymbol{F}\mathfrak{A}(E, F)$, and $B \in \boldsymbol{FB}(F, G)$. $\boldsymbol{F}\mathfrak{A}$ has nonzero fuzzy bounded linear operators, as implied by the constraint $(\mathbf{I_0})$.*

One fundamental important example of Definition 21 defined by [6] as follows. Let $1 < p < \infty$. When a fuzzy real number $\eta \in \mathscr{F}^+$ exists such that for every $(x_j)_{j=1}^m$ in E and $m \in \mathbb{N}$, the fuzzy bounded linear operator T from $(E, \widehat{\|\cdot\|})$ into $(F, \widehat{\|\cdot\|})$ is referred to as absolutely fuzzy p-summing, and the partial ordering

$$\left[\sum_{j=1}^{m} \widehat{\|Tx_j\|}^p\right]^{\frac{1}{p}} \preceq \eta \odot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^{m} \widehat{|f(x_j)|}^p\right]^{\frac{1}{p}} \tag{5}$$

holds. From $(E, \widehat{\|\cdot\|})$ into $(F, \widehat{\|\cdot\|})$, the class of all absolutely fuzzy p-summing operators is represented by $\mathbf{F}\mathfrak{P}_{\psi,p}(E, F)$. In this case, the absolutely fuzzy p-summing norm $\widehat{\mathsf{FP}_{\psi,p}(T)}$ of T is defined by, $[\widehat{\mathsf{FP}_{\psi,p}(T)}]_\alpha = [\mathsf{FP}_{\psi,p}(T)_\alpha^-, \mathsf{FP}_{\psi,p}(T)_\alpha^+]$ for all $\alpha \in (0, 1]$, where

$$\mathsf{FP}_{\psi,p}(T)_\alpha^- := \sup_{\beta<\alpha} \sup_{\substack{x_j\in E \\ x_j\neq 0}} \frac{\left[\sum\limits_{j=1}^{m} \|Tx_j\|_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum\limits_{j=1}^{m} |f(x_j)|_\beta^{p,-}\right]^{\frac{1}{p}}},$$

and

$$\mathsf{FP}_{\psi,p}(T)_\alpha^+ := \inf\left\{\eta_\alpha^+ : \text{Partial ordering (5) holds}\right\}.$$

3.1. **Basic examples of fuzzy operator ideals.**

3.1.1. *Interpolative simple fuzzy operator ideals.*

Definition 22. *Let* $\mathbf{F}\mathfrak{A}$ *be a fuzzy operator ideal, and let* $0 \le \theta < 1$. *A fuzzy bounded linear operator* T *from* E *into* F *is called interpolative simple fuzzy if there exist fuzzy real number* $\eta \in \mathscr{F}^+$, *fuzzy bounded linear operator* $S \in \mathbf{F}\mathfrak{A}(E,G)$ *and complete fuzzy normed space* G, *such that*

$$\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \preceq \eta^\theta \odot \widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}, \forall(\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph. \quad (6)$$

The class of all interpolative simple fuzzy operators from E *into* F *is denoted by* $\mathbf{F}\mathfrak{A}_\psi^\theta(E,F)$.

Proposition 23. *Let* $0 \le \theta < 1$. *If* $\mathbf{F}\mathfrak{A}$ *be a fuzzy operator ideal, then* $\mathbf{F}\mathfrak{A}_\psi^\theta$ *be also a fuzzy operator ideal.*

Proof. First to show that the algebraic condition of $(\mathbf{I_0})$. Let $(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$, $e \in F$ and $f \in \mathfrak{B}_{\mathbf{F}E^*}$ we get

$$\widehat{\left\|f \boxdot e\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \preceq \widehat{\|e\|}^\theta \odot \widehat{\left\|f \boxdot e\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}. \quad (7)$$

Since the fuzzy bounded linear operator $S := f \boxdot e \in \mathbf{F}\mathfrak{A}(E,F)$ and $\eta := \widehat{\|e\|} \in \mathscr{F}^+$, we have $f \boxdot e \in \mathbf{F}\mathfrak{A}_\psi^\theta(E,F)$. To prove the algebraic condition of $(\mathbf{I_1})$. Let T_1 and T_2 be in $\mathbf{F}\mathfrak{A}_\psi^\theta(E,F)$. There are fuzzy real numbers $\eta_i \in \mathscr{F}^+$, complete fuzzy normed spaces G_i and fuzzy bounded linear operators $S_i \in \mathbf{F}\mathfrak{A}(E,G_i)$, $i = 1,2$ such that

$$\widehat{\left\|T_i\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \preceq \eta_i^\theta \odot \widehat{\left\|S_i\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}, \forall(\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph. \quad (8)$$

for all $(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$. Put $\widehat{G} := G_1 \boxplus_1 G_2$ and the fuzzy bounded linear operator $S := J_1S_1 + J_2S_2 \in \mathbf{F}\mathfrak{A}(E,\widehat{G})$ and applying Theorem 5, we get for all $(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$

$$\begin{aligned}
\widehat{\left\|(T_1+T_2)\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} &\preceq \widehat{\left\|T_1\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \oplus \widehat{\left\|T_2\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \\
&\preceq \sum_{i=1}^{2}\eta_i^\theta \odot \widehat{\left\|S_i\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta} \\
&\preceq \left(\sum_{i=1}^{2}\widehat{\left\|S_i\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\right)^{1-\theta} \odot \left(\sum_{i=1}^{2}\eta_i\right)^{\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta} \\
&= (\eta_1 \oplus \eta_2)^\theta \odot \widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|_1}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}.
\end{aligned} \quad (9)$$

Since the fuzzy bounded linear operator $S \in \mathbf{F}\mathfrak{A}(E,\tilde{G})$ and $\eta := \eta_1 \oplus \eta_2 \in \mathscr{F}^+$, we obtain $T_1 + T_2 \in \mathbf{F}\mathfrak{A}_\psi^\theta(E,F)$. To show the algebraic condition of $(\mathbf{I_2})$. Suppose that $T \in \mathbf{F}\mathfrak{A}_\psi^\theta(E,F)$ and $B \in \mathbf{FB}(F,W)$.

