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ATOMIC SOLUTIONS OF NON-HOMOGENEOUS PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS VIA TENSOR PRODUCT THEORY

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Abstract. In this paper, we develop and apply the atomic solution method to obtain exact solutions of a class of non-homogeneous parabolic partial differential equations (PPDEs). The approach is based on the tensor product theory in Banach spaces together with key properties of atomic operators. By exploiting these structures, the PDE is decomposed into simpler components, allowing us to construct exact solutions in terms of atomic functions. Several cases are examined, leading to nine distinct atomic solutions under different assumptions on the source terms. The method highlights the effectiveness of the atomic framework in handling non-homogeneous PDEs and demonstrates its potential as a systematic tool for solving problems that arise in heat flow, diffusion, and related physical processes.

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1. Introduction

Partial differential equations (PDEs) play a major role in, almost, all scientific domains [5], [6]. They provide a potential window for studying many complex physical facts and phenomena with spatial behavior that changes in time. Natural processes like heat conduction, fluid motion, quantum mechanical systems, and general relativity are typical examples of mathematical modeling by PDEs. In fact PDEs are present everywhere not only physics, but also in all natural domains such as engineering, chemistry, biology, and social studies. There are many types of PDEs and also, there are many ways at which they can be classified such as order, linearity, homogeneity, and the coefficients type [17].

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The general form of a second order non-homogeneous PDE with constant coefficients is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = F(x_1,\ x_2,\ \cdots,\ x_n), \tag{1.1}$$

where $u = u(x_1,\ x_2,\ \cdots,\ x_n) : \mathbb{R}^n \to \mathbb{R}$ is the unknown function, a_{ij}, b_i, c, and d $(1 \le i \le n)$ are constants, and F is a given function of $x_1, x_2, \cdots, x_n$.

When modeling some physical problems in engineering, we usually come across one of the, well known, three types of second-order PDEs, namely, elliptic, hyperbolic, and parabolic PDEs. Each type of PDEs has certain characteristics and can represent specific dynamical regime. Parabolic partial differential equations (PPDEs) model phenomena that describe how physical quantities change slowly over a small interval of time such as heat flow, diffusion, and fluid flow [8].

One important property of PPDEs is that they have an initial value problem associated with them. This means that the solution is specified at some initial time. PPDEs also have a maximum principle, which states that the maximum value of the solution is attained at the given initial time.

In 2010, [13], a new procedure for solving ordinary and fractional differential equations was presented. This new method utilizes the concept of tensor product of two Banach spaces as well as some properties of the atoms operators in order to solve differential equations. The obtained solution in this case is called an atomic solution that is named for atoms operators [1–4,7,10,18].

In the current paper, our target is to determine the exact atomic solutions of a certain type of PPDEs that is

$$u_{ss}(s,t) + 2u_{st}(s,t) + u_{tt}(s,t) + u(s,t) = f(s)g(t),$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are given functions in s and t, respectively. Before we present the detailed procedure for obtaining the atomic solutions of the above PPDE, we commence with some related definitions and theorems.

2. Atoms Operators and Tensor Product of Banach Spaces

Definition 2.1. *Let V and W be any two Banach spaces where V^* is the dual space of V. Assume that $v \in V$ and $w \in W$. Then the operator $A : V^* \to W$ defined by*

$$Av^* = v^*(v)w = \langle v, v^* \rangle\, w,$$

is said to be an atom and is denoted by $v \otimes w$. Moreover, since the range of $v \otimes w$ is the span of w, the atom $v \otimes w$ is a bounded rank-one linear operator.

For example, let V and W be two Banach spaces such that $V = W = C\,[0,1]$, with the norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$ for any $f \in C\,[0,1]$. Define V^* to be the space of all regular Borel measures on $[0,1]$. So, if $\mu \in V^*$, then $\mu(f) = \int_0^1 f\, d\mu$. Now, let $f \in V$, $g \in W$, and $\mu \in V^*$, then $(f \otimes g)\,\mu = \mu(f)g = g\int_0^1 f(t)$

$d\mu(t)$. Further, $f \otimes g$ is a bounded linear operator. Also, indeed, $\|(f \otimes g)(\mu)\| = |\mu(f)| \|g\| \leq \|\mu\| \|f\| \|g\|$ for all $\mu \in V^*$.

