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ORDER MULTIVALUED INTEGRABILITY IN BANACH LATTICES

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Abstract. The McShane-type integral is utilized in the analysis of multifunctions with values in weakly compact and convex subsets of a weakly distributive Banach lattice Y, endowed with an order-continuous norm. This study investigates several properties and distinctions related to this integral. It is shown that, for simple multifunctions, the integral coincides with the usual integral obtained from order McShane-type selections in the Aumann framework. Furthermore, the approach allows us to obtain results analogous to those known in the norm-based setting, while highlighting genuinely order-theoretic phenomena.

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1. Introduction

In recent decades, significant advances have been made in the study of integrable multifunctions, driven primarily by the pioneering contributions of Aumann and Debreu. This research trajectory has rigorously examined the multivalued context, culminating in the formulation of diverse integral frameworks applicable to multivalued functions within Banach spaces and other vector spaces. Multivalued analysis has proven to be indispensable in various fields, facilitating the resolution of challenges in domains such as optimal control theory, set-valued stochastic processes, and economic theory [,13]. The impetus for this exploration stems from the limitations of the Bochner integral in managing vector-valued functions, as emphasized by classic examples that reveal its shortcomings. Furthermore, the applicability of selection theorems related to the Aumann–Bochner integral often requires the underlying Banach space to be separable [3,6,7,12,14].

This paper builds on extensive comparative studies of norm- and order-based multivalued integration within both Banach and vector lattices, as detailed for instance in [1,2,8]. Focusing on the phenomenon

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of order convergence within these structures, we introduce a new set-valued integral, denoted by $\Phi(G, A)$, for functions taking values in Banach lattice spaces. This integral, fundamentally grounded in the theoretical framework developed in [1,3], is defined through the collection of all order-convergence limits of Riemann sums computed from δ_n-fine McShane partitions. The study explores the attributes of convexity and boundedness within this integral framework. Moreover, it is shown that for simple multifunctions, this integral aligns with the conventional integral derived via McShane-type selections following the Aumann approach, while revealing new insights analogous to those observed in normed contexts.

2. MATERIALS AND METHODS

Consider a compact metric space (S, d), and let μ be a measure that is regular, nonatomic, and σ-additive, mapping the σ-algebra $\mathfrak{B}$, which comprises Borel sets of S, to the non-negative real numbers $\mathbb{R}_0^+$, denoted as

$$\mu : \mathfrak{B} \to \mathbb{R}_0^+.$$

Assume Y represents a weakly (σ)-distributive Banach lattice equipped with an order-continuous norm $\|\cdot\|$. Define $P_0(Y)$ as the collection of all nonempty subsets of the Banach space Y, $bg(Y)$ as the collection of all nonempty, bounded, and closed subsets of Y, and $cwk(Y)$ (or $ck(Y)$) as the set comprising all convex, weakly compact, non-empty subsets of Y. For all $E, F \in P_0(Y)$, the Minkowski sum is defined by

$$E + F = \{e + f : e \in E,\ f \in F\}.$$

The operation $\oplus$ on $bg(Y)$ is specified by

$$E \oplus F = \mathrm{cl}(E + F).$$

If $E_i \in bg(Y)$ for $i = 1, \dots, n$, then

$$\sum_{i=1}^{n} E_i = \mathrm{cl}(E_1 + \cdots + E_n).$$

Note that if $E, F \in bg(Y)$, then $E \oplus F \in bg(Y)$. If E and F belong to $cwk(Y)$ (or to $ck(Y)$), then the Minkowski sum is already closed, so no further closure is needed.

A sequence $(p_n)_n$ in S is order-convergent (o-convergent) to p if there exists a sequence $(z_n)_n \subset \mathbb{R}$ such that $z_n \downarrow 0$ and $|p_n - p| \leq z_n$ for all $n \in \mathbb{N}$, expressed as

$$(o)\text{-}\lim_n p_n = p.$$

A gauge on S is any map $\delta : S \to \mathbb{R}^+$.

A partition of S, denoted by Π, is a finite collection of pairs

$$\{(A_1, \xi_1), \dots, (A_r, \xi_r)\}$$

where A_i are disjoint and $\bigcup_{i=1}^{r} A_i = S$, with ξ_i referred to as tags. A partition is a Henstock-type (H-partition) if each tag ξ_i resides within its corresponding set A_i; otherwise, it is called a free or McShane partition (M-partition).

