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THE NON-CENTRAL BELL NUMBERS WITH COMPLEX INDICES: SOME PROPERTIES AND RECURRENCES

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Abstract. In this paper, we introduce a more general version of the classical Bell numbers, referred to as non-central Bell numbers, for complex indices, using the concept of Hankel contours. We explore the key properties of these generalized numbers and show that several important characteristics of the classical Bell numbers still hold in this extended context. Specifically, we derive a Dobinski-type formula and establish recurrence relations that resemble those in the classical case. These findings enhance our understanding of Bell numbers in the framework of complex analysis and combinatorics.

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1. Introduction

For a long time, the goal of combinatorial analysis was to count the different ways of arranging objects under given circumstances. But now, combinatorial analysis is also relevant to the problem of existence, estimation, and structuration.

The Bell numbers contribute a lot in developing the Stirling numbers of the second kind in terms of their applications in combinatorics and statistics. This present study may be able to determine whether the distribution of non-central Stirling numbers for real arguments is asymptotically normal. The possible results in this study may be used as a tool to obtain such asymptotic approximation.

The Bell numbers, denoted by B_n is defined as the sum of all Stirling numbers of the second kind $S(n,k)$, as k ranges from 0 to n, that is,

$$B_n = \sum_{k=0}^{n} S(n,k).$$

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The numbers B_n count the number of distinct partitions of an n-set. The expression $k!S(n,k)$ represents the number of surjections from an n-set onto a k-set. Additionally, known properties of Bell numbers B_n are listed here.

i) The Bell numbers B_n have the following generating function

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{e^t-1}.$$

ii) The Bell numbers B_n satisfy the recurrence relation

$$B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} B_r.$$

iii) [Dobinski] The Bell numbers B_n can be written in the form of a convergent series

$$B_n = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^n}{r!}.$$

Parallel to the definition of ordinary Bell numbers, Corcino, Jaylo, and Ringia [4] introduced the non-central Bell numbers, denoted by $B_a(n)$, as the sum of the non-central Stirling numbers of the second kind as k ranges from 0 to n, that is,

$$B_a(n) = \sum_{k=0}^{n} S_a(n,k).$$

The following are some properties of the non-central Bell numbers $B_a(n)$.

iv) The non-central Bell numbers $B_a(n)$ have the following generating function

$$\sum_{n=0}^{\infty} B_a(n) \frac{u^n}{n!} = e^{(e^u-1)-au}.$$

v) The non-central Bell numbers $B_a(n)$ satisfy the recurrence relation

$$B_a(n+1) = \sum_{k=0}^{n} \binom{n}{k} B_a(k) S_a(n,k).$$

vi) [Dobinski] The non-central Bell numbers $B_a(n)$ can be written in the form of a convergent series

$$B_a(n) = \frac{1}{e} \sum_{j=0}^{n} \frac{(j-a)^n}{j!}.$$

Note that if $a=0$, the classical Bell numbers, B_n, are obtained.

Generalizing Stirling numbers to real and complex arguments not only extends numerous classical combinatorial identities but also enhances their relevance and utility across various branches of mathematics. In [5], Flajolet and Prodinger defined an extension of the classical Stirling numbers $S(n,k)$ of the second kind to complex arguments. Similarly, in [3], the Bell numbers B_n were extended to complex arguments preserving the definition of Bell numbers as the sum of Stirling numbers of the second kind. In [6], Koutras introduced the non-central Stirling numbers $S_a(n,k)$ of the second kind through a natural extension of the definition of $S(n.k)$. Furthermore, in [1], the non-central Stirling numbers $S_a(x, y)$ were extended to complex arguments x and y where key properties, including the recurrence relations of the classical Stirling numbers, are preserved under this complex generalization.

Looking ahead, extending the non-central Bell numbers (as defined in [2]) to complex arguments, building on the framework established in [3], holds great promise. These extended Bell numbers are integral to the study of partitions and other combinatorial structures. Such an extension could not only reveal new identities and recurrence relations but also offer profound insights into fields like number theory, asymptotics, and complex analysis.

