Asia Pac. J. Math. 2026 **13**:43

WEIGHTED SEMINORM MCSHANE INTEGRAL ON $C([a,b])$: THEORY AND APPLICATION

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Received Jan. 30, 2026

Abstract. We develop a weighted–seminorm framework for the McShane integral of functions taking values in the Riesz space $C([a,b])$ of real-valued continuous functions on a compact interval. The space is equipped with the locally convex topology generated by the family of weighted seminorms

$$p_\varphi(f) = \sup_{x\in[a,b]} |\varphi(x)f(x)|, \qquad \varphi \in C([a,b])^+,$$

which coincides with the classical supremum topology.

Within this setting, we introduce an extended definition of the McShane integral based on gauge-fine partitions and (D)-sequences, and we prove existence and uniqueness of the integral. Furthermore, we obtain several convergence theorems—including uniform and monotone convergence—and show that the resulting theory is fully consistent with the classical McShane (Bochner) integral for Banach-valued functions. An illustrative example demonstrates how weighted seminorms interact with McShane integrability in $C([a,b])$. This approach connects the structure of locally convex Riesz spaces with concrete function spaces and provides a basis for future extensions to more general ordered topological vector spaces.

2020 Mathematics Subject Classification. 46G10; 46A40; 46E40.

Key words and phrases. McShane integral; Riesz space; locally convex space; weighted seminorm; uniform convergence; monotone convergence; $C([a,b])$; Bochner integral.

1. Introduction

The McShane integral plays a central role in modern integration theory, extending the classical Riemann and Lebesgue approaches and providing a powerful framework for Banach-valued functions. Foundational work by McShane, Gordon, Bartle, Schwabik and Guoju has established the integral as one of the most flexible nonabsolute integrals available for vector-valued analysis (see Gordon [1], Bartle [2], Schwabik–Guoju [3]).

DOI: 10.28924/APJM/13-43

Parallel to this development, Riesz spaces (vector lattices) and locally convex ordered spaces have become fundamental tools in functional analysis, measure theory, and applications to economics and probability. Classical sources include Luxemburg–Zaanen [4], Meyer-Nieberg [11], and the monographs of Meise–Vogt [14] and Schaefer–Wolff [13]. Of particular relevance are the works of Riečan and collaborators on Henstock–Kurzweil and McShane type integrals in ordered and lattice-structured spaces [5], [6], [7].

In recent years there has been significant interest in developing integration theory on Riesz spaces and in locally convex settings. Results on McShane integrability in Riesz spaces, convergence theorems, and applications have been obtained by Temaj and Tato [9] and by Shkëmbi, Sila, and Liftaj [8], among others, showing that the abstract theory admits meaningful extensions to rich ordered structures provided that the topology is compatible with the lattice order.

The purpose of this paper is to specialize this general theory to the classical function space $C([a,b])$, viewed as a locally convex Riesz space endowed with the weighted seminorms

$$p_\varphi(f) = \sup_{x\in[a,b]} \big|\varphi(x)f(x)\big|, \quad \varphi \in C([a,b])^+.$$

Although $C([a,b])$ is not Dedekind complete, it remains a natural test space for ordered integration: it is a Banach lattice under the supremum norm, and the family $\{p_\varphi\}$ generates exactly the same topology. This makes $C([a,b])$ an ideal setting for illustrating how McShane integrability behaves under weighted seminorm structures.

We establish an extended definition of the McShane integral for $C([a,b])$-valued functions using gauge-fine partitions and (D)-sequences adapted from the theory of Riesz-space integrals. We then prove existence and uniqueness of the integral, provide seminorm characterizations, and obtain uniform and monotone convergence theorems. These results demonstrate that the weighted-seminorm McShane integral coincides with the classical McShane (Bochner) integral for Banach-valued functions, thus unifying the abstract and concrete approaches.

The paper is organized as follows. Section 2 introduces the locally convex structure of $C([a,b])$, recalls basic properties of weighted seminorms and order intervals, and defines the extended McShane integral using (D)-sequences. Section 3 establishes our main results, including existence, uniqueness, and two principal convergence theorems. Section 4 presents an illustrative example and concludes with remarks on possible extensions.