A partition is δ-fine if the distance between any point $\tau \in A_i$ and its corresponding tag ξ_i is less than $\delta(\xi_i)$ for all $i = 1, \dots, r$. Alternatively, a gauge δ can also be defined as a mapping that associates to each point $\xi \in S$ an open ball centered at ξ. The symbols $|\cdot|$ and $\|\cdot\|$ refer to the modulus and norm of X, respectively; for the relation between them, see for example [1,4,5,15].

An element e of the positive cone X^+ is an order unit in X if X is the solid linear subspace of itself generated by e; that is, for every $u \in X$ there is an $n \in \mathbb{N}$ such that $|u| \le ne$. An M-space is a Banach lattice M in which the norm is an order-unit norm, namely there is an order unit $e \in M$ and an equivalent Riesz norm $\|\cdot\|_e$, defined on M by the formula

$$\|u\|_e = \min\{\alpha > 0 : |u| \le \alpha e\}$$

for every $u \in M$. In this case, one also has $\|u + v\| = \|u\| \vee \|v\|$ for positive $u, v \in M$.

An L-space is a Banach lattice L such that $\|u + v\| = \|u\| + \|v\|$ whenever $u, v \in L^+$. For further details on L- and M-spaces, see also [15].

Definition 2.1. A function $g : S \to Y$ is said to be order McShane integrable (or order Henstock integrable) on S if there exists an $L \in Y$ such that for every (o)-sequence $(p_n)_n$ in Y there is a corresponding sequence $(\delta_n)_n$ of gauges $\delta_n : S \to (0, +\infty)$ such that, for every n and every δ_n-fine M-partition (or H-partition)

$$\Pi = \{(A_1, \xi_1), \dots, (A_r, \xi_r)\}$$

of S, the following inequality holds:

$$\Big|\sum_{\Pi} g - L\Big| \le p_n,$$

where

$$\sum_{\Pi} g = \sum_{i=1}^{r} g(\xi_i)\, \mu(A_i).$$

We denote

$$L = (oM)\int_S g\, d\mu \quad (\text{respectively, } L = (oH)\int_S g\, d\mu),$$

where L is the (o)-McShane integral (respectively (o)-Henstock integral) of g on S. The function g is (oM)-integrable (respectively (oH)-integrable) on a subset $I \subset S$ if the function $g \cdot \chi_I$ is (oM)-integrable (respectively (oH)-integrable) on S, where χ_I is the characteristic function of I. The (oM)-integral (respectively (oH)-integral) of g on I is denoted by

$$(oM)\int_I g\, d\mu, \qquad (oH)\int_I g\, d\mu.$$

Definition 2.2. Let the set $U(C,p)$ be defined by

$$U(C,p)=\{v\in Y:\exists x\in C \text{ such that } |v-x|\le p\}.$$

A set $C\subset Y$ is said to be closed if it satisfies $C=\mathrm{cl}(C)$, where the closure of C, denoted $\mathrm{cl}(C)$, is given by

$$\mathrm{cl}(C)=\bigcup_{(p_n)_n,(\delta_n)_n}\bigcap_n U(C,p_n),$$

with the union taken over all (o)-sequences $(p_n)_n$ and associated gauge sequences $(\delta_n)_n$.

Definition 2.3 ([2]). Let $G:S\to 2^Y$ be a multifunction with non-empty values, and let $A\in\mathfrak{B}$. The (o)-integral of G on A is the set

$$\Phi^{\circ}(G,A)=\bigcup_{(p_n)_n,(\delta_n)_n}\bigcap_n\bigcap_{\Pi_{\delta_n}}\mathcal{U}\Big(\sum_{i=1}^{q}G(\xi_i)\mu(A_i),p_n\Big),$$

where $(p_n)_n$ denotes any (o)-sequence, $(\delta_n)_n$ any sequence of gauges, Π_{δ_n} the family of all δ_n-fine M-partitions of A, and

$$\mathcal{U}(C,p):=\{v\in Y:|v-x|\le p \text{ for some } x\in C\}.$$

Proposition 2.1. *If $\Phi(G,A)$ is non-empty and the multifunction G has convex values, then $\Phi(G,A)$ is convex.*