Then

$$\begin{aligned}\widehat{\left\|BT\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} &\preceq \widehat{\|B\|}\odot\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\\ &\preceq \widehat{\|B\|}\odot\eta^{\theta}\odot\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}\\ &= \left(\widehat{\|B\|}^{\frac{1}{\theta}}\odot\eta\right)^{\theta}\odot\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta}\end{aligned}$$

Since $\gamma := \widehat{\|B\|}^{\frac{1}{\theta}}\odot\eta\in\mathscr{F}^{+}$ and a fuzzy bounded linear operator $S\in\mathbf{F}\mathfrak{A}(E,G)$, we obtain $BT\in\mathbf{F}\mathfrak{A}_{\psi}^{\theta}(E,W)$. □

Question 24. *Under which fuzzy norm the interpolative simple fuzzy operator ideal* $\mathbf{F}\mathfrak{A}_{\psi}^{\theta}$ *be a complete fuzzy normed space ?*

Proposition 25. *Let* $0\le\theta,\theta_1,\theta_2<1$. *Then the following holds.*

(**a**) *If* $\theta_1\le\theta_2$ *and* $\theta_2\neq 0$, *then* $\mathbf{F}\mathfrak{A}_{\psi}^{\theta_1}\subset\mathbf{F}\mathfrak{A}_{\psi}^{\theta_2}$.
(**b**) $\left(\mathbf{F}\mathfrak{A}_{\psi}^{\theta_1}\right)^{\theta_2}\subset\mathbf{F}\mathfrak{A}_{\psi}^{\theta_1+\theta_2-\theta_1\cdot\theta_2}$.

Proof. To prove (**a**), assume $T\in\mathbf{F}\mathfrak{A}_{\psi}^{\theta_1}(E,F)$ and $\epsilon>0$. Then

$$\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\preceq\eta^{\theta_1}\odot\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta_1}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta_1},\forall(\beta_k)_{k=1}^{\mathcal{N}}\text{ in }\Re\text{ and }\mathcal{N}\in\aleph,$$

holds for a suitable fuzzy real number $\eta\in\mathscr{F}^{+}$, fuzzy bounded linear operator $S\in\mathbf{F}\mathfrak{A}(E,G)$ and complete fuzzy normed space G. From our hypothesis we obtain

$$\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\preceq\left(\eta^{\frac{\theta_1}{\theta_2}}\odot\widehat{\|S\|}^{\frac{\theta_2-\theta_1}{\theta_2}}\right)^{\theta_2}\odot\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta_2}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta_2},$$

$\forall(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N}\in\aleph$. Since $\gamma:=\eta^{\frac{\theta_1}{\theta_2}}\odot\widehat{\|S\|}^{\frac{\theta_2-\theta_1}{\theta_2}}\in\mathscr{F}^{+}$ and a fuzzy bounded linear operator $S\in\mathbf{F}\mathfrak{A}(E,G)$, hence $T\in\mathbf{F}\mathfrak{A}_{\psi}^{\theta_2}(E,F)$.

To verify (**b**), let $T\in\left(\mathbf{F}\mathfrak{A}^{\theta_1}\right)_{\psi}^{\theta_2}(E,F)$. Then

$$\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\preceq\eta^{\theta_2}\odot\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta_2}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta_2},\forall(\beta_k)_{k=1}^{\mathcal{N}}\text{in }\Re\text{and }\mathcal{N}\in\aleph.\quad(10)$$

holds for a fuzzy real number $\eta\in\mathscr{F}^{+}$, a fuzzy bounded linear operator $S\in\mathbf{F}\mathfrak{A}_{\psi}^{\theta_1}(E,G)$ and suitable complete fuzzy normed space G

$$\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}\preceq\gamma^{\theta_1}\odot\widehat{\left\|R\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|}^{1-\theta_1}\odot\widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right\|}^{\theta_1},\forall(\beta_k)_{k=1}^{\mathcal{N}}\text{in }\Re\text{and }\mathcal{N}\in\aleph.\quad(11)$$

holds for a fuzzy real number $\gamma \in \mathscr{F}^{+}$, suitable complete fuzzy normed space $\widehat{G}$ and a fuzzy bounded linear operator $R \in \mathbf{F}\mathfrak{A}(E, \widehat{G})$. From (10) and (11) we have

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|} \preceq \eta^{\theta_2} \odot \gamma^{\theta_1 \cdot (1-\theta_2)} \odot \widehat{\left\| R\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|}^{(1-\theta_1)\cdot(1-\theta_2)} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta_1 \cdot (1-\theta_2)}$$
$$\odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta_2} \preceq \eta^{\theta_2+\theta_1-\theta_1\cdot\theta_2} \odot \widehat{\left\| R\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|}^{1-\theta_2-\theta_1+\theta_1\cdot\theta_2} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta_2+\theta_1-\theta_1\cdot\theta_2},$$

hence $T \in \mathbf{F}\mathfrak{A}_{\psi}^{\theta_1+\theta_2-\theta_1\cdot\theta_2}(E, F)$. □

