Asia Pac. J. Math. 2026 **13**:38

ON THE GEODETIC DOMINATION OF COMPLETE DEGREE SPLITTING GRAPHS

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Received Jan. 9, 2026

Abstract. Let $G = (V, E)$ be a graph whose vertex set is partitioned into degree classes $X_1, X_2, \ldots, X_{\pi(G)}$, where vertices in each class have the same degree. The *complete degree splitting graph* $\mathcal{CDS}(G)$ is obtained from G by adding a new vertex corresponding to each degree class and joining it to all vertices in that class. A set $S \subseteq V$ is called a *geodetic dominating set* if it is both a geodetic set and a dominating set. The minimum cardinality of such a set is called the *geodetic domination number*, denoted by $\gamma_g(G)$.

In this paper, we investigate the geodetic domination number of complete degree splitting graphs. Explicit values of $\gamma_g(\mathcal{CDS}(G))$ are determined for several special classes of graphs. In addition, we establish results for the complete degree splitting graphs arising from the *corona* of two graphs. These results contribute to the understanding of how degree-based graph transformations affect geodetic domination parameters.

2020 Mathematics Subject Classification. 05C12; 05C69.

Key words and phrases. geodetic; domination; geodetic domination; complete degree splitting.

1. Introduction

Throughout this paper, we consider finite simple graphs, that is, graphs without loops and multiple edges. For a graph G, the vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The order of G is $n(G) = |V(G)|$ and its size is $p(G) = |E(G)|$.

For a vertex $v \in V(G)$, the open neighborhood of v is

$$N(v) = \{u \in V(G) : uv \in E(G)\},$$

and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $\deg(v)$, is given by $|N(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Given two graphs G_1 and G_2, their union $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Their join $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding all edges joining each vertex of

DOI: 10.28924/APJM/13-38

G_1 to every vertex of G_2. The corona $G_1 \circ G_2$ is constructed by taking one copy of G_1 and, for each vertex of G_1, attaching a copy of G_2 and joining that vertex to every vertex in the corresponding copy.

We use standard notation for classical graph families: P_n for the path of order $n \geq 2$, C_n for the cycle of order $n \geq 3$, K_n for the complete graph of order n, $K_{m,n}$ for the complete bipartite graph, S_n for the star of order n, and W_n for the wheel of order $n \geq 4$.

If G is connected, the distance $d(x, y)$ between vertices x and y is the length of a shortest x–y path. The diameter of G is defined by

$$\operatorname{diam}(G) = \max_{x,y \in V(G)} d(x, y).$$

A shortest x–y path is called an x–y geodesic. The interval between vertices u and v, denoted $I[u, v]$, consists of all vertices lying on some u–v geodesic. For $S \subseteq V(G)$, define

$$I[S] = \bigcup_{u,v \in S} I[u, v].$$

A set $S \subseteq V(G)$ is a geodetic set if $I[S] = V(G)$. The minimum cardinality of a geodetic set is called the geodetic number and is denoted by $g(G)$ [3,9]. Further developments appear in [3].

A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of such a set is the domination number, denoted by $\gamma(G)$ [11].

A set that is both geodetic and dominating is called a geodetic dominating set. The minimum cardinality of such a set is the geodetic domination number, denoted by $\gamma_g(G)$ [6,10].

Complete Degree Splitting Graphs. Let $G = (V, E)$ be a connected graph of order $n \geq 2$. Let $\pi(G)$ denote the number of distinct vertex degrees in G. If $p_1, p_2, \ldots, p_{\pi(G)}$ are the distinct degrees, define

$$X_i = \{x \in V(G) : \deg_G(x) = p_i\}, \qquad 1 \leq i \leq \pi(G).$$

Note that $X_i \cap X_j = \varnothing$ whenever $i \neq j$, and $V(G) = \bigcup_{i=1}^{\pi(G)} X_i$. The collection $\mathcal{P} = \{X_i : 1 \leq i \leq \pi(G)\}$ is called the degree partition of G.

The complete degree splitting graph of G, denoted by $CDS(G)$, is obtained by adding a new vertex w_i corresponding to each degree class X_i and joining w_i to every vertex in X_i. This construction is based on the degree splitting graph [13] and its complete version [5].