Now, if we assume two atoms, namely, $v_1 \otimes w_1$ and $v_2 \otimes w_2$ such that $v_1 \otimes w_1 = v_2 \otimes w_2$, then for any $v^* \in V^*$ we have

$$\langle v_1, v^* \rangle w_1 = \langle v_2, v^* \rangle w_2.$$

Thus, we can assume $w_1 = w_2$. Similarly, one can prove $v_1 = v_2$.

Now, let us present the following result about atoms.

Lemma 1. [11] *Let $v_1 \otimes w_1$ and $v_2 \otimes w_2$ be two nonzero atoms in $V \otimes W$ such that $v_1 \otimes w_1 + v_2 \otimes w_2 = v_3 \otimes w_3$. Then either $v_1 = v_2 = v_3$ or $w_1 = w_2 = w_3$.*

Finally, one of the old and notable theorems in the field of applied functional analysis as well as approximation theory in tensor product [12] which guarantees that any continuous function of more than one variable can be expressed as a sum of products of continuous separable functions.

Theorem 2.1. [9] *Let N, M be two compact intervals, and $C(N)$, $C(M)$, and $C(N \times M)$ be the spaces of continuous functions on N, M, and $N \times M$, respectively. Then for every $h \in C(N \times M)$ we have $h(x, y) = \sum_{i=1}^{\infty} q_i(x) r_i(y)$, where $q_i(x) \in C(N)$ and $r_i(y) \in C(M)$.*

For more details about tensor product theory of Banach space, we refer the readers to [9, 14–16].

3. Main Result

Consider the following PPDEs in two variables

$$\begin{aligned} &u_{ss}(s,t) + 2u_{st}(s,t) + u_{tt}(s,t) + u(s,t) = f(s)g(t), \\ &u(0,0) = u_x(0,0) = u_y(0,0) = 1. \end{aligned} \tag{3.1}$$

where $u = u(s,t) : \mathbb{R}^2 \to \mathbb{R}$ is the unknown function and $f, g : \mathbb{R} \to \mathbb{R}$ are given functions in s and t, respectively.

A solution to (3.1) is said to be atomic if it has the form $u(s,t) = P(s) \otimes Q(t)$ where $P(s)$ and $Q(t)$ are twice differentiable functions. In this section, we are interested in obtaining the atomic solution of (3.1).

Now, from Theorem (1), we begin our solution procedure with considering that

$$u(s,t) = P(s) Q(t). \tag{3.2}$$

Moreover, from the initial conditions in (3.1), we can assume, without loss of generality, that

$$P(0) = P'(0) = 1 \text{ and } Q(0) = Q'(0) = 1. \tag{3.3}$$

Thus, by substituting (3.2) into the PPDE (3.1) we get

$$P''(s)Q(t) + 2P'(s)Q'(t) + P(s)Q''(t) + P(s)Q(t) = f(s)g(t), \tag{3.4}$$

Clearly, each term of (3.4) is just a product of two functions one of them is pure in s and the other is pure in t. Therefore, in tensor product form, (3.4) can be written as

$$P'' \otimes Q + P' \otimes 2Q' + P \otimes Q'' + P \otimes Q = f \otimes g. \tag{3.5}$$

Equivalently,

$$P'' \otimes Q + P' \otimes 2Q' + P \otimes \left[Q'' + Q\right] = f \otimes g. \tag{3.6}$$

By utilizing Lemma (1), there are three different cases:

$$\begin{aligned} &(\mathbf{A})\ P'' \otimes Q + P' \otimes 2Q' \text{ is an atom,}\\ &(\mathbf{B})\ P'' \otimes Q + P \otimes (Q'' + Q)\ \text{ is an atom,}\\ &(\mathbf{C})\ P' \otimes 2Q' + P \otimes (Q'' + Q)\ \text{ is an atom.} \end{aligned} \tag{3.7}$$