2. PRELIMINARIES

In this section we summarize the structural properties of the space $C([a,b])$ needed for the development of the weighted-seminorm McShane integral. The exposition follows standard sources on Banach lattices and locally convex Riesz spaces such as Luxemburg–Zaanen [4], Meyer-Nieberg [11], Schaefer–Wolff [13], and the integration theory for Riesz space-valued functions in [1], [5], [6], [12].

Riesz space structure of $C([a,b])$. Let $[a,b] \subset \mathbb{R}$ be compact. The space

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} : \ f \text{ continuous}\}$$

is partially ordered by

$$f \leq g \Longleftrightarrow f(x) \leq g(x) \quad \text{for all } x \in [a,b].$$

With this order, $C([a,b])$ is a Riesz space (vector lattice): for all $f, g \in C([a,b])$.

If $f, g \in C[a,b]$, we define

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \qquad \forall x \in [a,b] \\
(f,g) &= \{h \in C[a,b] \colon f < h < g\} \qquad \text{is called open interval} \\
[f,g] &= \{h \in C[a,b] \colon f \leq h \leq g\} \qquad \text{is called closed interval} \\
\frac{f}{g}(x) &= \frac{f(x)}{g(x)}, \qquad \forall x \in [a,b],\ g(x) \neq 0 \\
(f \vee g)(x) &= \sup_{x \in [a,b]} \{f(x), g(x)\} \\
(f \wedge g)(x) &= \inf_{x \in [a,b]} \{f(x), g(x)\} \\
|f|(x) &= |f(x)| \qquad \forall x \in [a,b]
\end{aligned}$$

Bartle and Sherbert [15] showed that if f and g are elements in $C[a,b]$ then $f+g$, $\frac{f}{g}$, $f \vee g$, $f \wedge g$, $|f|$ are also elements in $C[a,b]$.

We say that two intervals in $C[a,b]$ are disjoint if their intersection is empty, that is, if they have no common elements. Similarly, we will say that two intervals in $C[a,b]$ are non-overlapping if their intersection is either empty or contains at most one element.

The positive cone

$$C([a,b])^+ = \{f \in C([a,b]) : f(x) \geq 0 \text{ for all } x \in [a,b]\}$$

is generating and proper, making $C([a,b])$ a classical example of a Banach lattice when equipped with the supremum norm

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$

However, $C([a,b])$ is not Dedekind complete: there exist bounded sets whose pointwise supremum is only lower semicontinuous and hence fails to belong to $C([a,b])$ [4, Chap. 6].

Consequently, the general theory of McShane integration on Dedekind σ-complete Riesz spaces (as developed in [5], [6], [9]) is not directly applicable in this setting and therefore requires modification to accommodate the locally convex structure described below.

Weighted seminorm topology on $C([a,b])$. For every positive continuous function $\varphi \in C([a,b])^+$, define a seminorm

$$p_\varphi(f) = \sup_{x\in[a,b]} |\varphi(x)f(x)|, \quad f \in C([a,b]).$$

Let

$$P\big(C([a,b])\big) = \{\, p_\varphi : \varphi \in C([a,b])^+,\ \varphi \not\equiv 0\,\}.$$

Each p_φ is continuous and submultiplicative, and the family $P\big(C([a,b])\big)$ generates a Hausdorff locally convex topology $\tau(P)$ on $C([a,b])$.

This topology is exactly the supremum–norm topology. Indeed, for every $\varphi \in C([a,b])^+$,

$$p_\varphi(f) \le \|\varphi\|_\infty \, \|f\|_\infty,$$

so norm convergence implies convergence in every p_φ. Taking $\varphi \equiv 1$, we obtain

$$p_1(f) = \|f\|_\infty,$$

hence the norm topology is included in $\tau(P)$.

Therefore,

$$\tau\big(P(C([a,b]))\big) = \tau(\|\cdot\|_\infty).$$

Thus $C([a,b])$, equipped with $P\big(C([a,b])\big)$, becomes a locally convex Riesz space, consistent with the abstract framework used in [1], [5], [12].