2. Preliminaries

Special functions, such as the Gamma function, play a crucial role in extending combinatorial identities and providing insights into various mathematical fields. The Gamma function [7], denoted by Γ, is a generalization of the factorial function to real and complex numbers. It is defined through the improper integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad \text{for} \quad \Re(z) > 0.$$

For $z = 1$, we have:

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

When $z = n \in \mathbb{Z}^+$, applying integration by parts gives:

$$\Gamma(n) = (n-1) \int_0^\infty t^{n-2} e^{-t}\, dt,$$

and repeating this process $(n-1)$ times leads to:

$$\Gamma(n) = (n-1)(n-2)\cdots(1) \int_0^\infty e^{-t}\, dt = (n-1)!.$$

Thus, the Gamma function $\Gamma(z)$ serves as an extension of the factorial function with its argument shifted by 1.

The following theorem is a key property of the Gamma function:

Theorem 2.1. [8] *The Gamma function* $\Gamma(z)$ *has the integral representation:*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma e^t t^{-z}\, dt,$$

where γ *is a Hankel contour.*

This formula, known as Hankel's contour integral, involves a path of integration γ that starts at $-\infty - i0$ on the real axis, moves to $-\epsilon - i0$, circles the origin counterclockwise with radius ϵ, and returns to $-\infty + i0$.

Before presenting the main result, we first highlight several key theorems from [1] that are essential for establishing fundamental properties of non-central Bell numbers with complex indices. Notably, the second theorem provides the explicit formula for $S_a(x, k)$.

Theorem 2.2. *For complex numbers x and a with $Re(a) < 0$, we have*

$$\frac{x!}{2\pi i}\int_{\mathcal{H}} e^{(j-a)u}\frac{du}{u^{x+1}} = (j-a)^x.$$

Theorem 2.3. *For complex numbers x and a with $Re(a) < 0$, and a nonnegative integer k, we have*

$$S_a(x,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j-a)^x.$$

Theorem 2.4. *For complex numbers x and a with $\mathrm{Re}(a) < 0$ and $\mathrm{Re}(x) > 0$, the non-central Stirling numbers $S_a(x,y)$ satisfy the following recurrence relation:*

$$S_a(x,y) = S_a(x-1,y-1) + (y-a)S_a(x-1,y).$$

In the next section, we introduce the definition of the non-central Bell numbers for complex indices and derive several important properties. This section also presents the main result of our work, which we anticipate will provide a solid foundation for future investigations and potential applications.

3. The Non-central Bell Numbers with Complex Indices

The exponential generating function for the non-central Bell numbers, $B_a(n)$, is given by

$$\sum_{n=0}^{\infty}\frac{B_a(n)}{n!}u^n = e^{(e^u-1)-au}.$$

This means that

$$\frac{B_a(n)}{n!} = [u^n]\left(e^{(e^u-1)-au}\right),$$

where $f(u) = e^{(e^u-1)-au}$ is an entire function. By Cauchy's Integral Formula,

$$B_a(n) = \frac{n!}{2\pi i}\int_{\gamma}\frac{e^{(e^u-1)-au}}{u^{n+1}}\,du.$$

where γ is a small contour encircling the origin.

We can deform γ into a Hankel contour which starts from $-\infty$ below the negative $x-axis$ surrounding the origin couterclockwise and returns to $-\infty$ above the negative $x-axis$. This suggests the following definition. Here, we assume that $\mathcal{H}$ is at a distance ≤ 1 from the real axis.

Definition 1. *The non-central Bell numbers $B_a(x)$ of complex arguments are defined for any complex number x by*

$$B_a(x) = \frac{x!}{2\pi i}\int_{\mathcal{H}}\frac{e^{(e^u-1)-au}du}{u^{x+1}},$$

where a is a complex number with $Re(a) < 0,\ x! = \Gamma(x+1)$ and the logarithm involved in the function u^{x+1} is taken to be the principal branch.

Based on this definition, we now explore a foundational identity in the theory of Bell numbers. The first theorem establishes the usual expression used in defining the Bell-type numbers, that is, the sum over Stirling numbers of the second kind.

Theorem 3.1. *The non-central Bell numbers $B_a(x)$ for complex arguments x and a, where $\mathrm{Re}(a) < 0$, are given by*

$$B_a(x) = \sum_{k=0}^{\infty} S_a(x,k),$$

where $S_a(x,k)$ denotes the non-central Stirling numbers of the second kind.