Case $(\mathbf{A})$: This case reveals that the sum of two atomes is an atom. Thus, again by Lemma (1), this case gives the following two situations, namely

$$\begin{aligned} &(\mathbf{Ai})\ P'' = P',\\ &(\mathbf{Aii})\ Q = 2Q'. \end{aligned} \tag{3.8}$$

Situation $(\mathbf{Ai})$**:** For this situation, (3.6) can be reduced as follows

$$P'' \otimes \left[Q + 2Q'\right] + P \otimes \left[Q'' + Q\right] = f \otimes g. \tag{3.9}$$

This implies, by lemma (1), either

$$\begin{aligned} &(\mathbf{Ai}, \mathbf{1})\ P'' = P = f,\\ &\text{or}\\ &(\mathbf{Ai}, \mathbf{2})\ \ [Q + 2Q'] = [Q'' + Q] = g. \end{aligned} \tag{3.10}$$

By considering $(\mathbf{Ai}, \mathbf{1})$ together with appropriate initial conditions from (3.3)we have

$$P'' = P' = P = f = e^s. \tag{3.11}$$

Hence, for an atomic solution to exist, $f(s)$ must be equal to e^s.

Now, on substituting (3.11) into (3.9), we have

$$Q''(t) + 2Q'(t) + 2Q(t) = g(t), \tag{3.12}$$

which is a non-homogeneous second order ordinary differential equation with constant coefficients. So, its general solution has the form $Q(t) = Q_h(t) + Q_p(t)$, where $Q_h(t) = e^{-t}\left[2\sin t + \cos t\right]$ is the complementary solution that can be obtained by considering the companion homogeneous equation, namely, $Q''(t) + 2Q'(t) + 2Q(t) = 0$ together with $Q(0) = 1$ and $Q'(0) = 1$ (3.3). While the particular solution $Q_p(t)$ can be obtained by the method of variation of parameters as follows

$$\begin{aligned} Q_p(t) &= e^{-t} \sin t \int \frac{-e^{-t}\cos(t) g(t)}{W\left[e^{-t}\sin t, e^{-t}\cos t\right]} dt \\ &+ e^{-t}\cos(t) \int \frac{e^{-t}\sin(t) g(t)}{W\left[e^{-t}\sin t, e^{-t}\cos t\right]} dt \\ &= e^{-t}\left[\sin(t)\int e^{t}\cos(t) g(t) dt - \cos(t)\int e^{t}\sin(t) g(t) dt\right], \end{aligned} \tag{3.13}$$

where $W\left[e^{-t}\sin t, e^{-t}\cos t\right]$ is the Wronskian of $e^{-t}\sin t$ and $e^{-t}\cos t$. Therefore, the general solution to (3.12) is given by

$$\begin{aligned} Q(t) &= Q_h(t) + Q_p(t) \\ &= e^{-t}\left[\sin t\left(2 + \int e^{t}\cos t g(t) dt\right) + \cos t\left(1 - \int e^{t}\sin t g(t) dt\right)\right. \\ &\left. + \cos t\left(1 - \int e^{t}\sin t g(t) dt\right)\right]. \end{aligned} \tag{3.14}$$

Hence, by assuming (3.2), (3.11), and (3.14) the first atomic solution to (3.1) that is associated with $(\mathbf{Ai}, \mathbf{1})$ is

$$u_{A,1}(s,t) = e^{s-t}\left[\sin t\left(2 + \int e^{t}\cos t g(t) dt\right) + \cos t\left(1 - \int e^{t}\sin t g(t) dt\right)\right]. \tag{3.15}$$