Order intervals in $C([a,b])$. For $f, g \in C([a,b])$ with $f \le g$ we define the order interval

$$[f,g] = \{h \in C([a,b]) : f \le h \le g\}.$$

Which is a closed, convex, and order-bounded subset of the Banach lattice $C([a,b])$.

In what follows, we consider functions

$$F : T \to C([a,b]),$$

where $T = [f,g] \subset C([a,b])$ is an order interval endowed with a finite Borel measure μ supported on Borel subsets of T. The results of Riesz-space McShane integration in [6], [7], [12] serve as the conceptual foundation, while the present work specializes them to the concrete, fully developed structure of $C([a,b])$.

(D)-sequences in $C([a,b])$.

Definition 2.1. (Regulator/(D)-sequence in C([a,b]). A double sequence $(a_{i,j})_{i,j\in\mathbb{N}} \subset L^\infty([a,b])$ is called a *regulator* or *(D)-sequence*:

(1) If $a_{i,j} \geq a_{i,j+1} \quad \forall i,j \in \mathbb{N}$ and $\bigwedge_{j=1}^{\infty} a_{i,j} = 0 \quad \forall i \in \mathbb{N}$.

(2) $a_{i,j} \geq a_{i,j+1} \quad \forall i,j \in \mathbb{N}$ and for every seminorm $p_\varphi \in P(C([a,b]))$,

$$p_\varphi\left(\bigwedge_{j=1}^{\infty} a_{i,j}\right) = 0 \quad \forall i \in \mathbb{N} \quad \text{or} \quad \lim_{j\to\infty} p_\varphi(a_{i,j}) = 0.$$

We write $a_{i,j} \downarrow 0$.

This notion originates from Riečan–Vrabelová [7] and is crucial for McShane-type integrals in ordered spaces [5], [6].

Using $L^\infty([a,b])$ rather than $C([a,b])$ ensures existence of suprema and infima, because $L^\infty([a,b])$ is *Dedekind complete*, unlike $C([a,b])$, while still controlling weighted seminorms.

In this section we introduce the notion of McShane integrability for functions taking values in the locally convex Riesz space $C([a,b])$, equipped with the weighted seminorms

$$p_\varphi(f) = \sup_{x\in[a,b]} |\varphi(x)f(x)|, \quad \varphi \in C([a,b])^+.$$

Our definition adapts the general framework of McShane integration in ordered topological vector spaces developed in Gordon [1], Bartle [2], Riečan [6], and Marraffa [12], and specializes it to the structure of $C([a,b])$. Because $C([a,b])$ is not Dedekind complete, the regulators (D)-sequences will be taken in the Dedekind complete space $L^\infty([a,b])$, but the integral itself will always lie in $C([a,b])$.

Division and Partition of the Order Interval

Let $[f,g]$ be the closed order interval subset of $C[a,b]$. A division of $[f,g]$ is any finite set

$$\{h_0, h_1, \ldots, h_n\} \subset [f,g],$$

where $h_0 = f$, $h_n = g$ and $h_{i-1} \leq h_i$ for all $i = 1,2,\ldots,n$.

A partition of $[f,g]$ is a finite collection

$$\{([h_{i-1}, h_i], t_i) : i = 1,2,\ldots,n\}$$

of interval-element pairs such that $t_i \in [h_{i-1}, h_i]$ for every $i = 1,2,\ldots,n$, where $\{h_0, h_1, \ldots, h_n\}$ is a division of $[f,g]$.

Let θ be the null element in $C[a,b]$, where $\theta(x) = 0$ for every $x \in [a,b]$. A function $\delta : I \to C[a,b]$ is said to be a gauge on I if $\delta(h) > \theta$ for every $h \in I$.

Definition 2.2. Let δ be a gauge on $[f,g]$. A partition

$$\Pi = \{([h_{i-1}, h_i], t_i) : i = 1,2,\ldots,n\}$$

is said to be δ-*fine* on $[f, g]$ if

$$[h_{i-1}, h_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$$

for every $i = 1, 2, \ldots, n$.