Motivation and Context. The geodetic number and domination number have been extensively studied due to their theoretical importance and practical relevance. The geodetic number was introduced in [3,9] and further developed in [3], while domination theory has been systematically presented in [11]. The geodetic domination number, introduced in [6,10], integrates these two notions by requiring simultaneous domination and shortest-path coverage.

In practical network settings such as communication and sensor networks, monitoring schemes must ensure complete vertex coverage while also maintaining control over shortest-path routing structures.

Wireless sensor network design emphasizes coverage and routing efficiency [1]. Moreover, studies of real-world networks highlight the importance of shortest-path organization and community structure in understanding connectivity and clustering [7,12]. These considerations naturally align with the concept of geodetic domination.

Graph transformations that encode structural heterogeneity are also of interest. Degree-based clustering and modular organization have been widely studied in complex networks [7,12]. The degree splitting graph and its complete version provide a framework for analyzing how classical graph parameters behave under degree-based augmentation [5,13].

Despite substantial work on geodetic and domination parameters in classical graph families [4,6,10], their behavior under degree-based transformations, particularly in the complete degree splitting graph, remains largely unexplored. This motivates our investigation of the geodetic domination number of $CDS(G)$ for several standard graph families and for graphs arising from the corona operation.

In Figure 1, a graph G and its $\mathcal{CDS}(G)$ are shown.

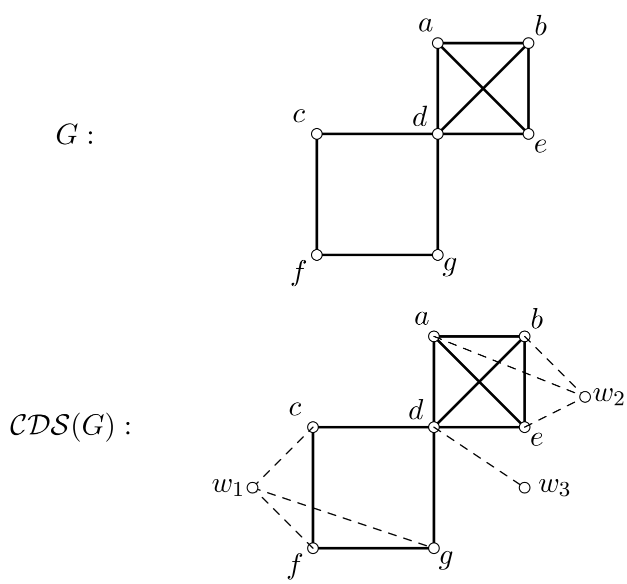

FIGURE 1. Here, $X_1 = \{c, f, g\}$, $X_2 = \{a, b, e\}$, and $X_3 = \{d\}$.

2. Some Basic Results

The bounds in the following observation are immediate by the definitions.

Observation 2.1. [6,10] If G is a connected graph of order $n \geq 2$, then

$$2 \leq \max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n.$$

Theorem 2.1. [6] *Let G be a connected graph of order $n \geq 2$. Then:*

(a) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *of G such that* $d(u, v) \leq 3$*,*

(b) *$\gamma_g(G) = n$ if and only if G is the complete graph on n vertices,*

(c) *$\gamma_g(G) = n-1$ if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and $G - v$ is the union of at least two complete graphs.*

The geodetic domination numbers of some standard graphs can be easily found, and are given in [4] as follows:

2.1 The path P_n of n vertices has $\gamma_g(P_n) = \left\lceil \frac{n+2}{3} \right\rceil$.

2.2 The cycle C_n of n vertices has $\gamma_g(C_n) = \left\lceil \frac{n}{3} \right\rceil, \quad n \geq 6$.

2.3 The complete graph K_n of n vertices has $\gamma_g(K_n) = n$.

2.4 The complete bipartite graph $K_{m,n}$ on $m+n$ vertices with $m, n \geq 2$ has $\gamma_g(K_{m,n}) = \min\{m, n, 4\}$.

2.5 The star graph S_n of n vertices has $\gamma_g(S_n) = n-1$.

2.6 The wheel graph W_n of n vertices has $\gamma_g(W_n) = \left\lceil \frac{n-1}{2} \right\rceil, \quad n \geq 5$.

3. Geodetic Domination of Complete Degree Splitting Graph of Some Special Graphs

In this section, we present results on the geodetic domination number of the complete degree splitting graph for certain classes of graphs.