Now, we move to $(\mathbf{Ai}, \mathbf{2})$ and consider $[Q(t) + 2Q'(t)] = [Q''(t) + Q(t)]$ which can be solved by taking into account the related initial conditions from (3.3) as

$$Q(t) = \frac{1}{2}\left[e^{2t} + 1\right]. \tag{3.16}$$

Indeed, we require $[Q(t) + 2Q'(t)] = g(t)$ and $[Q''(t) + Q(t)] = g(t)$ in order to obtain an atomic solution related to $(\mathbf{Ai}, \mathbf{2})$. Thus,

$$g(t) = \frac{1}{2}\left[5e^{2t} + 1\right]. \tag{3.17}$$

Therefore, on substituting both (3.16) and (3.17) into (3.9), we get $P''(s) + P(s) = f(s)$ which is again a non-homogeneous second order ordinary differential equation with constant coefficients that can be solved by similar argument as in (3.12). So, by utilizing the related initial conditions from (3.3) we have

$$P(s) = \sin s\left(1 + \int f(s)\cos s ds\right) + \cos s\left(1 - \int f(s)\sin s ds\right). \tag{3.18}$$

Consequently, the atomic solution that is associated to $(\mathbf{Ai}, \mathbf{2})$, can be formulated by assuming (3.2), (3.16), and (3.18) as follows:

$$u_{A,2}(s,t) = \frac{1}{2}\left[e^{2t} + 1\right]\left[\sin s\left(1 + \int f(s)\cos s ds\right) + \cos s\left(1 - \int f(s)\sin s ds\right)\right]. \tag{3.19}$$

Situation $(\mathbf{Aii})$**:** Here, (3.6) can be reduced as follows:

$$\left[P'' + P'\right] \otimes Q + P \otimes \left[Q'' + Q\right] = f \otimes g. \tag{3.20}$$

This implies, by lemma (1), either

$$
\begin{aligned}
&(\mathbf{Aii}, \mathbf{1})\ P'' + P' = P = f,\\
&\text{or}\\
&(\mathbf{Aii}, \mathbf{2})\ Q = [Q'' + Q] = g.
\end{aligned} \tag{3.21}
$$

Hence, from $(\mathbf{Aii}, \mathbf{1})$, we have $P(s) = f(s)$ such that $P''(s) + P'(s) - P(s) = 0$, this argument together with appropriate initial conditions from (3.3) gives

$$
P(s) = f(s) = \left(\frac{3\sqrt{5}+5}{10}\right) e^{\frac{\sqrt{5}+1}{2}s} - \left(\frac{3\sqrt{5}-5}{10}\right) e^{\frac{1-\sqrt{5}}{2}s}. \tag{3.22}
$$

Therefore, on substituting (3.22) into (3.20), we have $Q''(t) + 2Q(t) = g(t)$ which is a non-homogeneous second order ordinary differential equation with constant coefficients that can be solved by similar argument as in (3.12). Thus, by considering the related initial conditions from (3.3) we get

$$
\begin{aligned}
Q(t) = &\frac{\sin\left(\sqrt{2}t\right)}{\sqrt{2}}\left(1 + \int g(t)\cos\left(\sqrt{2}t\right) dt\right)\\
&+ \frac{\cos\left(\sqrt{2}t\right)}{\sqrt{2}}\left(\sqrt{2} - \int g(t)\sin\left(\sqrt{2}t\right) dt\right).
\end{aligned} \tag{3.23}
$$

Hence, by assuming (3.2), (3.22), and (3.23) an atomic solution of (3.1) exists and given by

$$
\begin{aligned}
u_{A,3}(s,t) = &\left[\left(\frac{3\sqrt{5}+5}{10}\right) e^{\frac{\sqrt{5}+1}{2}s} - \left(\frac{3\sqrt{5}-5}{10}\right) e^{\frac{1-\sqrt{5}}{2}s}\right]\\
&\left[\frac{\sin\left(\sqrt{2}t\right)}{\sqrt{2}}\left(1 + \int g(t)\cos\left(\sqrt{2}t\right) dt\right)\right.\\
&\left.+\frac{\cos\left(\sqrt{2}t\right)}{\sqrt{2}}\left(\sqrt{2} - \int g(t)\sin\left(\sqrt{2}t\right) dt\right)\right].
\end{aligned} \tag{3.24}
$$