Proof. Consider any two elements $v_1,v_2\in\Phi(G,A)$. Let $(p_n)_n$ and $(r_n)_n$ be (o)-sequences in Y associated with v_1 and v_2, respectively. Define $s_n=2(p_n+r_n)$ for each $n\in\mathbb{N}$. Given $v_1,v_2\in\Phi(G,A)$, for every $n\in\mathbb{N}$ there exist gauges $\delta_{n,1}$ and $\delta_{n,2}$ such that any M-partition (or H-partition)

$$\Pi=\{(F_1,\zeta_1),\dots,(F_r,\zeta_r)\}$$

that is simultaneously $\delta_{n,1}$-fine and $\delta_{n,2}$-fine yields points $b_1,b_2\in\sum_{j=1}^{r}G(\zeta_j)\mu(F_j)$ satisfying

$$|v_1-b_1|\le p_n,\qquad |v_2-b_2|\le r_n.$$

Introduce $\delta_n(x)=\min\{\delta_{n,1}(x),\delta_{n,2}(x)\}$ for each $x\in S$. Any δ_n-fine M-partition is therefore both $\delta_{n,1}$-fine and $\delta_{n,2}$-fine. Since G is convex-valued, the set $\sum_{j=1}^{r}G(\zeta_j)\mu(F_j)$ is convex. Thus, for any $\beta\in[0,1]$,

$$\beta b_1+(1-\beta)b_2\in\sum_{j=1}^{r}G(\zeta_j)\mu(F_j),$$

and

$$\big|\beta b_1+(1-\beta)b_2-\beta v_1-(1-\beta)v_2\big|\le\beta|b_1-v_1|+(1-\beta)|b_2-v_2|\le s_n.$$

Hence $\beta v_1+(1-\beta)v_2\in\Phi(G,A)$, which proves convexity. □

Proposition 2.2. *Assume that G is bounded and that Y is super Dedekind-complete. Then $\Phi(G,A)$ is closed.*

Proof. We aim to establish the identity

$$\Phi(G,A) = \bigcup_{(p_n)_n,(\delta_n)_n} \bigcap_n U(\Phi(G,A),p_n).$$

The inclusion

$$\Phi(G,A) \subset \bigcup_{(p_n)_n,(\delta_n)_n} \bigcap_n U(\Phi(G,A),p_n)$$

is trivial. Conversely, consider any

$$v \in \bigcup_{(p_n)_n,(\delta_n)_n} \bigcap_n U(\Phi(G,A),p_n).$$

This implies the existence of an (o)-sequence $(p_n)_n$ such that for each n there exists $v_n \in \Phi(G,A)$ satisfying

$$|v - v_n| \le p_n, \qquad \forall n \in \mathbb{N}.$$

Given that Y is super Dedekind-complete and weakly σ-distributive, we may choose $(p_n)_n$ so that

$$0 = \bigwedge_n p_n,$$

and it follows that $v = (o)\text{-}\lim_n v_n$, that is,

$$\bigwedge_n \sup_{m \ge n} |v_m - v| = 0.$$

We prove $v \in \Phi(G,A)$. The sequence $(v_n)_n$ is bounded. Since $v_n \in \Phi(G,A)$, an (o)-sequence $(s_n)_n$ can be found such that, for every n, there is a gauge δ_n with the property that for every δ_n-fine M-partition (or H-partition)

$$\{(A_1,\xi_1),\ldots,(A_q,\xi_q)\}$$

of A there exists $u_n \in \sum_{i=1}^q G(\xi_i)\mu(A_i)$ with

$$|v_n - u_n| \le s_n, \qquad \forall n \in \mathbb{N}.$$

Assuming G is bounded, $(u_n)_n$ is also bounded, and thus $(v_n - u_n)_n$ is bounded. Since $(v_n)_n$ converges to v in the (o)-sense, there exists an (o)-sequence $(t_n)_n$ such that for every $n \in \mathbb{N}$ we can find n_1 with

$$|v - v_n| \le t_n, \qquad n \ge n_1.$$

Define $h_n = 2(t_n + s_n)$ for all $n \in \mathbb{N}$; then $(h_n)_n$ is an (o)-sequence. For each n,

$$|v - u_{n_1}| \le |v - v_{n_1}| + |v_{n_1} - u_{n_1}| \le h_n,$$

and this shows that v belongs to the closure generated by such approximations. By definition of $\Phi(G,A)$, this implies $v \in \Phi(G,A)$, so $\Phi(G,A)$ is closed. $\square$