Theorem 3.1. *For $n \geq 6$, the geodetic domination number $\gamma_g(\mathcal{CDS}(P_n))$ of the complete degree splitting graph of a path P_n is given by:*

$$\gamma_g(\mathcal{CDS}(P_n)) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Let P_n be a path with $V(P_n) = \{v_i : 1 \leq i \leq n\}$, $n \geq 6$. Let $\mathcal{P} = \{X_1, X_2\}$ be the degree partition of P_n where $X_1 = \{v_1, v_n\}$ and $X_2 = \{v_i : 2 \leq i \leq n-1\}$. The complete degree splitting graph of a path P_n, denoted by $\mathcal{CDS}(P_n)$, is obtained by adding vertices w_1 and w_2 such that $V(\mathcal{CDS}(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $E(\mathcal{CDS}(P_n)) = E(P_n) \cup \{w_1v_1, w_1v_n\} \cup \{w_2u : u \in X_2\}$.

Let $S = \{v_1, v_n\} \cup \{v_{2m} : 2 \leq m \leq \left\lfloor \frac{n-2}{2} \right\rfloor\}$.

Let $x \in V(\mathcal{CDS}(P_n)) - S$. Then $x \in \{v_2, v_3, w_1, w_2\} \cup \{v_{2m+1} : 2 \leq m \leq \left\lfloor \frac{n-2}{2} \right\rfloor\} \cup \{v_{n-1}\}$.

Suppose $x = v_2$ or $x = v_3$. Then $[v_1, v_2, v_3, v_4]$ is a geodesic path containing x with end vertices in S. If $x = w_1$ or $x = w_2$, then $[v_1, w_1, v_n]$ and $[v_1, v_2, w_2, v_4]$ are geodesic paths containing w_1 and w_2, respectively. Next, suppose $x \in \{v_{2m+1} : 2 \leq m \leq \left\lfloor \frac{n-2}{2} \right\rfloor\}$. Then $[v_{2m}, v_{2m+1}, v_{2m+2}]$ is a geodesic path containing x with end vertices in S. Lastly, if n is odd and $x = v_{n-1}$, then $[v_{2p}, w_2, v_{n-1}, v_n]$ where $p = \left\lfloor \frac{n-2}{2} \right\rfloor$ is a geodesic path containing x with end vertices in S. Hence, $I[S] = V(\mathcal{CDS}(P_n))$. In effect, S is a geodetic set in $\mathcal{CDS}(P_n)$.

Moreover, suppose $x = v_2$. Then v_2 is dominated by $v_1 \in S$. If $x = v_3$, then v_3 is dominated by $v_4 \in S$. If $x = w_1$ or $x = w_2$ or $x = v_{n-1}$, then v_1, v_{2m} and $v_n \in S$ dominates w_1, w_2 and v_{n-1}, respectively. Next, suppose $x \in \{v_{2m+1} : 2 \leq m \leq \lfloor \frac{n-2}{2} \rfloor\}$. Note that $d(v_{2m}, v_{2m+1}) = 1$ and $v_{2m} \in S$ for all $2 \leq m \leq \lfloor \frac{n-2}{2} \rfloor$. This implies that v_{2m+1} is dominated by $v_{2m} \in S$ for all $m, 2 \leq m \leq \lfloor \frac{n-2}{2} \rfloor$. Hence, S is a dominating set.

Accordingly, S is a geodetic and dominating set. Thus,

$$\gamma_g(\mathcal{CDS}(P_n)) \leq |S| = 2 + \left(\left\lfloor \frac{n-2}{2} \right\rfloor - 1 \right) = \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor.$$

Suppose S^* is a geodetic dominating set in $\mathcal{CDS}(P_n)$ such that $|S^*| < |S| = \lfloor \frac{n}{2} \rfloor$. Let $m = \lfloor \frac{n}{2} \rfloor$. Let $A_i = \{v_{2i-1}, v_{2i}\}$ for $1 \leq i < m$, and let $A_m = \{v_{n-1}, v_n\}$ if n is even and $A_m = \{v_{n-2}, v_{n-1}, v_n\}$ if n is odd. Consider the following cases:

Case 1. $A_j \cap S^* = \emptyset$, $1 < j < m$.