Remark 26. *The following is a generalization of Definition 22. Suppose that $0 \leq \theta < 1$, let $\mathbf{F}\mathfrak{A}_{\psi}$ and $\mathbf{F}\mathfrak{B}_{L}$ be fuzzy operator ideals. A fuzzy bounded linear operator T belongs to $(\mathbf{F}\mathfrak{A}^{\theta}, \mathbf{F}\mathfrak{B}_{L}^{\theta})_{\psi}(E, F)$, from E into F, if there exist fuzzy real numbers $\eta, \gamma \in \mathscr{F}^{+}$, complete fuzzy normed spaces G_1, G_2, and fuzzy bounded linear operators $S_1 \in \mathbf{F}\mathfrak{A}(E, G_1)$, $S_2 \in \mathbf{F}\mathfrak{B}(E, G_2)$ such that*

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|} \preceq \eta^{\theta} \odot \widehat{\left\| S_1\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|}^{1-\theta} \odot \gamma^{1-\theta} \odot \widehat{\left\| S_2\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right) \right\|}^{\theta},$$

$\forall (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$.

3.1.2. *Absolutely simple fuzzy (p, θ)-summing operators.*

Definition 27. *Suppose that $0 \leq \theta < 1$ and $1 < p < \infty$. A fuzzy bounded linear operator T from $(E, \widehat{\|\cdot\|})$ into $(F, \widehat{\|\cdot\|})$ is called absolutely simple fuzzy (p, θ)-summing if there exists a fuzzy real number $\eta \in \mathscr{F}^{+}$ such that $\forall ((\beta_k^j)_{k=1}^{\mathcal{N}})_{j=1}^{m} \in \Re$, $\mathcal{N} \in \aleph$ and m belongs to $\mathbb{N}$, the partial ordering*

$$\left[\sum_{j=1}^{m} \widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right) \right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \preceq \eta \odot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^{m} \left(\widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right) \right|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|}^{\theta} \right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \tag{12}$$

holds. The notation $\mathbf{F}\mathfrak{P}_{\psi,p}^{\theta}(E, F)$ refers to the class of all absolutely simple fuzzy (p, θ)-summing operators that map from the space $(E, \widehat{\|\cdot\|})$ into the space $(F, \widehat{\|\cdot\|})$. In this case, the absolutely simple fuzzy (p, θ)-summing norm $\widehat{\mathbf{FP}_{\psi,p}^{\theta}(T)}$ of T is defined by, $[\widehat{\mathbf{FP}_{\psi,p}^{\theta}(T)}]_{\alpha} = [\mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{-}, \mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{+}]$ for all $\alpha \in (0, 1]$, where

$$\mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{-} := \sup_{\beta < \alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \\ 0 \neq \beta^j \in \Re \\ 1 \leq j \leq m}} \frac{\left[\sum_{j=1}^{m} \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right) \right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^{m} \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right) \right|_{\beta}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\beta}^{\theta,-} \right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}},$$

and

$$\mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{+} := \inf\left\{\eta_{\alpha}^{+} : \textit{Partial ordering (12) holds}\right\}.$$

Proposition 28. *Suppose that $0 \leq \theta < 1$ and $1 < p < \infty$. If $T \in \mathbf{F}\mathfrak{P}_{\psi,p}^{\theta}(E, F)$, then $\widehat{\mathbf{FP}_{\psi,p}^{\theta}(T)} \in \mathscr{F}$.*

Proof. First, we prove that for all $\alpha \in (0,1]$, $[\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}]_\alpha$ is a nonempty interval. Suppose that $\alpha \in (0,1]$, $\beta < \alpha$ and $((\beta^j_k)^{\mathcal{N}}_{k=1})^m_{j=1}$ in $\Re$, $m \in \mathbb{N}$ and $\mathcal{N} \in \aleph$. Based on our hypothesis, we have:

$$\frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_\beta^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \le \eta^-_\beta. \tag{13}$$

and

$\eta^-_\beta \le \eta^-_\alpha$. Since $\eta^-_\alpha \le \eta^+_\alpha$ we get $\dfrac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_\beta^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \le \eta^+_\alpha.$

Therefore

$$\begin{aligned}\mathbf{FP}^{\theta}_{\psi,p}(T)^-_\alpha &:= \sup_{\beta<\alpha}\sup_{\substack{\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\\0\neq\beta^j\in\Re\\1\le j\le m}}\frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_\beta^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\
&\le \inf\left\{\eta^+_\alpha : \text{Partial ordering (12) holds}\right\}\\
&=: \mathbf{FP}_{\psi,p}(T)^+_\alpha.\end{aligned}$$

Now, we demonstrate that the expression $[\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}]_\alpha$ satisfies the requirements outlined in [3, Lemma 2.9]:

(**i**) Suppose that $0 < \alpha_1 \le \alpha_2 \le 1$. To prove that $[\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}]_{\alpha_2} \subset [\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}]_{\alpha_1}$. First, we have

$$\begin{aligned}\mathbf{FP}^{\theta}_{\psi,p}(T)^-_{\alpha_1} &:= \sup_{\beta<\alpha_1}\sup_{\substack{\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\\0\neq\beta^j\in\Re\\1\le j\le m}}\frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_\beta^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\
&\le \sup_{\beta<\alpha_2}\sup_{\substack{\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\\0\neq\beta^j\in\Re\\1\le j\le m}}\frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_\beta^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_\beta^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_\beta^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\\
&=: \mathbf{FP}_{\psi,p}(T)^-_{\alpha_2}.\end{aligned} \tag{14}$$