3. Results and Discussion

This study introduces the concept of the order-McShane type integral for multivalued mappings, in the spirit of [2], where the Henstock integral framework is employed to define a multivalued integral in Banach lattices. The research provides a rigorous examination of order-McShane (and Henstock) multivalued integrability within the framework of order theory, under the assumption that Y is a weakly distributive Banach lattice equipped with an order-continuous norm. Furthermore, the properties of this integral are analyzed, and its relationship to the Aumann approach is evaluated. While many results align with existing findings in normed spaces, significant distinctions are identified.

Definition 3.1. Let $G : S \to cwk(Y)$ be a multifunction. We say that G is (oM)-integrable (or (oH)-integrable) if, for every measurable subset $A \subset S$, there exists an element $L_A \in cwk(Y)$, an (o)-sequence $(p_n)_n$ in Y, and a corresponding sequence of gauges $(\delta_n)_n$ on S such that, for each n and every δ_n-fine M-partition (or H-partition)

$$\Pi = \{(A_1, \xi_1), \dots, (A_q, \xi_q)\}$$

of A, the following conditions hold:

$$\sum_{i=1}^{q} G(\xi_i)\mu(A_i) \subset U(L_A, p_n),$$

and

$$L_A \subset U\Big(\sum_{i=1}^{q} G(\xi_i)\mu(A_i), p_n\Big).$$

Proposition 3.1. *Suppose that the multifunction $G : S \to cwk(Y)$ is (oM)-integrable (or (oH)-integrable). Then the corresponding element L_A associated with any measurable subset $A \subset S$ is uniquely determined.*

Proof. Assuming without loss of generality that $A = S$, let $L = L_S$ and $L' = L'_S$ be two elements of $cwk(Y)$ satisfying the integrability conditions in the previous definition. Consider $(p_n)_n$ and $(p'_n)_n$, the (o)-sequences corresponding to L and L', respectively, along with their associated gauge sequences $(\delta_n)_n$ and $(\delta'_n)_n$. For a fixed n, and any $(\delta_n \wedge \delta'_n)$-fine M-partition (or H-partition) Π, we have

$$L \subset U\Big(\sum_{i=1}^{q} G(\xi_i)\mu(A_i), p_n\Big) \subset U(L', p_n + p'_n),$$

and similarly,

$$L' \subset U\Big(\sum_{i=1}^{q} G(\xi_i)\mu(A_i), p'_n\Big) \subset U(L, p_n + p'_n).$$

Consequently, for each $b \in L$ and any n, there exists $b'_n \in L'$ such that

$$|b - b'_n| \le p_n + p'_n.$$

This implies that $(b'_n)_n$ is (o)-convergent (and in particular norm convergent) to b. Since L' is closed, it follows that $b \in L'$, hence $L \subset L'$. By symmetry, $L' \subset L$, so $L = L'$. □

Theorem 3.1. *Consider a multifunction $G : S \to cwk(Y)$ that is (oM)-integrable. Assume further that for each $\xi \in S$ one has $\sup G(\xi) \in G(\xi)$. Then G can be decomposed into the sum of an (oM)-integrable single-valued function $h : S \to Y$ and an (oM)-integrable multifunction $H : S \to cwk(Y)$. Moreover, for every functional $y^* \in Y^*$, the support mapping $\xi \mapsto s(y^*, H(\xi))$ is non-negative and it vanishes for every positive functional $y^* \geq 0$.*

Proof. Define $h(\xi) = \sup G(\xi)$ for each $\xi \in S$, and let $H(\xi) = G(\xi) - h(\xi)$, namely we translate each set by $-h(\xi)$. It is immediate that $\sup H(\xi) = 0$, and, given that $\sup G(\xi) \in G(\xi)$, we deduce $0 \in H(\xi)$. For any fixed ξ and functional $y^* \in Y^*$, the support mapping defined by

$$s(y^*, H(\xi)) = \sup\{y^*(v) : v \in H(\xi)\}$$

is non-negative. If y^* is positive, then $y^*(0) \geq y^*(v)$ for all $v \in H(\xi)$, so $s(y^*, H(\xi)) = 0$ for all positive $y^* \in Y^*$.