Now, we move to $(\mathbf{Aii}, \mathbf{2})$ and consider both $Q(t) = Q''(t) + Q(t)$ and $Q(t) = g(t)$. From these two equations and by taking into account the related initial conditions from (3.3), we get

$$
Q(t) = g(t) = t + 1. \tag{3.25}
$$

Therefore, on substituting (3.25) into (3.20), we get $P''(s) + P'(s) + P(s) = f(s)$. Following similar argument as in (3.12) together with assuming the related initial conditions from (3.3) we have

$$
\begin{aligned}
P(s) = &e^{-\frac{s}{2}}\sin\left(\frac{\sqrt{3}}{2}s\right)\left(\sqrt{3} + \frac{2}{\sqrt{3}}\int f(s)\cos\left(\frac{\sqrt{3}}{2}s\right) ds\right)\\
&+ e^{-\frac{s}{2}}\cos\left(\frac{\sqrt{3}}{2}s\right)\left(1 - \frac{2}{\sqrt{3}}\int f(s)\sin\left(\frac{\sqrt{3}}{2}s\right) ds\right).
\end{aligned} \tag{3.26}
$$

Consequently, an atomic solution of (3.1) that is associated to $(\mathbf{Aii}, \mathbf{2})$ exists and can be formulated by assuming (3.2), (3.25), and (3.26) as follows

$$u_{A,4}(s,t) = [t+1]\left[e^{-\frac{s}{2}}\sin\left(\frac{\sqrt{3}}{2}s\right)\left(\sqrt{3}+\frac{2}{\sqrt{3}}\int f(s)\cos\left(\frac{\sqrt{3}}{2}s\right)ds\right)\right.$$
$$\left.+e^{-\frac{s}{2}}\cos\left(\frac{\sqrt{3}}{2}s\right)\left(1-\frac{2}{\sqrt{3}}\int f(s)\sin\left(\frac{\sqrt{3}}{2}s\right)ds\right)\right]. \tag{3.27}$$

So far, we have four different atomic solutions to (3.1), namely, $u_{A,1}$ (3.15), $u_{A,2}$ (3.19), $u_{A,3}$ (3.24), and $u_{A,4}$ (3.27). More atomic solutions can be obtained via considering case $(\mathbf{B})$ and case $(\mathbf{C})$ (3.7).

Case $(\mathbf{B})$: This case reveals that the sum of two atoms is an atom. Thus, by Lemma (1), this case gives the following two situations, namely

$$\begin{aligned} &(\mathbf{Bi})\ P'' = P, \\ &(\mathbf{Bii})\ Q = Q'' + Q. \end{aligned} \tag{3.28}$$

Situation $(\mathbf{Bi})$**:** When considering this situation, (3.6) can be written as follows

$$P'' \otimes \left(Q'' + 2Q\right) + P' \otimes 2Q' = f \otimes g. \tag{3.29}$$

This implies, by lemma (1), either

$$\begin{aligned} &(\mathbf{Bi},\mathbf{1})\ P'' = P' = f, \\ &\text{or} \\ &(\mathbf{Bi},\mathbf{2})\ \ (Q'' + 2Q) = 2Q' = g. \end{aligned} \tag{3.30}$$

By $(\mathbf{Bi}, \mathbf{1})$ together with the appropriate initial conditions from (3.3)we have

$$P''(s) = P'(s) = P(s) = f(s) = e^s. \tag{3.31}$$

Therefore, on substituting (3.31) into (3.29), we have $Q''(t) + 2Q'(t) + 2Q(t) = g(t)$. Now, by referring to situation $(\mathbf{Ai}, \mathbf{1})$, one can easily recognize that both situations $(\mathbf{Bi}, \mathbf{1})$ and $(\mathbf{Ai}, \mathbf{1})$ are the same and hence, they give the same atomic solution. This implies $u_{B,1} = u_{A,1}$ (3.15) .