Since $0 < \alpha_1 \leq \alpha_2 \leq 1$ we obtain $\eta^+_{\alpha_2} \leq \eta^+_{\alpha_1}$ and then

$$\mathsf{FP}^{\theta}_{\psi,p}(T)^+_{\alpha_2} := \inf\left\{\eta^+_{\alpha_2} : \text{Partial ordering (12) holds}\right\} \leq \inf\left\{\eta^+_{\alpha_1} : \text{Partial ordering (12) holds}\right\} =: \mathsf{FP}^{\theta}_{\psi,p}(T)^+_{\alpha_1}.$$

(ii) Suppose that $(\alpha_k)_{k\in\mathbb{N}}$ a sequence that increases in $(0,1]$ and converges to α. To prove that $[\lim\limits_{k\to\infty}\mathsf{FP}^{\theta}_{\psi,p}(T)^-_{\alpha_k}, \lim\limits_{k\to\infty}\mathsf{FP}^{\theta}_{\psi,p}(T)^+_{\alpha_k}] = [\mathsf{FP}^{\theta}_{\psi,p}(T)^-_{\alpha}, \mathsf{FP}^{\theta}_{\psi,p}(T)^+_{\alpha}]$. Since $\alpha_k \leq \alpha_{k+1} \leq \alpha$, we may conclude that

$$\sup_k \sup_{\beta<\alpha_k} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$
$$\leq \sup_{\beta<\alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}. \tag{15}$$

Assume that $\epsilon > 0$. Then $\beta_0 < \alpha$ exists in such a way that:

$$\sup_{\beta<\alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} - \epsilon$$
$$< \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_{\beta_0}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_{\beta_0}^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_{\beta_0}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.$$

Because $\alpha_k \nearrow \alpha$, there are $0 < k_0$, where $\beta_0 < \alpha_{k_0} \leq \alpha$. Then

$$\sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum\limits_{j=1}^{m}\left\|T\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right\|_{\beta_0}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup\limits_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum\limits_{j=1}^{m}\left(\left|f\left(\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right)\right|_{\beta_0}^{1-\theta,-}\cdot\left\|\sum\limits_{k=1}^{\mathcal{N}}\psi(\beta^j_k)b_k\right\|_{\beta_0}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$\leq \sup_{\beta<\alpha_{k_0}} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$\leq \sup_{k}\sup_{\beta<\alpha_{k}} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.$$

Therefore

$$\sup_{\beta<\alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} - \epsilon$$

$$< \sup_{k}\sup_{\beta<\alpha_{k}} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.$$

As $\epsilon \longrightarrow 0$, we have

$$\sup_{\beta<\alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$\leq \sup_{k}\sup_{\beta<\alpha_{k}} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}. \tag{16}$$

From Inequalities (15) and (16), we obtain

$$\lim_{k\to\infty}\sup_{\beta<\alpha_{k}} \sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\le j\le m}} \frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$= \sup_{k} \sup_{\beta<\alpha_k} \sup_{\substack{\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum_{j=1}^{m} \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum_{j=1}^{m} \left(\left|f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$= \sup_{\beta<\alpha} \sup_{\substack{\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k \\ 0\neq\beta^j\in\Re \\ 1\leq j\leq m}} \frac{\left[\sum_{j=1}^{m} \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum_{j=1}^{m} \left(\left|f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}.$$

Since $\alpha_k \leq \alpha$, we have

$$\begin{aligned}\mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{+} &:= \inf\left\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\right\} \leq \inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\} \\ &=: \mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha_k}^{+},\end{aligned} \tag{17}$$

hence

$$\mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{+} := \inf\left\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\right\} \leq \inf_{k}\inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\}. \tag{18}$$

Let $\epsilon > 0$. Then there is $\eta_0 \in \mathscr{F}$, where $\eta_{0\alpha}^{+} \leq \inf\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\} + \epsilon$. Since $\alpha_k \nearrow \alpha$, it follows that $\inf_{k} \eta_{0\alpha_k}^{+} = \eta_{0\alpha}^{+}$. So, there is $0 < k_0$ such that $\eta_{0\alpha_{k_0}}^{+} \leq \eta_{0\alpha}^{+} + \epsilon$. Therefore

$$\inf_{k}\inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\} \leq \inf\left\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\right\} + 2\epsilon.$$

As $\epsilon \longrightarrow 0$, we get

$$\inf_{k}\inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\} \leq \inf\left\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\right\}. \tag{19}$$

From Inequalities (18) and (19), we have

$$\begin{aligned}\lim_{k\to\infty}\inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\} &= \inf_{k}\inf\left\{\eta_{\alpha_k}^{+} : \text{Partial ordering (12) holds}\right\} \\ &= \inf\left\{\eta_{\alpha}^{+} : \text{Partial ordering (12) holds}\right\}.\end{aligned}$$

(iii) To prove that $-\infty < \mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{-} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_{\alpha}^{+} < \infty$, for all $\alpha \in (0,1]$.