Then $v_{2j-1}, v_{2j} \notin S^*$. Since S^* is a geodetic set, there exist $a, b \in S^*$ such that $v_{2j-1},\ v_{2j} \in I[a, b]$. Suppose $a = v_{i_1}$ and $b = v_{i_2}$, where $i_1 < 2j - 1$ and $i_2 > 2j$. This is not possible since if $a \in A_1$ and $b \in A_m$, then $I[a, b] \subseteq A_1 \cup A_m \cup \{w_1, w_2\}$; if $a \in A_1$ and $b \notin A_m$, then $I[a, b] \subseteq A_1 \cup \{w_2, b\}$; if $a \notin A_1$ and $b \in A_m$, then $I[a, b] \subseteq A_m \cup \{a, w_2\}$; if $a \notin A_1$ and $b \notin A_m$, then $I[a, b] = \{a, w_2, b\}$. Suppose $a = w_2$. Then $b = v_{2j}$. This is also not possible since $b \in S^*$ and $v_{2j} \notin S^*$. Suppose $b = w_2$. Then $a = v_{2j-1}$. Again, this is not possible since $a \in S^*$ and $v_{2j-1} \notin S^*$. Suppose $a = w_1$ and $b = v_k$, $1 \leq k \leq n$. Note that $d(a, v_{2j}) = 4$. Since $v_{2j} \in I[a, b]$, $d(a, b) > 4$. This is a contradiction.

Case 2. $A_1 \cap S^* = \emptyset$.

Since S^* is a dominating set, we must have $w_1, w_2 \in S^*$. Moreover, because $|S^*| < \lfloor \frac{n}{2} \rfloor = m$ and $w_1, w_2 \in S^*$, there exists some j with $1 < j < m$ such that $A_j \cap S^* = \emptyset$. By Case 1, this is not possible.

Case 3. $A_m \cap S^* = \emptyset$.

This case is similar to Case 2.

Therefore, for $n \geq 6$, $\gamma_g(\mathcal{CDS}(P_n)) = \lfloor \frac{n}{2} \rfloor$ indeed. □

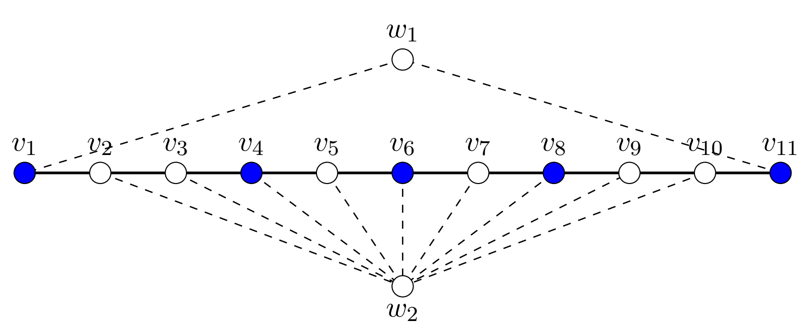

Figure 2. The complete degree splitting graph of path of order 11 $\mathcal{CDS}(P_{11})$ with geodetic domination number $\gamma_g(\mathcal{CDS}(P_{11})) = 5$.

Remark 3.2. *It is known from Section 2 that the geodetic domination number of the wheel graph is* $\gamma_g(W_n) = \lceil \frac{n-1}{2} \rceil$ *for* $n \geq 5$. *Since* $\mathcal{CDS}(C_n) \cong W_{n+1}$, *it follows immediately that*

$$\gamma_g(\mathcal{CDS}(C_n)) = \lceil \tfrac{n}{2} \rceil, \quad n \geq 4.$$

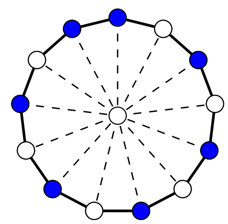

FIGURE 3. The complete degree splitting graph of a cycle of order 13 $\mathcal{CDS}(C_{13})$ with geodetic domination number $\gamma_g(\mathcal{CDS}(C_{13})) = 7$.