Assuming integrability of G, there exists an (o)-sequence $(p_n)_n$ in Y and a gauge sequence $(\delta_n)_n$ on S such that for any n and every δ_n-fine M-partition $\Pi = \{(A_1, \xi_1), \dots, (A_q, \xi_q)\}$ we have

$$\sum_{i=1}^{q} G(\xi_i)\mu(A_i) \subset U(L, p_n),$$

and

$$L \subset U\Big(\sum_{i=1}^{q} G(\xi_i)\mu(A_i), p_n\Big),$$

where $L = (oM) \int_S G \, d\mu$. Writing

$$\sum_{\Pi} h = \sum_{i=1}^{q} h(\xi_i)\mu(A_i),$$

standard arguments in order integration theory yield

$$\Big|\sum_{\Pi} h - \sup L\Big| \leq p_n,$$

so h is (oM)-integrable with integral $\sup L$. Consequently,

$$\sum_{i=1}^{q} H(\xi_i)\mu(A_i) = \sum_{i=1}^{q} G(\xi_i)\mu(A_i) - \sum_{i=1}^{q} h(\xi_i)\mu(A_i) \subset U(L - \sup L, 2p_n),$$

and similarly

$$L - \sup L \subset U\Big(\sum_{i=1}^{q} H(\xi_i)\mu(A_i), 2p_n\Big).$$

These inclusions show that H is (oM)-integrable (and similarly (oH)-integrable), with integral $L - \sup L$, where

$$L = (oM) \int_S G \, d\mu \quad (\text{or } L = (oH) \int_S G \, d\mu).$$

□

We proceed by defining the Aumann integral employing order McShane (Kurzweil–Henstock) integrable selections, to facilitate a comparison with the integral defined earlier for multifunctions. Consider a multifunction $G : S \to P_0(Y)$. Define S^1_G as the set of all (oM)-integrable selections of G, represented by

$$S^1_G = \{g : g(\xi) \in G(\xi) \text{ for } \mu\text{-almost every } \xi, \text{ and } g \text{ is } (oM)\text{-integrable}\}.$$

Definition 3.2. Suppose that a multifunction $G : S \to P_0(Y)$ is such that the set S^1_G of (oM)-integrable selections is non-empty. For every $A \in \mathfrak{B}$ we define the Aumann–McShane integral (abbreviated as (AM)) by

$$(AM) \int_A G \, d\mu = \left\{ \int_A g \, d\mu : g \in S^1_G \right\}.$$

Theorem 3.2. *Let $G = \sum_{r \le n} C_r 1_{A_r}$ be a simple multifunction, where each $C_r \in cwk(Y)$ and the sets $\{A_r\}$ are pairwise disjoint. For any $B \in \mathfrak{B}$ one has*

$$\mathrm{cl}\big(\Phi(G, B)\big) = \mathrm{cl}\left((A) \int_B G \, d\mu \right).$$

Moreover,

$$\Phi^\circ(G, B) = \left\{ (oM) \int_B g \, d\mu : g \in S^1_G \right\}.$$

Proof. Assume $G = \sum_{r \le n} C_r 1_{A_r}$ is a simple multifunction, where $C_r \in cwk(Y)$ and $\{A_r\}$ are pairwise disjoint measurable sets. Let $(\delta_n)_n$ be a sequence of gauges, $\delta_n : S \to (0, +\infty)$, and for each n consider a δ_n-fine M-partition (or H-partition)

$$\Pi = \{(B_1, \xi_1), \ldots, (B_q, \xi_q)\}$$

of B. Then

$$\sum_{i=1}^{q} G(\xi_i)\mu(B_i) = \sum_{i=1}^{q} \Big(\sum_{r \le n} C_r 1_{A_r} \Big)(\xi_i)\mu(B_i) = \sum_{r \le n} C_r \sum_{i=1}^{q} 1_{A_r}(\xi_i)\mu(B_i) = \sum_{r \le n} \sum_{\Pi} G|_{A_r},$$

which expresses the decomposition of the sums along the simple components.