Since

$$0 \leq \frac{\left[\sum_{j=1}^{m} \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right\|_{\beta}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum_{j=1}^{m} \left(\left|f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}},$$

for all nonzero $((\beta_k^j)_{k=1}^{\mathcal{N}})_{j=1}^m$ in $\Re$, $\mathcal{N} \in \aleph$ and all $\beta \in (0,1]$. Then $0 \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^-$. Let $\eta \in \mathscr{F}$ such that Partial ordering (12) holds $\forall((\beta_k^j)_{k=1}^{\mathcal{N}})_{j=1}^m$ in $\Re$, $\mathcal{N} \in \aleph$ and $m \in \mathbb{N}$. Consequently, for any α in the interval $(0,1]$, $\eta_\alpha^+ < \infty$. Hence $\mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^+ < \infty$. $\widehat{\mathbf{FP}_{\psi,p}^{\theta}(T)}$ is a fuzzy real number, as a result. □

Proposition 29. *Suppose that* $0 \leq \theta < 1$ *and* $1 < p < \infty$. *When* $T \in \mathbf{F}\mathfrak{P}_{\psi,p}^{\theta}(E,F)$, *we get*

$$\left[\sum_{j=1}^m \overline{\left\| T \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \preceq \widehat{\mathbf{FP}_{\psi,p}^{\theta}(T)} \odot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\overline{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|}^{1-\theta} \odot \overline{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|}^{\theta}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}},$$

$\forall((\beta_k^j)_{k=1}^{\mathcal{N}})_{j=1}^m \in \Re$, $m \in \mathbb{N}$ *and* $\mathcal{N} \in \aleph$.

Proof. Assume that the sequence $(\beta_k)_{k\in\mathbb{N}}$ increases in $(0,1]$ and converges to $\alpha \in (0,1]$. Since

$$\frac{\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|_{\beta_k}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}$$

$$\leq \sup_{\substack{\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \\ 0 \neq \beta^j \in \Re \\ 1 \leq j \leq m}} \frac{\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}}{\sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|_{\beta_k}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^-.$$

Then

$$\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^- \cdot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|_{\beta_k}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$$

Given that $\beta_k \nearrow \alpha$, [3, Lemma 2.9 (b)] implies that

$$\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_{\alpha}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}} = \lim_{k\to\infty} \left[\sum_{j=1}^m \left\| T \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right\|_{\beta_k}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}}$$

$$\leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^- \cdot \lim_{k\to\infty} \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|_{\beta_k}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\beta_k}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}$$

$$\leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^- \cdot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|_{\alpha}^{1-\theta,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k \right\|_{\alpha}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$$

Then

$$\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\alpha}^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}} \tag{20}$$

$$\leq \mathbf{FP}^{\theta}_{\psi,p}(T)^{-}_{\alpha}\cdot \sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\alpha}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\alpha}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$$

From our hypothesis we have

$$\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\alpha}^{\frac{p}{1-\theta},+}\right]^{\frac{1-\theta}{p}}\leq$$

$$\eta_{\alpha}^{+}\cdot \sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\alpha}^{1-\theta,+}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\alpha}^{\theta,+}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}.$$

Then

$$\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\alpha}^{\frac{p}{1-\theta},+}\right]^{\frac{1-\theta}{p}}\leq \inf\left\{\eta_{\alpha}^{+}: \text{Partial ordering (12) holds}\right\}$$

$$\cdot \sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\alpha}^{1-\theta,+}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\alpha}^{\theta,+}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}. \tag{21}$$

From Inequalities (20) and (21), we fulfill the requirement. □

Proposition 30. *Suppose that $1<p<\infty$ and $0\leq\theta<1$. If $T\in\mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$, then $\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}\preceq\eta$, where η defined in (12).*

Proof. Let $\alpha\in(0,1]$ and $\beta<\alpha$. From Inequality (13) we have

$$\mathbf{FP}^{\theta}_{\psi,p}(T)^{-}_{\alpha}:=\sup_{\beta<\alpha}\sup_{\substack{\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\\ 0\neq\beta^j\in\Re\\ 1\leq j\leq m}}\frac{\left[\sum_{j=1}^{m}\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|_{\beta}^{p,-}\right]^{\frac{1}{p}}}{\sup_{\widehat{\|f\|}\preceq\hat{1}}\left[\sum_{j=1}^{m}\left(\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|_{\beta}^{1-\theta,-}\cdot\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|_{\beta}^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}}\leq\eta_{\alpha}^{-}.$$

Since $\mathbf{FP}^{\theta}_{\psi,p}(T)^{+}_{\alpha}:=\inf\{\eta_{\alpha}^{+}: \text{Partial ordering (12) holds}\}\leq\eta_{\alpha}^{+}$, then $\mathbf{FP}^{\theta}_{\psi,p}(T)^{+}_{\alpha}\leq\eta_{\alpha}^{+}$. Thus, we conclude that $\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T)}\preceq\eta$. □

Proposition 31. *Let $0 \leq \theta < 1$ and $1 < p < \infty$. If $T \in \mathbf{F}\mathfrak{A}^{\theta}_{\psi}(E,F)$, then a regular probability measure μ defined on $\mathfrak{B}_{\mathbf{F}E^*}$ and a fuzzy real number $\eta \in \mathscr{F}^+$ exist such that:*

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \eta \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta} \right)^{\frac{p}{1-\theta}} d\mu(f) \right)^{\frac{1-\theta}{p}},$$

for all $(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$.

Proof. From our hypothesis there are fuzzy real number $\eta \in \mathscr{F}^+$, fuzzy bounded linear operator $S \in \mathbf{F}\mathfrak{A}_{\psi}(E,G)$ and complete fuzzy normed space G such that

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \eta^{\theta} \odot \widehat{\left\| S\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta}, \forall (\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph. \quad (22)$$

From [Zabalwy] there is a regular probability measure μ defined on $\mathfrak{B}_{\mathbf{F}E^*}$ such that

$$\widehat{\left\| S\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \widehat{\mathbf{FP}_{\psi,p}(S)} \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^{p} d\mu(f) \right)^{\frac{1}{p}}, \forall (\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph.$$

From (22) we have

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \eta^{\theta} \odot \left(\widehat{\mathbf{FP}_{\psi,p}(S)} \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^{p} d\mu(f) \right)^{\frac{1}{p}} \right)^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta},$$

$\forall (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$.