Remark 3.3. *It is known from Section 2 that the geodetic domination number of the complete graph is* $\gamma_g(K_n) = n$. *Since* $\mathcal{CDS}(K_n) \cong K_{n+1}$, *it follows immediately that*

$$\gamma_g(\mathcal{CDS}(K_n)) = n + 1.$$

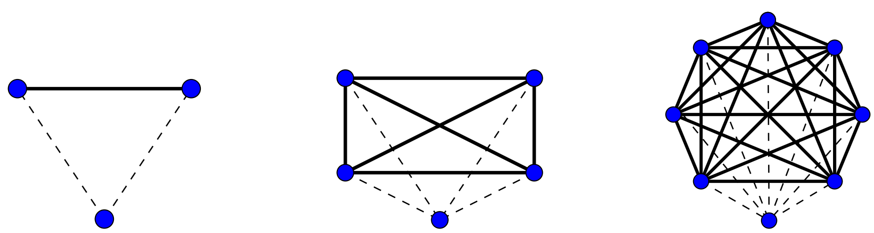

FIGURE 4. The complete degree splitting graph of complete graphs $\mathcal{CDS}(K_2)$, $\mathcal{CDS}(K_4)$, and $\mathcal{CDS}(K_7)$ with geodetic domination number 3, 5, and 8, respectively.

Theorem 3.4. *For non-negative integers* m *and* n,

$$\gamma_g(\mathcal{CDS}(K_{m,n})) = \begin{cases} 3 & \text{if } m = n = 1, \\ \min\{m, n, 4\} & \text{if } m = n \text{ and } m, n \neq 1, \\ 2 & \text{if } m \neq n. \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph. Let $Y = \{y_1, y_2, ..., y_m\}$ and $Z = \{z_1, z_2, ..., z_n\}$ be the partite sets of $K_{m,n}$, where $m \leq n$.

If $m = n = 1$, then $\mathcal{CDS}(K_{1,1})$ is obtained by adding one vertex w_1 and each vertex y_1, z_1 is joined to w_1. Hence, $\mathcal{CDS}(K_{1,1})$ is a complete graph of order 3. By virtue of Theorem 2.1(b), $\gamma_g(\mathcal{CDS}(K_{1,1})) = 3$.

If $m = n$ and $m, n \neq 1$, then $\mathcal{CDS}(K_{m,n})$ is obtained by adding one vertex w_1 and each vertex in Y and Z is joined to w_1. First, consider $2 \leq m \leq 3$. When $m = 2$ or 3, S=Y is a minimum geodetic dominating set of G. Now, let $m \geq 4$ and let $S' = \{y_1, y_2, z_1, z_2\}$. Since $I[S'] = V(\mathcal{CDS}(K_{m,n})) = V$, it follows that S' is a geodetic set of $\mathcal{CDS}(K_{m,n}),\ m = n \geq 4$. It will also follow that if $x \in V - S'$, then S' dominates x. Hence, S' is a geodetic dominating set of $\mathcal{CDS}(K_{m,n})$ where $m = n$ and $m \geq 4$. It remains to show that if S^* is a 3-element subset of V, then $I[S^*] \neq V$. Assume that S^* is a subset of Y or Z, say the former. Then $I[S^*] = S^* \cup Z \cup \{w_1\} \neq V$. In effect, we may take that $S^* \cap Y = \{y_i, y_j\}$ and $S^* \cap Z = \{z_k\}$. Then, $I[S^*] = \{y_i, y_j\} \cup Z \cup \{w_1\} \neq V$. Consequently, S' is the minimum geodetic dominating set of $\mathcal{CDS}(K_{m,n})$, where $m = n$ and $m \geq 4$. Therefore, the geodetic domination number of $\mathcal{CDS}(K_{m,n})$ where $m = n$ and $m, n \neq 1$ is $\gamma_g(\mathcal{CDS}(K_{m,n})) = \min\{m, n, 4\}$.

Lastly, suppose $m \neq n$. Then $\mathcal{CDS}(K_{m,n})$ is obtained by adding two vertices w_1, w_2 and each vertex in Y is joined to w_1 and each vertex in Z is joined to w_2. Let $S'' = \{w_1, w_2\}$. Then $I[S''] = V$. If $x \in V - S''$, then S'' dominates x. Thus, S'' is a geodetic dominating set of $\mathcal{CDS}(K_{m,n})$ where $m \neq n$. Since $d(w_1, w_2) = 3$, by virtue of Theorem 2.1(a), $\gamma_g(\mathcal{CDS}(K_{m,n})) = |S''| = 2$. □