Now, we move to $(\mathbf{Bi}, \mathbf{2})$ from which we have, $Q''(t) - 2Q'(t) + 2Q(t) = 0$. Therefore, by (3.3) we get

$$Q(t) = e^t \cos t. \tag{3.32}$$

Indeed, we require $Q''(t) + 2Q(t) = 2Q'(t) = g(t)$ in order to obtain an atomic solution related to situation $(\mathbf{Bi}, \mathbf{2})$. Thus,

$$g(t) = 2e^t\left(\cos t - \sin t\right). \tag{3.33}$$

Now, on substituting both (3.32) and (3.33) into (3.29), we get $P''(s) + P'(s) = f(s)$. This is a non-homogeneous second order ordinary differential equation with constant coefficients. So, its general solution has the form $P(s) = P_h(s) + P_p(s)$ and can be solved by considering related initial conditions from (3.3) as

$$P(s) = \left(2 - e^{-s}\right) + \left(1 - e^{-s}\right) \int f(s) e^{s} ds. \tag{3.34}$$

Consequently, the atomic solution that is associated to $(\mathbf{Bi}, \mathbf{2})$, can be formulated by assuming (3.2), (3.32), and (3.34) is given by

$$u_{B,2}(s,t) = e^{t} \cos t \left[\left(2 - e^{-s}\right) + \left(1 - e^{-s}\right) \int f(s) e^{s} ds\right]. \tag{3.35}$$

Situation $(\mathbf{Bii})$**:** Here, (3.6) can be reduced as follows

$$\left[P'' + P\right] \otimes Q + P' \otimes 2Q' = f \otimes g. \tag{3.36}$$

This implies, by lemma (1), either

$$\begin{aligned} &(\mathbf{Bii}, \mathbf{1})\ P'' + P = P' = f, \\ &\text{or} \\ &(\mathbf{Bii}, \mathbf{2})\ Q = 2Q' = g. \end{aligned} \tag{3.37}$$

From $(\mathbf{Bii}, \mathbf{1})$, we have $P'(s) = f(s)$ such that $P''(s) - P'(s) + P(s) = 0$, this argument together with appropriate initial conditions from (3.3) gives

$$P(s) = e^{\frac{s}{2}} \left(\frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} s\right) + \cos\left(\frac{\sqrt{3}}{2} s\right)\right), \text{ where } f(s) = P'(s). \tag{3.38}$$

Therefore, on substituting (3.38) into (3.36), we have $2Q'(t) + Q(t) = g(t)$ which can be solved by considering both the complementary solution $Q_h(t)$ and the particular solution $Q_p(t)$ with the related initial conditions from (3.3) as

$$Q(t) = \frac{1}{2} e^{-\frac{1}{2}t} \int g(t) e^{\frac{1}{2}t} dt. \tag{3.39}$$

Hence, by assuming (3.2), (3.38), and (3.39) an atomic solution of (3.1) exists and given by

$$u_{B,3}(s,t) = \frac{1}{2} e^{\frac{1}{2}(s-t)} \int g(t) e^{\frac{1}{2}t} dt \left(\frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} s\right) + \cos\left(\frac{\sqrt{3}}{2} s\right)\right). \tag{3.40}$$

Now, we move to $(\mathbf{Bii}, \mathbf{2})$ where $Q(t) = 2Q'(t)$ and $Q(t) = g(t)$. Hence, from (3.3), we get

$$Q(t) = g(t) = e^{\frac{1}{2}t}. \tag{3.41}$$

Therefore, on substituting (3.41) into (3.36), we get $P''(s) + P'(s) + P(s) = f(s)$ which can be solved by considering the related initial conditions from (3.3) as

$$\begin{aligned} P(s) = {} & e^{-\frac{s}{2}} \sin\left(\frac{\sqrt{3}}{2} s\right) \left(\sqrt{3} + \frac{2}{\sqrt{3}} \int f(s) \cos\left(\frac{\sqrt{3}}{2} s\right) ds\right) \\ & + e^{-\frac{s}{2}} \cos\left(\frac{\sqrt{3}}{2} s\right) \left(1 - \frac{2}{\sqrt{3}} \int f(s) \sin\left(\frac{\sqrt{3}}{2} s\right) ds\right). \end{aligned} \tag{3.42}$$