To show

$$\Phi\Big(\sum_{r \le n} C_r 1_{A_r}, B \Big) \subset \sum_{r \le n} \Phi(C_r 1_{A_r}, B),$$

take $x \in \Phi\big(\sum_{r \le n} C_r 1_{A_r}, B\big)$. For every n there is an (o)-sequence $(p_n)_n$ and a sequence of gauges $(\delta_n)_n$ such that, for all δ_n-fine M-partitions $\Pi_1 = \{(A_1, \xi_1), \ldots, (A_r, \xi_r)\}$ of B, there exists

$$y \in \sum_{\Pi_1} \Big(\sum_{r \le n} C_r 1_{A_r} \Big)$$

with $|y-x| \le p_n$. A refinement $\Pi_2 = \{(A'_1, \zeta_1), \dots, (A'_t, \zeta_t)\}$ may be chosen so that, for each $r = 1, \dots, n$, the restriction $\Pi_2|_{A_r}$ is a partition of A_r. Then

$$y \in \sum_{j=1}^{t} \Big(\sum_{r \le n} C_r 1_{A_r} \Big)(\zeta_j)\mu(A'_j) = \sum_{r \le n} C_r \sum_{j=1}^{t} 1_{A_r}(\zeta_j)\mu(A'_j).$$

For each r we may select $y_r \in \Phi(C_r 1_{A_r}, B)$ and an (o)-sequence $(a_n)_n$ such that

$$|y - \sum_{r=1}^{n} y_r| \le a_n,$$

which yields

$$x \in \bigcap_n U\Big(\sum_{r \le n} \Phi(C_r 1_{A_r}, B), 2(p_n + a_n) \Big) \subset \bigcup_{(p_n)_n, (\delta_n)_n} \bigcap_n U\Big(\sum_{r \le n} \Phi(C_r 1_{A_r}, B), b_n \Big),$$

for another (o)-sequence $(b_n)_n$. Since $\sum_{r \le n} \Phi(C_r 1_{A_r}, B)$ is closed, we obtain

$$\sum_{r \le n} \Phi(C_r 1_{A_r}, B) = \sum_{r \le n} C_r \, \mu(B \cap A_r) \subset (A) \int_B G \, d\mu \subset \mathrm{cl} \left((A) \int_B G \, d\mu \right).$$

To complete the proof it remains to show $(A) \int_B G \, d\mu \subset \Phi(G, B)$. Suppose $x \in (A) \int_B G \, d\mu$, so $x = \int_B g \, d\mu$ for some $g \in S^1_G$. By the definition of the (oM)-integral there exists an (o)-sequence $(p_n)_n$ and, for each n, a gauge δ_n such that for every δ_n-fine M-partition $\Pi = \{(A_1, \xi_1), \dots, (A_r, \xi_r)\}$ of B we have

$$\Big| \sum_{i=1}^{r} g(\xi_i)\mu(A_i) - x \Big| \le p_n.$$

Since each $g(\xi_i) \in G(\xi_i)$, we have

$$\sum_{i=1}^{r} g(\xi_i)\mu(A_i) \in \sum_{\Pi} G,$$

and thus $x \in \Phi(G, B)$ by the very definition of Φ.

Now assume Y is super Dedekind-complete. Given G simple with closed and bounded values, Proposition 2.2 above implies that

$$\Phi\Big(\sum_{r \le n} C_r 1_{A_r}, B \Big) = \mathrm{cl} \left((A) \int_B G \, d\mu \right),$$

while for $C_r \in cwk(Y)$ we obtain the stated correspondence with the Aumann–McShane integral and its order-closure. □

4. Conclusion

This paper has delved into the order McShane (Henstock) integrability of multifunctions within a compact, regular metric space, where the multifunctions take values in a weakly σ-distributive Banach lattice endowed with an order-continuous norm. We have established that for simple multifunctions, the Φ°-integral aligns with the order-closure of the Aumann–McShane integral when the components C_i lie in $cbg(Y)$. When $C_i \in cwk(Y)$, there is a direct correspondence with the Aumann integral. This highlights a strong analytical congruence among various integral constructs under suitable structural constraints on the multifunctions.

The results significantly enrich the discussion in functional analysis by clarifying how different integrability conditions influence the adaptation of integration to order multivalued integrals. These insights emphasize the interplay between the lattice structure of the Banach space and the type of integrability feasible for multifunctions, and they suggest several lines of further inquiry.