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \left(\eta^{\theta} \odot \widehat{\mathbf{FP}_{\psi,p}(S)}^{1-\theta} \right) \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta} \right)^{\frac{p}{1-\theta}} d\mu(f) \right)^{\frac{1-\theta}{p}},$$

for all $(\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$. □

Question 32. *What are the sufficient conditions to verify Proposition 31 ?*

We now prove the fundamental characterization of absolutely fuzzy (p,θ)-summing operators.

Theorem 33. *Suppose that $0 \leq \theta < 1$ and $1 < p < \infty$. A fuzzy bounded linear operator $T \in \mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$ if and only if a regular probability measure μ defined on $\mathfrak{B}_{\mathbf{F}E^*}$ and a fuzzy real number $\eta \in \mathscr{F}^+$ exist such that*

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|} \preceq \eta \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta} \right)^{\frac{p}{1-\theta}} d\mu(f) \right)^{\frac{1-\theta}{p}},$$

for all $(\beta_k)_{k=1}^{\mathcal{N}}$ *in* $\Re$ *and* $\mathcal{N} \in \aleph$.

Proof. Assume that the existence of such a measure μ and $\eta \in \mathscr{F}^+$. Then, given $m \in \mathbb{N}$, $((\beta_k^j)_{k=1}^{\mathcal{N}})_{j=1}^m$ in $\Re$, $\mathcal{N} \in \aleph$ and $\alpha \in (0,1]$,

$$\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_\alpha^{\frac{p}{1-\theta},-}\right]^{\frac{1-\theta}{p}} \leq \eta_\alpha^{p,-} \cdot \sum_{j=1}^m \int_{\mathfrak{B}_{\mathbf{F}E^*}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|^{1-\theta} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right\|_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} d\mu(f)$$

$$\leq \eta_\alpha^{p,-} \cdot \int_{\mathfrak{B}_{\mathbf{F}E^*}} \sum_{j=1}^m \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|^{1-\theta} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right\|_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} d\mu(f)$$

$$\leq \eta_\alpha^{p,-} \cdot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|^{1-\theta} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right\|_\alpha^{\theta,-}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}. \quad (23)$$

In the same fashion of Inequality (23), we obtain

$$\left[\sum_{j=1}^m \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right\|_\alpha^{\frac{p}{1-\theta},+}\right]^{\frac{1-\theta}{p}} \leq \eta_\alpha^{p,+} \cdot \sup_{\widehat{\|f\|} \preceq \hat{1}} \left[\sum_{j=1}^m \left(\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right)\right|^{1-\theta} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k^j) b_k\right\|_\alpha^{\theta,+}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}. \quad (24)$$

From Inequalities (23) and (24), we obtain $T \in \mathbf{F}\mathfrak{P}_{\psi,p}(E,F)$ with $\widehat{\mathsf{FP}^\theta_{\psi,p}(T)} \preceq \eta$. Conversely, suppose that $T \in \mathbf{F}\mathfrak{P}^\theta_{\psi,p}(E,F)$. For every finite subset M of E, the fuzzy real number-valued function Ψ_M on $\mathfrak{B}_{\mathbf{F}E^*}$ defined by

$$\Psi_M(f) := \sum_{x\in M}\left(\widehat{\mathsf{FP}^\theta_{\psi,p}(T)}^{\frac{p}{1-\theta}} \odot \widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|}^p \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right\|}^{\frac{p\theta}{1-\theta}} \ominus \widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|}^{\frac{p}{1-\theta}}\right).$$

For each $\alpha \in (0,1]$, the fuzzy number $\Psi_M(f)$'s α-level set is defined by:

$$[\Psi_M(f)]_\alpha = [\Psi^-_{M,\alpha}(f), \Psi^+_{M,\alpha}(f)],$$

where

$$\Psi^-_{M,\alpha}(f) := \sum_{x\in M}\left(\mathsf{FP}^\theta_{\psi,p}(T)_\alpha^{\frac{p}{1-\theta},-} \cdot \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|_\alpha^{p,-} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right\|_\alpha^{\frac{p\theta}{1-\theta},-} - \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|_\alpha^{\frac{p}{1-\theta},+}\right), \quad (25)$$

and

$$\Psi^+_{M,\alpha}(f) := \sum_{x\in M}\left(\mathsf{FP}^\theta_{\psi,p}(T)_\alpha^{\frac{p}{1-\theta},+} \cdot \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right|_\alpha^{p,+} \cdot \left\|\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right\|_\alpha^{\frac{p\theta}{1-\theta},+} - \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k\right)\right\|_\alpha^{\frac{p}{1-\theta},-}\right). \quad (26)$$

Since for every $x \in M$ the functions $R_x : \mathfrak{B}_{\mathbf{F}E^*} \longrightarrow \mathbb{R}$, $R_x(f) := |f(x)|^p$, are continuous on $\mathfrak{B}_{\mathbf{F}E^*}$, it is plain that the functions $\Psi^-_{M,\alpha}$ and $\Psi^+_{M,\alpha}$ defined in (25) and (26) belong to $C(\mathfrak{B}_{\mathbf{F}E^*})$, respectively. Since $T \in \mathbf{F}\mathfrak{P}^\theta_{\psi,p}(E,F)$ and $\Psi^-_{M,\alpha}(f) \leq \Psi^+_{M,\alpha}(f)$ hence $\sup_{\widehat{\|f\|} \preceq \hat{1}} \Psi^-_{M,\alpha}(f) \geq 0$ and $\sup_{\widehat{\|f\|} \preceq \hat{1}} \Psi^+_{M,\alpha}(f) \geq 0$. Note that $B^-_\alpha := \left\{\Psi^-_{M,\alpha} : M \subset E\right\}$ and $B^+_\alpha := \left\{\Psi^+_{M,\alpha} : M \subset E\right\}$ be the convex subsets of $C(\mathfrak{B}_{\mathbf{F}E^*})$ for every