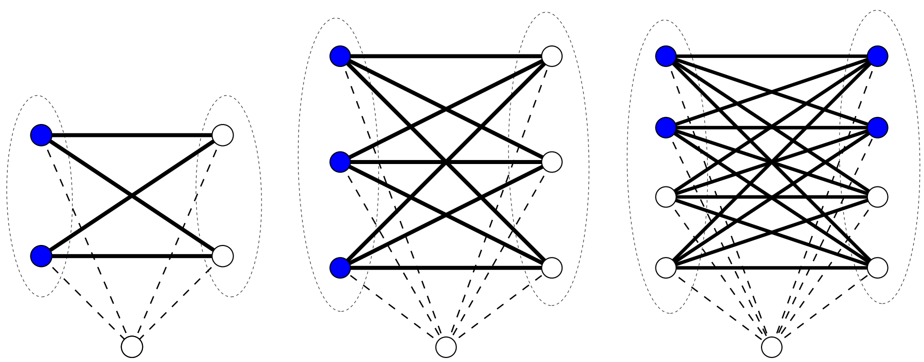

FIGURE 5. The complete degree splitting graph of complete bipartite graphs $\mathcal{CDS}(K_{2,2})$, $\mathcal{CDS}(K_{3,3})$, and $\mathcal{CDS}(K_{4,4})$ with geodetic domination number $\gamma_g(\mathcal{CDS}(K_{2,2})) = 2$, $\gamma_g(\mathcal{CDS}(K_{3,3})) = 3$, and $\gamma_g(\mathcal{CDS}(K_{4,4})) = 4$, respectively.

Theorem 3.5. *Let G be a connected graph of order $n \geq 4$ and let $\mathcal{P} = \{X_i : 1 \leq i \leq \pi(G)\}$ be the degree partition of G. Suppose there exists a vertex v of G such that $\deg(v) = n - 1$ and $X_1 = \{v\}$. Then*

$$\gamma_g\big(\mathcal{CDS}(G)\big) = \pi(G).$$

Proof. We obtain the complete degree splitting graph of G, $\mathcal{CDS}(G)$, by adding new vertices $w_i, 1 \leq i \leq \pi(G)$ and edges w_1v, and w_ix where $i \in \{2, \ldots, \pi(G)\}$ and $x \in X_i$. In effect,

$$V(\mathcal{CDS}(G)) = V(G) \cup \{w_i : 1 \leq i \leq \pi(G)\} \text{ and}$$

$$E(\mathcal{CDS}(G)) = E(G) \cup \{w_1v\} \cup \{w_iu : u \in X_i, 2 \leq i \leq \pi(G)\}$$

Let $S = \{w_1, w_2, \cdots w_{\pi(G)}\}$. Let $x \in V(\mathcal{CDS}(G)) - S$. Then $x \in X_1 \cup X_2 \cup \cdots \cup X_{\pi(G)}$. Without loss of generality, suppose that $x \in X_2$. Then $[w_2, x, v, w_1]$ is a geodesic path with end vertices in S. Thus, each vertex in X_i, $1 \leq i \leq \pi(G)$, lies on some geodesic paths with end vertices in S. That is, $I[S] = V(\mathcal{CDS}(G))$. Hence, S is a geodetic set in $\mathcal{CDS}(G)$. Moreover, since every $x \in V(\mathcal{CDS}(G)) - S$ is adjacent to some vertices in S, S is a dominating set. Consequently, S is a geodetic dominating set. Therefore, $\gamma_g(\mathcal{CDS}(G)) \leq |S| = \pi(G)$.

Suppose S^* is a geodetic dominating set such that $|S^*| < |S| = \pi(G)$. Define, for $1 \leq i \leq \pi(G)$, the pairwise disjoint sets $B_1 = \{w_1,\, v\},$ and $B_i = \{w_i\} \cup X_i \;\; (2 \leq i \leq \pi(G)),$ Observe that the vertex w_1 is adjacent only to $v \in B_1$, and for each $2 \leq i \leq \pi(G)$, the vertex w_i is adjacent only to vertices of $X_i \subseteq B_i$. Hence, for $2 \leq i \leq \pi(G)$, to dominate w_i, any dominating set S^* must contain at least one vertex from B_i. Since the sets B_i are disjoint, every dominating set satisfies $|S^*| \geq \sum_{i=1}^{\pi(G)} 1_{S^* \cap B_i \neq \emptyset} = \pi(G)$. Because a geodetic dominating set is, in particular, a dominating set, we have $\gamma_g\big(\mathcal{CDS}(G)\big) \;\geq\; \pi(G)$, so equality holds. □

Corollary 3.6. *If S_n is a star graph of order $n \geq 4$, then $\gamma_g(\mathcal{CDS}(S_n)) = 2$.*

Corollary 3.7. *If W_n is a wheel graph of order $n \geq 5$, then $\gamma_g(\mathcal{CDS}(W_n)) = 2$.*

Corollary 3.8. *If F_n is a fan graph of order $n \geq 5$, then $\gamma_g(\mathcal{CDS}(F_n)) = 3$.*

4. Geodetic Domination of Complete Degree Splitting Graph of Corona of Graphs

In this section, we give results on the geodetic domination number of complete degree splitting graphs resulting from the corona of graphs.