Consequently, an atomic solution of (3.1) that is associated to $(\mathbf{Bii}, \mathbf{2})$ exists and can be formulated by assuming (3.2), (3.41), and (3.42) as follows

$$u_{B,4}(s,t) = e^{\frac{1}{2}t}\left[e^{-\frac{s}{2}}\sin\left(\frac{\sqrt{3}}{2}s\right)\left(\sqrt{3}+\frac{2}{\sqrt{3}}\int f(s)\cos\left(\frac{\sqrt{3}}{2}s\right)ds\right)\right.$$
$$\left.+e^{-\frac{s}{2}}\cos\left(\frac{\sqrt{3}}{2}s\right)\left(1-\frac{2}{\sqrt{3}}\int f(s)\sin\left(\frac{\sqrt{3}}{2}s\right)ds\right)\right]. \tag{3.43}$$

So far, we have seven different atomic solutions to (3.1), namely, $u_{A,1} = u_{B,1}$ (3.15), $u_{A,2}$ (3.19), $u_{A,3}$ (3.24), $u_{A,4}$ (3.27), $u_{B,2}$ (3.35), $u_{B,3}$ (3.40), and $u_{B,4}$ (3.43). Now, for more atomic solutions to (3.1), we consider case $(\mathbf{C})$ (3.7).

Case $(\mathbf{C})$: This case reveals that the sum of two atoms is an atom. Thus, again by Lemma (1), this case gives the following two situations, namely

$$\begin{aligned}&(\mathbf{Ci})\ P' = P,\\&(\mathbf{Cii})\ 2Q' = Q'' + Q.\end{aligned} \tag{3.44}$$

Situation $(\mathbf{Ci})$**:** For this situation, (3.6) can be reduced as follows

$$P'' \otimes Q + P'\left[Q'' + 2Q' + Q\right] = f \otimes g. \tag{3.45}$$

This implies, by lemma (1), either

$$\begin{aligned}&(\mathbf{Ci}, \mathbf{1})\ P'' = P' = f,\\&\text{or}\\&(\mathbf{Ci}, \mathbf{2})\ Q = Q'' + 2Q' + Q = g.\end{aligned} \tag{3.46}$$

By considering $(\mathbf{Ci}, \mathbf{1})$ together with appropriate initial conditions from (3.3)we have

$$P''(s) = P'(s) = P(s) = f(s) = e^s. \tag{3.47}$$

Therefore, on substituting (3.47) into (3.45), we have $Q''(t) + 2Q'(t) + 2Q(t) = g(t)$. By referring to both $(\mathbf{Ai}, \mathbf{1})$ and $(\mathbf{Bi}, \mathbf{1})$, clearly, $(\mathbf{Ci}, \mathbf{1})$ will give atomic solution $u_{C,1}$ that is similar to $u_{A,1}$ and $u_{B,1}$. So, $u_{A,1} = u_{B,1} = u_{C,1}$ (3.15).

Now, we move to $(\mathbf{Ci}, \mathbf{2})$ from which we have $Q''(t) + 2Q'(t) = 0$ and $Q(t) = g(t)$. Hence, by utilizing related initial conditions from (3.3) we get,

$$Q(t) = g(t) = \frac{3}{2} - \frac{1}{2}e^{-2t}. \tag{3.48}$$

On substituting both (3.48) into (3.45), we get $P''(s) + P'(s) = f(s)$ which can be solved by considering the related initial conditions from (3.3) as

$$P(s) = \left(2 - e^{-s}\right) + \left(1 - e^{-s}\right)\int f(s)e^s ds. \tag{3.49}$$

Consequently, the atomic solution that is associated to $(\mathbf{Ci}, \mathbf{2})$, can be obtained by assuming (3.2), (3.48), and (3.49) as follows:

$$u_{C,2}(s,t) = \left(\frac{3}{2} - \frac{1}{2}e^{-2t}\right)\left[\left(2 - e^{-s}\right) + \left(1 - e^{-s}\right)\int f(s)e^{s}ds\right]. \tag{3.50}$$