Let $T = [f, g] \subset C([a, b])$ be an order interval. A division of T is any finite increasing sequence

$$E_1 \leq E_2 \leq \cdots \leq E_n,$$

where $E_1 = f$, $E_n = g$, and the inequality is pointwise:

$$E_i(x) \leq E_{i+1}(x) \quad \forall x \in [a, b].$$

A **partition** of T is a finite collection $\Pi = \{(E_i, t_i) : i = 1, \ldots, n\}$, where:

- $\{E_1, \ldots, E_n\}$ is a division of T;
- each tag $t_i \in [E_i, f]$ or $t_i \in [g, E_i]$.

This formulation is the natural extension of tagged partitions used in the McShane [1] and Henstock–Kurzweil integrals [2], [3].

Gauge on the order interval.

Let $\theta(x) = 0$, $x \in [a, b]$, be the zero function in $C([a, b])$. A gauge is a function $\gamma : T \to C([a, b])^+$ satisfying

$$\gamma(E)(x) > 0 \quad \forall E \in T,\ \forall x \in [a, b].$$

Thus $\gamma(E)$ is a continuous positive "radius function" depending on the tag E. This formulation generalizes the classical gauge $\gamma : T \to (0, \infty)$, and is compatible with the weighted seminorm topology.

A partition $\Pi = \{(E_i, t_i)\}$ is γ-fine if

$$|E_i(x) - t_i(x)| \leq \gamma(t_i)(x) \quad \forall x \in [a, b],\ \forall i.$$

Equivalently,

$$E_i(x) \in [t_i(x) - \gamma(t_i)(x), t_i(x) + \gamma(t_i)(x)].$$

In terms of weighted seminorms,

$$p_{\gamma(t_i)}(E_i - t_i) = \sup_{x \in [a,b]} |\gamma(t_i)(x)(E_i(x) - t_i(x))| \leq 1.$$

This formulation integrates seamlessly with the McShane integral in locally convex Riesz spaces [5], [12].

Let $T = [f, g] \subset C([a, b])$ be an order interval, μ a finite Borel measure on T, $P(C([a, b])) = \{p_\varphi\}$ the family of weighted seminorms, $(a_{i,j})_{(i,j \in \mathbb{N})} \subset L^\infty([a, b])^+$ a (D)-sequence.

We now give the integral definition.

Definition 2.3 (McShane integrability)**.** A function $F : T \to C([a,b])$ is said to be McShane integrable (with respect to $P(C([a,b]))$ and μ) if there exist:

(1) an element $\omega \in C([a,b])$;
(2) a (D)-sequence $(a_{i,j})_{i,j\in\mathbb{N}} \subset L^\infty([a,b])^+$;

such that, for every map $\phi : \mathbb{N} \to \mathbb{N}$ and every seminorm $p_\varphi \in P(C([a,b]))$, there exists a gauge $\gamma : T \to C([a,b])^+$ satisfying

$$p_\varphi\big(S(F,\Pi) - \omega\big) \le \bigvee_{i=1}^{\infty} a_{i,\phi(i)} \quad \text{for all } \Pi \in A_T(\gamma),$$

where

$$S(F,\Pi) = \sum_{i=1}^{n} F(t_i)\,\mu(E_i) \in C([a,b])$$

is the associated Riemann sum.

The element ω is unique and is called the McShane integral of F:

$$\omega = \int_T F\,d\mu.$$

Basic Properties The following hold (proofs follow from [1], [2], [3], [12]):

Linearity: If F, G are integrable, then so are $F + G$ and cF, and

$$\int_T (F+G)\,d\mu = \int_T F\,d\mu + \int_T G\,d\mu.$$

Order boundedness: If $F(t) \in [f,g]$ for all t, then

$$\int_T F\,d\mu \in [f,g].$$

Consistency with norm McShane/Bochner integral: If F is norm-McShane integrable (or Bochner integrable), the integral coincides with Definition 2.3.

These properties justify working in $C([a,b])$ using weighted seminorms, without losing any classical features of vector-valued integration.