Theorem 4.1. *Let G and H be connected graphs such that $|G|, |H| \geq 2$. Then*

$$\gamma_g(\mathcal{CDS}(G \circ H)) = \pi(G) + \pi(H).$$

Proof. Let $\mathcal{P}_1 = \{X_i : 1 \leq i \leq \pi(G)\}$ and $\mathcal{P}_2 = \{T_j : 1 \leq j \leq \pi(H)\}$ be the degree partitions of G and H, respectively. Note the following:

(i) For each i, $1 \leq i \leq \pi(G)$,

$$\deg_{G \circ H}(x) = \deg_G(x) + |H|, \quad \forall x \in X_i.$$

(ii) For every $j = 1, 2, \ldots, \pi(H)$,

$$\deg_{G \circ H}(x) = \deg_{H^a}(x) + 1, \quad \forall x \in T_j^a,\ \forall a \in V(G).$$

Thus, the degree partition of $G \circ H$ are the following:

$$X_1, X_2, \ldots, X_{\pi(G)}, \quad T_1^*, T_2^*, \ldots, T_{\pi(H)}^*,$$

where

$$T_j^* = \bigcup_{a \in V(G)} T_j^a, \quad 1 \leq j \leq \pi(H).$$

Let w_i be the adjoining vertex corresponding to X_i, $1 \leq i \leq \pi(G)$. Moreover, let z_j be the adjoining vertex corresponding to T_j^*, $1 \leq j \leq \pi(H)$. Let $W = \{w_i : 1 \leq i \leq \pi(G)\}$ and $Z = \{z_j : 1 \leq j \leq \pi(H)\}$. In effect,

$$V(\mathcal{CDS}(G \circ H)) = V(G \circ H) \cup \{w_i : 1 \leq i \leq \pi(G)\} \cup \{z_j : 1 \leq j \leq \pi(H)\},$$

and

$$E(\mathcal{CDS}(G \circ H)) = E(G \circ H) \cup \{w_i u : u \in X_i, 1 \leq i \leq \pi(G)\} \cup \{z_j v : v \in T_j^*, 1 \leq j \leq \pi(H)\}.$$

Let $D = \{w_i : 1 \leq i \leq \pi(G)\} \cup \{z_j : 1 \leq j \leq \pi(H)\}$.

Claim 4.2. *D is a dominating set in $\mathcal{CDS}(G \circ H)$.*

Let $x \in V(\mathcal{CDS}(G \circ H)) - D$. Then $x \in V(G \circ H)$.

If $x \in V(G)$, then there exists $X_i, 1 \leq i \leq \pi(G)$ such that $x \in X_i$. This implies that x is adjacent to $w_i \in D$. If $x \in V(H)$, then there exists a vertex $a \in V(G)$ and T_j, $1 \leq j \leq \pi(H)$ such that $x \in T_j^a$. This implies that x is adjacent to $z_j \in D$. Thus, D is a dominating set in $\mathcal{CDS}(G \circ H)$.

Claim 4.3. *D is a geodetic set in $\mathcal{CDS}(G \circ H)$.*

Let $x \in V(\mathcal{CDS}(G \circ H)) - D$. That is, $x \in V(G \circ H)$.