Proof. As results for expanding the integrand in Definition 1, we get

$$\begin{aligned} e^{(e^u-1)-au} &= e^{-au}\sum_{k=0}^{\infty}\frac{(e^u-1)^k}{k!} \\ &= e^{-au}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{ju} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{(j-a)u}. \end{aligned}$$

Hence,

$$\begin{aligned} B_a(x) &= x!\cdot\frac{1}{2\pi i}\int_{\mathcal{H}}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{(j-a)u}\right)\frac{du}{u^{x+1}} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\cdot x!\cdot\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{e^{(j-a)u}}{u^{x+1}}\,du \end{aligned}$$

Using Theorem 2.2,

$$B_a(x) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j-a)^x$$

Thus, by Theorem 2.3,

$$B_a(x) = \sum_{k=0}^{\infty} S_a(x,k).$$

□

A key property of Bell-type numbers is their Dobinski-type formula. In contrast to the explicit formula given in Theorem 3.1, the formula presented in the following theorem stands out for its simplicity, featuring only a single summation.

Theorem 3.2. *The non-central Bell numbers $B_a(x)$ for complex numbers x and a, where $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(a) < 0$, satisfy the following relation:*

$$B_a(x) = \frac{1}{e}\sum_{j=0}^{\infty}\frac{(j-a)^x}{j!}.$$

Proof. Observe that

$$\begin{aligned} e^{(e^u-1)-au} &= e^{-au}\cdot e^{-1}\cdot e^{e^u} \\ &= e^{-au}\cdot e^{-1}\sum_{j=0}^{\infty}\frac{(e^u)^j}{j!} \\ &= \frac{1}{e}\sum_{j=0}^{\infty}\frac{e^{(j-a)u}}{j!}. \end{aligned}$$

Hence,

$$\begin{aligned} B_a(x) &= x!\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{1}{e}\sum_{j=0}^{\infty}\frac{e^{(j-a)u}}{j!}\frac{du}{u^{x+1}} \\ &= \frac{x!}{2\pi i}\sum_{j=0}^{\infty}\frac{1}{e\,j!}\int_{\mathcal{H}}\frac{e^{(j-a)u}}{u^{x+1}}\,du \\ &= \frac{1}{e}\sum_{j=0}^{\infty}\frac{1}{j!}\left(x!\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{e^{(j-a)u}}{u^{x+1}}\,du\right). \end{aligned}$$

Using Theorem 2.2, this simplifies to

$$B_a(x) = \frac{1}{e}\sum_{j=0}^{\infty}\frac{(j-a)^x}{j!}.$$

□

The explicit Dobinski-type formula provides a direct evaluation of $B_a(x)$, expressing it as a single infinite sum. To complement this, the following theorem introduces a recurrence relation that offers a more computationally efficient way to generate these numbers, revealing their inherent recursive structure.

Theorem 3.3. *The non-central Bell numbers for complex arguments x and a, where $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(a) < 0$, satisfy the following recurrence relation:*

$$B_a(x) = -a\,B_a(x-1) + B_{a-1}(x-1) + S_{a-1}(x-1,-1).$$

Proof. Summing both sides of the recurrence relation in Theorem 2.4 with $y = k$, a nonnegative integer, we have

$$\begin{aligned} \sum_{k=0}^{\infty}S_a(x,k) &= \sum_{k=0}^{\infty}S_a(x-1,k-1) + \sum_{k=0}^{\infty}(k-a)S_a(x-1,k) \\ &= \sum_{k=0}^{\infty}S_a(x-1,k) + S_a(x-1,-1) + \sum_{k=0}^{\infty}kS_a(x-1,k) - a\sum_{k=0}^{\infty}S_a(x-1,k) \\ &= (1-a)\sum_{k=0}^{\infty}S_a(x-1,k) + S_a(x-1,-1) + \sum_{k=0}^{\infty}kS_a(x-1,k). \end{aligned}$$

Hence, by Theorem 3.1,

$$B_a(x) = (1-a)B_a(x-1) + S_a(x-1,-1) + \sum_{k=0}^{\infty} kS_a(x-1,k). \tag{3.1}$$