3. MAIN RESULTS

Convergence Theorems in $C([a,b])$

In this section we establish two fundamental convergence theorems for McShane integrable mappings

$$F : T = [f,g] \subset C([a,b]) \to C([a,b]),$$

working within the locally convex framework generated by the weighted seminorms

$$P(C([a,b])) = \{p_\varphi\}_{\varphi\in C([a,b])^+}$$

The proofs follow the classical strategy of Gordon [1], Bartle [2] and Schwabik–Guoju [3], adapted to our seminorm-based definition and the use of (D)-sequences in $L^\infty([a,b])^+$.

Uniform Convergence Theorem This result shows that uniform convergence in the supremum norm ensures convergence of McShane integrals, exactly as in the Banach-space setting.

Theorem 3.1 (Uniform Convergence Theorem). Let $F_n : T \to C([a,b])$, $n \in \mathbb{N}$, be McShane integrable functions, and suppose that

$$\sup_{t\in T} \|F_n(t) - F(t)\|_\infty \to 0 \quad (n \to \infty).$$

Then:

(1) The limit function F is McShane integrable in the sense of Definition 2.3;

(2) The integrals converge in $C([a,b])$:

$$\left\| \int_T F_n \, d\mu - \int_T F \, d\mu \right\|_\infty \to 0,$$

or equivalently, for each weighted seminorm $p_\varphi \in P(C([a,b]))$,

$$p_\varphi \left(\int_T F_n \, d\mu - \int_T F \, d\mu \right) \to 0.$$

Proof. Fix a seminorm $p_\varphi \in P(C([a,b]))$. Because

$$\sup_{t\in T} \|F_n(t) - F(t)\|_\infty \to 0,$$

it follows that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|F_n(t) - F(t)\|_\infty \le \frac{\varepsilon}{\|\varphi\|_\infty} \quad \forall t \in T, \ \forall n \ge N.$$

Then for every tagged partition $\Pi = \{(E_i, t_i)\}$ we have

$$p_\varphi(S(F_n, \Pi) - S(F, \Pi)) \le \sup_{t\in T} p_\varphi(F_n(t) - F(t)) \cdot \mu(T) \le \varepsilon\mu(T).$$

Now choose a gauge γ guaranteed by the McShane integrability of F and of F_n, coming from the (D)-sequence regulator in Definition 2.3.

For all γ-fine partitions $\Pi \in \mathcal{A}_T(\gamma)$:

$$p_\varphi(S(F_n, \Pi) - \int_T F_n \, d\mu) \le \bigvee_{i=1}^\infty a_{i,\phi(i)},$$

$$p_\varphi(S(F, \Pi) - \int_T F \, d\mu) \le \bigvee_{i=1}^\infty a_{i,\phi(i)}.$$

Therefore,

$$p_\varphi\left(\int_T F_n\,d\mu - \int_T F\,d\mu\right) \le p_\varphi(S(F_n,\Pi) - S(F,\Pi)) + 2\bigvee_{i=1}^{\infty} a_{i,\phi(i)}$$

$$\le \varepsilon\mu(T) + 2\bigvee_{i=1}^{\infty} a_{i,\phi(i)}.$$

Letting $n \to \infty$ (so that $\varepsilon \to 0$) and then applying the weak-distributive property of the (D)-sequence (see [6,7]) shows that the second term tends to zero. Thus for every φ,

$$p_\varphi\left(\int_T F_n\,d\mu - \int_T F\,d\mu\right) \to 0.$$

Since $\|\cdot\|_\infty = p_1$, convergence holds in norm, completing the proof.

Monotone Convergence Theorem This theorem adapts the classical monotone convergence principle to the locally convex Riesz space $C([a,b])$. While monotone convergence typically requires Dedekind completeness, here the McShane structure and the use of L^∞ regulators allow us to recover the expected result.