Situation $(\mathbf{Cii})$**:** (3.6) can be reduced as follows:

$$P'' \otimes Q + \left(P' + P\right) \otimes 2Q' = f \otimes g. \tag{3.51}$$

This implies, by lemma (1), either

$$\begin{aligned} &(\mathbf{Cii}, \mathbf{1})\ P'' = P' + P = f, \\ &\text{or} \\ &(\mathbf{Cii}, \mathbf{2})\ Q = 2Q' = g. \end{aligned} \tag{3.52}$$

Hence, from $(\mathbf{Cii}, \mathbf{1})$, we have $P''(s) = f(s)$ such that $P''(s) - P'(s) - P(s) = 0$, this argument together with appropriate initial conditions from (3.3) gives

$$P(s) = \left(\frac{5+\sqrt{5}}{10}\right)e^{\frac{\sqrt{5}+1}{2}s} + \left(\frac{5-\sqrt{5}}{10}\right)e^{\frac{1-\sqrt{5}}{2}s}. \text{ such that } f(s) = P''(s). \tag{3.53}$$

Therefore, on substituting (3.53) into (3.51), we have $2Q'(t) + Q(t) = g(t)$ which can be solved by considering the related initial conditions from (3.3) as

$$Q(t) = \frac{1}{2}e^{-\frac{1}{2}t}\int g(t)e^{\frac{1}{2}t}dt. \tag{3.54}$$

Hence, by assuming (3.2), (3.53), and (3.54) an atomic solution to (3.1) exists and given by

$$u_{C,3}(s,t) = \frac{1}{2}e^{-\frac{1}{2}t}\int g(t)e^{\frac{1}{2}t}dt\left[\left(\frac{5+\sqrt{5}}{10}\right)e^{\frac{\sqrt{5}+1}{2}s} + \left(\frac{5-\sqrt{5}}{10}\right)e^{\frac{1-\sqrt{5}}{2}s}\right]. \tag{3.55}$$

Finally, we move to $(\mathbf{Cii}, \mathbf{2})$ by which we have $2Q'(t) - Q(t) = 0$ and $Q(t) = g(t)$. From these two equations and by taking into account the related initial conditions from (3.3), we get

$$Q(t) = g(t) = e^{\frac{1}{2}t}. \tag{3.56}$$

Therefore, on substituting (3.56) into (3.51), we get $P''(s) + P'(s) + P(s) = f(s)$. Referring to situation $(\mathbf{Bii}, \mathbf{2})$, one can easily recognize that both situations $(\mathbf{Cii}, \mathbf{2})$ and $(\mathbf{Bii}, \mathbf{2})$ are the same and hence, they give the same atomic solution. This implies $u_{C,4} = u_{B,4}$ (3.43) .

According to the atomic solution method, the PPDE (3.1) has the following nine atomic solutions: $u_{A,1} = u_{B,1} = u_{C,1}$ (3.15), $u_{A,2}$ (3.19), $u_{A,3}$ (3.24), $u_{A,4}$ (3.27), $u_{B,2}$ (3.35), $u_{B,3}$ (3.40), $u_{B,4} = u_{C,4}$ (3.43), $u_{C,2}$ (3.50), and $u_{C,3}$ (3.55).

4. Conclusion

In this study, we employed the atomic solution method to solve a non-homogeneous parabolic partial differential equation. By applying tensor product theory in Banach spaces and the properties of atomic operators, we systematically derived nine distinct exact solutions corresponding to different structural cases of the equation.

The main novelty of this work lies in showing that the atomic framework is not only capable of handling homogeneous problems, as in earlier studies, but also extends naturally to non-homogeneous parabolic PDEs. Each solution arises from precise conditions on the factor functions $P(s)$ and $Q(t)$, leading to either explicit closed forms or integral representations.

From an applications perspective, PPDEs model a wide range of time-dependent diffusion processes, including heat conduction, mass transfer, and fluid flow. The exact solutions obtained here provide useful benchmarks for validating numerical simulations and can guide further extensions of the atomic approach to more complex PDEs, such as higher-dimensional systems, fractional models, and equations with variable coefficients.

Future research may explore the application of this method to nonlinear PDEs or coupled systems, where traditional techniques often face significant challenges. The results of this paper underline the potential of atomic solutions as a robust analytical tool in both theoretical and applied mathematics.

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