Now, applying Theorem 2.3, we can express

$$\sum_{k=0}^{\infty} kS_a(x-1,k) = \sum_{k=0}^{\infty} k\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j-a)^{x-1}.$$

Rearranging,

$$\sum_{k=0}^{\infty} kS_a(x-1,k) = \sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\left[\binom{k-1}{j}+\binom{k-1}{j-1}\right](j-a)^{x-1}.$$

Splitting the sum,

$$\begin{aligned}\sum_{k=0}^{\infty} kS_a(x-1,k) = &\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j}(j-a)^{x-1} \\ &+\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j-1}(j-a)^{x-1}.\end{aligned}$$

Observe that

$$\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j}(j-a)^{x-1} = -\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-1-j}\binom{k-1}{j}(j-a)^{x-1}.$$

Consequently, by Theorem 2.3, we get

$$\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j}(j-a)^{x-1} = -\sum_{k=0}^{\infty} S_a(x-1,k-1). \tag{3.2}$$

On the other hand,

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j-1}(j-a)^{x-1} \\ &= -\sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-1-j}\binom{k-1}{j-1}(j-a)^{x-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-1-(j-1)}\binom{k-1}{j-1}(j-1-(a-1))^{x-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=-1}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(j-(a-1))^{x-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-1-j}\binom{k-1}{j}(j-(a-1))^{x-1}.\end{aligned}$$

Again, by Theorem 2.3,

$$\sum_{k=0}^{\infty}\frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j-1}(j-a)^{x-1}=\sum_{k=0}^{\infty}S_{a-1}(x-1,k-1). \tag{3.3}$$

Thus, by equations (3.2) and (3.3), we have

$$\begin{aligned}\sum_{k=0}^{\infty}k\,S_a(x-1,k) &= -\sum_{k=0}^{\infty}S_a(x-1,k-1)+\sum_{k=0}^{\infty}S_{a-1}(x-1,k-1)\\ &= -S_a(x-1,-1)-\sum_{k=0}^{\infty}S_a(x-1,k)+S_{a-1}(x-1,-1)+\sum_{k=0}^{\infty}S_{a-1}(x-1,k).\end{aligned}$$

Then, by Theorem 3.1,

$$\sum_{k=0}^{\infty}k\,S_a(x-1,k)=-S_a(x-1,-1)-B_a(x-1)+S_{a-1}(x-1,-1)+B_{a-1}(x-1). \tag{3.4}$$

Finally, substitution of equation (3.4) into equation (3.1) yields:

$$\begin{aligned}B_a(x) &= (1-a)B_a(x-1)+S_a(x-1,-1)+\sum_{k=0}^{\infty}k\,S_a(x-1,k)\\ &= (1-a)B_a(x-1)+S_a(x-1,-1)\\ &\quad -S_a(x-1,-1)-B_a(x-1)+S_{a-1}(x-1,-1)+B_{a-1}(x-1)\\ &= -aB_a(x-1)+B_{a-1}(x-1)+S_{a-1}(x-1,-1).\end{aligned}$$

□

It is important to note that, for complex case,

$$S_{a-1}(x-1,-1)\neq 0,$$

however, for the integral case,

$$S_{a-1}(x-1,-1)=0.$$

4. Conclusion

The study of non-central Bell numbers $B_a(x)$ for complex values of x and a (where the real part of a is negative) uncovers an extended structure beyond classical Bell numbers. Using Cauchy's Integral Formula and Hankel contours, these numbers are represented as contour integrals in the complex plane. This leads to key results: a representation using Stirling numbers (Theorem 3.1), a simplified infinite sum akin to Dobinski's formula (Theorem 3.2), and a recurrence relation for computing values (Theorem 3.3). These findings generalize known combinatorial results and offer new tools for analysis and computation.

Future research can explore how $B_a(x)$ behaves for large x providing insight into their growth and links to special functions. Developing fast, accurate methods to compute these values using the recurrence and infinite sum formulas could be useful for practical applications. Additionally, since combinatorial numbers like these often appear in physics and probability, further work could investigate their roles in quantum mechanics, statistical systems, and probabilistic models.

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