$\alpha \in (0,1]$. Let $A := \left\{ \Psi \in C(\mathfrak{B}_{\mathbf{F}E^*}) : \sup_{\widehat{\|f\|} \preceq \hat{1}} \Psi(f) < 0 \right\}$ of $C(\mathfrak{B}_{\mathbf{F}E^*})$ be an open convex subset. Since $A \cap B_\alpha^- = \phi$ and $A \cap B_\alpha^+ = \phi$ for every $\alpha \in (0,1]$, First Separation Theorem, and Riesz Representation Theorem ensure the existence of regular Borel measure μ on $\mathfrak{B}_{\mathbf{F}E^*}$, and of constants $r_1, r_2 \in \mathbb{R}$ such that for each $\alpha \in (0,1]$ we have

$$\langle \mu, \Psi \rangle < r_1 \leq \left\langle \mu, \Psi_{M,\alpha}^- \right\rangle, \forall\, (\Psi, \Psi_{M,\alpha}^-) \in A \times B_\alpha^-, \tag{27}$$

and

$$\langle \mu, \Psi \rangle < r_2 \leq \left\langle \mu, \Psi_{M,\alpha}^+ \right\rangle, \forall\, (\Psi, \Psi_{M,\alpha}^+) \in A \times B_\alpha^+. \tag{28}$$

Because of 0 belongs to B_α^- and B_α^+, then we have $r_1 \leq 0$ and $r_2 \leq 0$, respectively. Given that A encompasses the constant functions represented by the negative real numbers, we obtain $r_1 = r_2 = 0$ and $\mu(\mathfrak{B}_{\mathbf{F}E^*}) > 0$. Without losing generality, one can also presume $\|\mu\| = 1$. From (27) one gets

$$0 \leq \left\langle \mu, \Psi_{\{x\},\alpha}^- \right\rangle =$$

$$\int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},-} \cdot \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right|_\alpha^{p,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|_\alpha^{\frac{p\theta}{1-\theta},-} - \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},+} \right) d\mu(f),$$

$\forall\, (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$. Since $\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},-} \leq \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},+}, \forall\, \alpha \in (0,1]$ we obtain

$$\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},-} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},-} \int_{\mathfrak{B}_{\mathbf{F}E^*}} \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right|_\alpha^{p,-} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|_\alpha^{\frac{p\theta}{1-\theta},-} d\mu(f), \tag{29}$$

$\forall\, (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$. Also from (28) we have $0 \leq \left\langle \mu, \Psi_{\{x\},\alpha}^+ \right\rangle$

$$\left\langle \mu, \Psi_{\{x\},\alpha}^+ \right\rangle = \int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},+} \cdot \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right|_\alpha^{p,+} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|_\alpha^{\frac{p\theta}{1-\theta},+} - \left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},-} \right) d\mu(f),$$

$\forall\, (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$. Since $\mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},-} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},+}, \forall\, \alpha \in (0,1]$ we obtain

$$\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|_\alpha^{\frac{p}{1-\theta},+} \leq \mathbf{FP}_{\psi,p}^{\theta}(T)_\alpha^{\frac{p}{1-\theta},+} \int_{\mathfrak{B}_{\mathbf{F}E^*}} \left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right|_\alpha^{p,+} \cdot \left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|_\alpha^{\frac{p\theta}{1-\theta},+} d\mu(f), \tag{30}$$

$\forall\, (\beta_k)_{k=1}^{\mathcal{N}}$ in $\Re$ and $\mathcal{N} \in \aleph$.
From Inequalities (29) and (30), we get

$$\widehat{\left\| T\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right\|} \preceq \eta \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \left(\widehat{\left| f\left(\sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right) \right|}^{1-\theta} \odot \widehat{\left\| \sum_{k=1}^{\mathcal{N}} \psi(\beta_k) b_k \right\|}^{\theta} \right)^{\frac{p}{1-\theta}} d\mu(f) \right)^{\frac{1-\theta}{p}}, \forall\, (\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph.$$

□

From Theorem 33 we obtain the following inclusion result.

Proposition 34. *If $p_1 \leq p_2$, then* $\left[\boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p_1}, \widehat{\boldsymbol{F}\mathsf{P}^{\theta}_{\psi,p_1}(\cdot)}\right] \subseteq \left[\boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p_2}, \widehat{\boldsymbol{F}\mathsf{P}^{\theta}_{\psi,p_2}(\cdot)}\right]$.