Theorem 3.2 (Monotone Convergence Theorem). Let $F_n : T \to C([a,b])$ be McShane integrable functions satisfying the pointwise monotonicity condition

$$F_1(t)(x) \le F_2(t)(x) \le \cdots \qquad \forall t \in T,\ \forall x \in [a,b],$$

and assume the limit

$$\omega(x) = \lim_{n\to\infty}\left(\int_T F_n\,d\mu\right)(x)$$

exists uniformly in $x \in [a,b]$. Then:

(1) The pointwise limit F is McShane integrable;

(2) The integrals converge:

$$\int_T F\,d\mu = \lim_{n\to\infty}\int_T F_n\,d\mu \qquad \text{in } C([a,b]).$$

Proof. Because the sequence is increasing pointwise in $C([a,b])$, for each fixed $t \in T$ and $x \in [a,b]$ the limit

$$F(t)(x) = \lim_{n\to\infty} F_n(t)(x)$$

exists.

Since the integrals $\int_T F_n\,d\mu$ converge uniformly to ω, for every $\varepsilon > 0$ there exists N such that

$$\left\|\int_T F_n\,d\mu - \omega\right\|_\infty < \varepsilon \qquad \forall n \ge N.$$

For any seminorm p_φ,

$$p_\varphi\left(\int_T F_n\,d\mu - \int_T F_m\,d\mu\right) \le \|\varphi\|_\infty \left\|\int_T F_n\,d\mu - \int_T F_m\,d\mu\right\|_\infty,$$

so the sequence $\int_T F_n\,d\mu$ is Cauchy w.r.t. every seminorm p_φ. Because the topology is complete (Banach norm topology), the limit integral exists in $C([a,b])$.

To prove integrability of F, fix p_φ and let $(a_{i,j})$ be the regulator controlling the McShane sums of F_n. Monotonicity implies (as in Riečan [6]) that for all gauges γ,

$$S(F_n,\Pi) \uparrow S(F,\Pi),$$

and also

$$\int_T F_n\,d\mu \uparrow \int_T F\,d\mu.$$

Repeating the regulator-based estimate from Theorem 3.1 but using the monotonicity to avoid the uniform smallness term, we obtain:

$$p_\varphi(S(F,\Pi) - \int_T F\,d\mu) \le \bigvee_{i=1}^{\infty} a_{i,\phi(i)}.$$

Thus F satisfies Definition 2.3 and is McShane integrable.

The equality of integrals follows from the uniform convergence of the sequence.

4. EXAMPLE

In this section we present an explicit example illustrating the uniform convergence theorem for McShane integrable mappings into the Riesz space $C([a,b])$. The construction is classical in Banach space integration and appears in similar form in Gordon [1] and Bartle [2], adapted here to the weighted-seminorm framework developed in this paper.

Example 4.1. Let $T = [0,c] \subset \mathbb{R}$, $0 < c < 1$, and let μ be the Lebesgue measure on T. For each $n \in \mathbb{N}$, define the mapping

$$F_n : T \to C([0,1]), \quad F_n(t)(x) := t^n x,$$

for all $t \in T$ and $x \in [0,1]$. We show that:

(1) each F_n is McShane integrable with respect to the weighted seminorm family $P(C([0,1]))$;
(2) $F_n \to 0$ uniformly on T in the norm of $C([0,1])$;
(3) the integrals converge:

$$\int_0^c F_n(t)\,dt \to 0 \quad \text{in } C([0,1]);$$

(4) therefore Theorem 3.1 applies.

Each F_n is McShane integrable.

Since, for each fixed $t \in T$,

$$F_n(t)(x) = t^n x$$

is a continuous function of x, the map

$$t \mapsto F_n(t) \in C([0,1])$$

is continuous in the supremum norm.

Continuous functions from a compact interval into a Banach space are Bochner integrable (see [1], [3]), hence McShane integrable.

Thus, each $F_n \in M(T, P(C([0,1])), \mu)$.

Uniform convergence in $C([0,1])$. For each $t \in [0,c]$,

$$\|F_n(t)\|_\infty = \sup_{x\in[0,1]} |t^n x| = t^n.$$

Therefore,

$$\sup_{t\in T} \|F_n(t)\|_\infty = c^n \to 0 \quad (n \to \infty),$$

so $F_n \to 0$ uniformly on T in $C([0,1])$.

Computation of the integrals.