Suppose $x \in V(G)$. Then there exists i with $1 \leq i \leq \pi(G)$ such that $x \in X_i$. Note x and w_i are adjacent in $\mathcal{CDS}(G \circ H)$. Let $y \in V(H^x)$. Then $y \in T_j^x$, for some j, with $1 \leq j \leq \pi(H)$. Now, $[w_i, x, y, z_j]$ is a geodesic path and $[w_i, x, y, z_j] \subseteq I[D]$, since $w_i, z_j \in D$. Suppose $x \in V(H^a)$ for some $a \in V(G)$. That is, $x \in T_j^a$ for some j', with $1 \leq j' \leq \pi(H)$. Since $a \in V(G)$, there exists i' with $1 \leq i' \leq \pi(G)$ such that $a \in X_{i'}$. Note $[w_{i'}, a, x, z_{j'}]$ is a geodesic path containing x with endpoints in w_j', $z_j' \in D$. Hence, D is a geodetic set in $\mathcal{CDS}(G \circ H)$.

By virtue of Claims 4.2 and 4.3, D is a geodetic dominating set in $\mathcal{CDS}(G \circ H)$.

Accordingly, $\gamma_g(\mathcal{CDS}(G \circ H)) \leq |D| = \pi(G) + \pi(H)$.

Suppose there exists a geodetic dominating set D^* in $\mathcal{CDS}(G \circ H)$ such that $|D^*| < |D|$.

Claim 4.4. $|D^* \cap [V(G) \cup W]| = |D^* \cap [\bigcup_{i=1}^{\pi(G)}(\{w_i\} \cup X_i)]| \geq \pi(G)$.

Suppose $|D^* \cap [\bigcup_{i=1}^{\pi(G)}(\{w_i\} \cup X_i)]| < \pi(G)$. That is, $\sum_{i=1}^{\pi(G)} |D^* \cap [\{w_i\} \cup X_i]| < \pi(G)$. Consequently, there exist $j = 1, 2, \ldots, \pi(G)$ such that $|D^* \cap [\{w_i\} \cup X_j]| = 0$, that is, $D^* \cap [\{w_i\} \cup X_j] = \emptyset$. This implies $w_j \notin D^*$. Since D^* is a dominating set in $\mathcal{CDS}(G \circ H)$, there exists $x \in D^*$ such that x

and w_j are adjacent in $\mathcal{CDS}(G \circ H)$. Now, $D^* \cap X_j = \emptyset$, and $x \in D^*$, so $x \notin X_j$. This contradicts the definition of $\mathcal{CDS}(G \circ H)$. Hence,

$$\left| D^* \cap \left[\bigcup_{i=1}^{\pi(G)} (\{w_i\} \cup X_i) \right] \right| \geq \pi(G).$$

Claim 4.5. $|D^* \cap [\bigcup_{a \in V(G)} (V(H^a) \cup Z)]| = |D^* \cap [\bigcup_{i=1}^{\pi(H)} (\{z_i\} \cup T_i^*)]| \geq \pi(H)$.

Suppose $|D^* \cap [\bigcup_{i=1}^{\pi(H)} (\{z_i\} \cup T_i^*)]| < \pi(H)$. That is, $\sum_{i=1}^{\pi(H)} |D^* \cap [\{z_i\} \cup T_i^*]| < \pi(H)$. Then there exist $j = 1, 2, \ldots, \pi(H)$ such that $|D^* \cap (\{z_j\} \cap T_j^*)| = 0$, that is, $D^* \cap (\{z_j\} \cap T_j^*) = \emptyset$. This implies that, $z_j \notin D^*$. Since D^* is a dominating set in $\mathcal{CDS}(G \circ H)$, there exists $y \in D^*$ such that z_j and y are adjacent in $\mathcal{CDS}(G \circ H)$. Now, $D^* \cap T_j^* = \emptyset$, and $y \in D^*$ implies $y \notin T_j^*$. This contradicts the definition of $\mathcal{CDS}(G \circ H)$. Hence,

$$\left| D^* \cap \left[\bigcup_{i=1}^{\pi(H)} (\{z_i\} \cup T_i^*) \right] \right| \geq \pi(H).$$

Now, by Claims 4.4 and 4.5, we have,

$$|D^*| = \left| D^* \cap \left[\bigcup_{i=1}^{\pi(G)} (\{w_i\} \cup X_i) \right] \right| + \left| D^* \cap \left[\bigcup_{j=1}^{\pi(H)} (\{z_j\} \cup T_j^*) \right] \right| \geq \pi(G) + \pi(H).$$

Therefore, $\gamma_g(\mathcal{CDS}\,(G \circ H)) = \pi(G) + \pi(H)$.

□

Acknowledgments. The first author acknowledges the support of the Department of Science and Technology – Science Education Institute (DOST-SEI) under its graduate scholarship program.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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