An unresolved challenge remains to elucidate the connections between order multivalued integrability and Aumann integrals for not necessarily simple multivalued functions. This issue opens up a wide range of investigations concerning the extension of integration principles into non-linear frameworks, with potential impact on both theoretical mathematics and applications in different scientific fields.

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REFERENCES

[1] A. Boccuto, A.M. Minotti, A.R. Sambucini, Set-Valued Kurzweil–Henstock Integral in Riesz Space Setting, PanAmer. Math. J. 23 (2013), 57–74.

[2] A. Boccuto, D. Candeloro, A.R. Sambucini, Henstock Multivalued Integrability in Banach Lattices With Respect to Pointwise Non Atomic Measures, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. 26 (2015), 363–383. https://doi.org/10.4171/RLM/710.

[3] A. Boccuto, A.R. Sambucini, A McShane Integral for Multifunctions, J. Concr. Appl. Math. 2 (2004), 307–325.

[4] A.V. Bukhvalov, A.I. Veksler, V.A. Geiler, Normed Lattices, J. Sov. Math. 18 (1982), 516–551. https://doi.org/10.1007/BF01084868.

[5] A.V. Bukhvalov, A.I. Veksler, G.Y. Lozanovskii, Banach Lattices–Some Banach Aspects of Their Theory, Russ. Math. Surv. 34 (1979), 159–212. https://doi.org/10.1070/rm1979v034n02abeh002909.

[6] B. Cascales, V. Kadets, J. Rodriguez, Measurable Selectors and Set-Valued Pettis Integral in Non-Separable Banach Spaces, J. Funct. Anal. 256 (2009), 673–699. https://doi.org/10.1016/j.jfa.2008.10.022.

[7] B. Cascales, V. Kadets, J. Rodriguez, Measurability and Selections of Multifunctions in Banach Spaces, J. Convex Anal. 17 (2010), 229–240.

[8] D. Candeloro, L. Di Piazza, K. Musiał, A.R. Sambucini, Relations among Gauge and Pettis Integrals for Cwk(x)-Valued Multifunctions, Ann. Mat. Pura Appl. 197 (2018), 171–183. https://doi.org/10.1007/s10231-017-0674-z.

[9] D. Candeloro, L. Di Piazza, K. Musiał, A.R. Sambucini, Some New Results on Integration for Multifunction, Ric. Mat. 67 (2018), 361–372. https://doi.org/10.1007/s11587-018-0376-x.

[10] D. Candeloro, A.R. Sambucini, Comparison Between Some Norm and Order Gauge Integrals in Banach Lattices, PanAmer. Math. J. 25 (2015), 1–16.

[11] K. Cichoń, M. Cichoń, Some Applications of Nonabsolute Integrals in the Theory of Differential Inclusions in Banach Spaces, in: G.P. Curbera, G. Mockenhaupt, W.J. Ricker (Eds.), Vector Measures, Integration and Related Topics, Birkhäuser, Basel, 2009: pp. 115–124. https://doi.org/10.1007/978-3-0346-0211-2_11.

[12] A. Croitoru, A. Gavriluţ, Comparison Between Birkhoff Integral and Gould Integral, Mediterr. J. Math. 12 (2014), 329–347. https://doi.org/10.1007/s00009-014-0410-5.

[13] L. Di Piazza, B. Satco, A New Result on Impulsive Differential Equations Involving Non-Absolutely Convergent Integrals, J. Math. Anal. Appl. 352 (2009), 954–963. https://doi.org/10.1016/j.jmaa.2008.11.048.

[14] L. Di Piazza, V. Marraffa, The McShane, PU and Henstock Integrals of Banach Valued Functions, Czechoslov. Math. J. 52 (2002), 609–633. https://doi.org/10.1023/a:1021736031567.

[15] D.H. Fremlin, Measure Theory, Vol. 3: Measure Algebras, Torres Fremlin, Colchester, 2002.

[16] M. Shkembi, S. Ceno, J. Shkembi, Some Convergence Results for the Strong Versions of Order-Integrals in Lattice Spaces, Math. Stat. 11 (2023), 528–533. https://doi.org/10.13189/ms.2023.110308.

[17] M. Shkembi, I. Temaj, A. Tato, A Note of the Mcshane Integral on the Riesz Space, Int. J. Math. Anal. 7 (2013), 807–819. https://doi.org/10.12988/ijma.2013.13079.