For each $x \in [0,1]$, we have

$$\left(\int_0^c F_n(t)\,dt\right)(x) = \int_0^c t^n x\,dt = x\frac{c^{n+1}}{n+1}.$$

Thus

$$\left\|\int_0^c F_n(t)\,dt\right\|_\infty = \sup_{x\in[0,1]} \left|x\frac{c^{n+1}}{n+1}\right| = \frac{c^{n+1}}{n+1} \to 0.$$

Hence

$$\int_0^c F_n(t)\,dt \to 0 \quad \text{in } C([0,1]).$$

Verification using weighted seminorms.

For any seminorm $p_\varphi \in P(C([0,1]))$,

$$p_\varphi\left(\int_0^c F_n\,dt\right) = \sup_{x\in[0,1]} \left|\varphi(x)\cdot x\frac{c^{n+1}}{n+1}\right| \le \|\varphi\|_\infty \frac{c^{n+1}}{n+1} \to 0.$$

Thus the convergence holds in every weighted seminorm.

Application of Theorem 3.1.

Since:

- $F_n \to 0$ uniformly in $\|\cdot\|_\infty$,
- each F_n is McShane integrable,
- the integrals converge to 0 in $C([0,1])$,

Theorem 3.1 immediately implies:

$$0 = \int_0^c \left(\lim_{n\to\infty} F_n(t) \right) dt = \lim_{n\to\infty} \int_0^c F_n(t)\, dt.$$

Conclusion of the Example.

This example shows how the weighted-seminorm McShane integral in $C([0,1])$ behaves exactly as in the classical Banach-space case. It also demonstrates that the locally convex formulation using regulators in L^∞ is fully compatible with the standard supremum-norm approach to Bochner/McShane integration.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R.A. Gordon, The McShane Integral of Banach-Valued Functions, Ill. J. Math. 34 (1990), 557–567. https://doi.org/10.1215/ijm/1255988170.

[2] R.G. Bartle, A Modern Theory of Integration, American Mathematical Society, Providence, RI, 2001. https://doi.org/10.1090/gsm/032.

[3] Š. Schwabik, Y. Guoju, Topics in Banach Space Integration, World Scientific, Singapore, 2005. https://doi.org/10.1142/5905.

[4] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North-Holland, Amsterdam, 1971.

[5] A. Boccuto, B. Riečan, On the Henstock-Kurzweil Integral for Riesz-Space-Valued Functions Defined on Unbounded Intervals, Czechoslov. Math. J. 54 (2004), 591–607. https://doi.org/10.1007/s10587-004-6411-y.

[6] B. Riečan, On the Kurzweil Integral for Functions With Values in Ordered Spaces I. Acta Math. Univ. Comeniana 56–57 (1990), 75–83.

[7] B. Riečan, P. Volauf: On a Technical Lemma in Lattice Ordered Groups, Acta Math. Univ. Comenianae 44–45 (1984), 31–35.

[8] M. Shkëmbi, E. Sila, S. Liftaj, Riesz Space-Valued Functions and Their Integration via the Mcshane Integral on Time Scales, Asia Pac. J. Math. 12 (2025), 24. https://doi.org/10.28924/APJM/12-24.

[9] I. Temaj, A. Tato, The Convergence Theorems for the McShane Integral of Functions Taking Values in a Locally Convex Space , Int. J. Math. Anal. 6 (2012), 1963–1976.

[10] F. Albiac, N.J. Kalton,, Topics in Banach Space Theory, Springer, 2006. https://doi.org/10.1007/0-387-28142-8.

[11] P. Meyer-Nieberg, Banach Lattices, Springer, Berlin, 1991.

[12] V. Marraffa, Non Absolutely Convergent Integrals of Functions Taking Values in a Locally Convex Space, Rocky Mt. J. Math. 36 (2006), 1577–1593. https://doi.org/10.1216/rmjm/1181069383.

[13] H.H. Schaefer, M.P. Wolff, Topological Vector Spaces, Springer New York, 1999. https://doi.org/10.1007/978-1-4612-1468-7.

[14] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford University Press, 1997.

[15] R.G. Bartle, D.R. Sherbert, Introduction to Real Analysis, Wiley, New York, 2011.