Proposition 35. *Suppose that $0 \leq \theta < 1$ and $1 < p < \infty$. If $\boldsymbol{F}\mathfrak{A}_{\psi}$ be a fuzzy operator ideal, then $\boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p}$ be fuzzy operator ideal.*

Proof. To prove that the conditions of Definition 21. From our hypothesis and by Proposition 31 and Theorem 33, Condition $(\mathbf{I_0})$ is satisfied. To prove that Condition $(\mathbf{I_1})$. Let S and $T \in \mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$. From Theorem 11 and our hypothesis we obtain

$$\left[\sum_{j=1}^{m}\widehat{\left\|T+S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \preceq \left[\sum_{j=1}^{m}\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \oplus \left[\sum_{j=1}^{m}\widehat{\left\|S\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}$$
$$\preceq (\eta_1 \oplus \eta_2) \odot \sup_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum_{j=1}^{m}\left(\widehat{\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|}^{\theta}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}$$

To show that Condition $(\mathbf{I_2})$. Let $T \in \mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$ and $B \in \mathbf{FB}(F,G)$ we have

$$\left[\sum_{j=1}^{m}\widehat{\left\|BT\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}} \preceq \widehat{\|B\|} \odot \left[\sum_{j=1}^{m}\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right\|}^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}$$
$$\preceq \widehat{\|B\|} \odot \eta \odot \sup_{\widehat{\|f\|}\preceq\hat{1}} \left[\sum_{j=1}^{m}\left(\widehat{\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right)\right|}^{1-\theta} \odot \widehat{\left\|\sum_{k=1}^{\mathcal{N}}\psi(\beta_k^j)b_k\right\|}^{\theta}\right)^{\frac{p}{1-\theta}}\right]^{\frac{1-\theta}{p}}. \tag{31}$$

□

Remarks 36. (1) *If $\theta = 0$, then $\boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$ coincides with the class $\boldsymbol{F}\mathfrak{P}_{\psi,p}(E,F)$ for $1 < p < \infty$.*

(2) *Let $0 \leq \theta < 1$ and $1 < p < \infty$. It can be easily checked that the inclusion $\boldsymbol{F}\mathfrak{P}_{\psi,p}(E,F) \subset \boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F) \subset \boldsymbol{FB}(E,F)$ with $\widehat{\|T\|} \preceq \widehat{\boldsymbol{F}\mathsf{P}^{\theta}_{\psi,p}(T)} \preceq \widehat{\boldsymbol{F}\mathsf{P}_{\psi,p}(T)}$.*

Proposition 37. *If $0 \leq \theta < 1$ and $1 < p < \infty$, then $\boldsymbol{F}\mathfrak{P}_{\psi,\frac{p}{1-\theta}} \subset \boldsymbol{F}\mathfrak{P}^{\theta}_{\psi,p}$.*

Proof. Suppose that $T \in \mathbf{F}\mathfrak{P}_{\psi,\frac{p}{1-\theta}}(E,F)$. Then a regular probability measure μ defined on $\mathfrak{B}_{\mathbf{F}E^*}$ exists so that

$$\widehat{\left\|T\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right\|} \preceq \widehat{\mathbf{F}\mathsf{P}_{\psi,\frac{p}{1-\theta}}(T)} \odot \left(\int_{\mathfrak{B}_{\mathbf{F}E^*}} \widehat{\left|f\left(\sum_{k=1}^{\mathcal{N}}\psi(\beta_k)b_k\right)\right|}^{\frac{p}{1-\theta}} d\mu(f)\right)^{\frac{1-\theta}{p}}, \ \forall\ (\beta_k)_{k=1}^{\mathcal{N}} \text{ in } \Re \text{ and } \mathcal{N} \in \aleph.$$

It follows that Theorem 31 holds with $\eta := \widehat{\mathbf{F}\mathsf{P}_{\psi,\frac{p}{1-\theta}}(T)}$ and $\frac{p}{1-\theta} := \frac{p\cdot\theta}{1-\theta} + p$. Thus $T \in \mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E,F)$ and $\widehat{\mathbf{F}\mathsf{P}^{\theta}_{\psi,p}(T)} \preceq \widehat{\mathbf{F}\mathsf{P}_{\psi,\frac{p}{1-\theta}}(T)}$. □

4. Conclusions

We have begun developing a new theory of fuzzy operator ideals between completely fuzzy normed spaces as a result of the theory of fuzzy functional analysis's steady rise in recent years. The primary idea of the manuscript is to connect simple concepts with the fuzzy-bounded linear operators and fuzzy operator ideal theories. We have studied new classes of fuzzy operator ideals that are interpolative simple fuzzy bounded linear operators and absolutely simple fuzzy (p, θ)-summing operators, $0 \leq \theta < 1$ and $1 < p < \infty$, between arbitrary complete fuzzy normed spaces. The domination theorem's fuzziness is of utmost importance, and its explanation utilizes the abstract fuzzy version of the Pietsch domination theorem. The theory of interpolative simple fuzzy bounded linear operators will lead to several generalizations to linear contexts, which will be beneficial. The classes listed above, such as fuzzy integral maps, fuzzy fuzzy nuclear maps, and duality for fuzzy p-summing maps, will soon catch the attention of several academics attempting to develop a parallel theory to the fuzzy one.

5. Open Problems

(1) Is the composition formula

$$\widehat{\mathbf{FP}^{\theta}_{\psi,p}(T \circ S)} \preceq \widehat{\mathbf{FP}^{\theta}_{\psi,r}(T)} \odot \widehat{\mathbf{FP}^{\theta}_{\psi,s}(S)} \tag{32}$$

true for arbitrary absolutely simple fuzzy (r, θ)-summing operators T, absolutely simple fuzzy (s, θ)-summing operators S and $\frac{1}{p} \leq (\frac{1}{r} + \frac{1}{s}) \wedge 1$?

(2) What is $\mathbf{F}\mathfrak{P}^{\theta}_{\psi,p}(E, F)$'s dual when F is a complete fuzzy normed space and E is a finite-dimensional fuzzy normed space?

(3) Describe and analyze an algorithm to obtain the absolutely simple fuzzy (p, θ)-summing norm of fuzzy bounded linear operators between finite-dimensional fuzzy normed spaces exactly.

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