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FOUNDATIONS AND SUBSTRUCTURES OF SHEFFER STROKE IUP-ALGEBRAS

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Abstract. In this study, we introduce the concept of Sheffer stroke IUP-algebras and investigate their fundamental properties. Within this framework, we define eight specialized subsets associated with the structure: derived Sheffer stroke IUP-subalgebras, derived Sheffer stroke IUP-filters, derived Sheffer stroke IUP-ideals, derived Sheffer stroke strong IUP-ideals, transformed Sheffer stroke IUP-subalgebras, transformed Sheffer stroke IUP-filters, transformed Sheffer stroke IUP-ideals, and transformed Sheffer stroke strong IUP-ideals. The interrelationships among these eight subsets are rigorously examined to uncover their algebraic dependencies and hierarchical organization. Moreover, the existence of Sheffer stroke IUP-algebras is established through concrete illustrative examples. We further define the Cartesian product of Sheffer stroke IUP-algebras and investigate the role of homomorphisms within this algebraic framework. These homomorphisms are subsequently extended to the aforementioned specialized subsets, thereby deepening the structural understanding and enhancing the potential theoretical and practical applications of Sheffer stroke IUP-algebras.

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Key words and phrases. Sheffer stroke IUP-algebra; derived Sheffer stroke IUP-subalgebra; transformed Sheffer stroke IUP-subalgebra; transformed Sheffer stroke IUP-ideal; transformed Sheffer stroke IUP-filter.

1. Introduction

BCK-algebras and BCI-algebras were defined by Imai and Iséki, respectively, as algebraic structures among many others. These abstract algebras have attracted considerable interest and extensive study from researchers, and have inspired the development of numerous other abstract algebraic systems, such as BE-algebras [12], pseudo-BE-algebras [3], eBE-algebras [26], pseudo-eBE-algebras [29], pseudo CI-algebras [27], eGE-algebras [2], PSRU-algebras [37], KU-algebras [25], UP-algebras [8], UP-bialgebras [14], extension of KU/UP-algebras [28], IUP-algebra [9], obic-algebra [7], and others.

Since the introduction of the IUP-algebra by Iampan et al. [9], this novel algebraic structure—characterized by four distinguished subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals—has sparked a rich and expanding body of research. Its elegant axiomatic foundation

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and versatile structural potential have drawn attention from researchers in abstract algebra, logic, and fuzzy systems. In 2023, Chanmanee et al. [6] extended the theory by investigating the direct products of infinite families of IUP-algebras, introducing the notion of weak direct products and pivotal findings on (anti-)IUP-homomorphisms. Complementing this, a parallel study by Chanmanee et al. [5] explored the structural behavior of external direct products in dual IUP-algebras, enhancing the algebraic toolkit for structural combination. Further developments came in 2024 when Kuntama et al. [13] incorporated fuzzy set theory into the framework, giving rise to fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals. This opened a pathway to modeling algebraic structures in the presence of vagueness and imprecision. Suayngam et al. [34] deepened this direction by applying Fermatean fuzzy sets to IUP-algebras, including an analysis of characteristic sets and t-(strong) level subsets. The conceptual expansion continued with the integration of intuitionistic fuzzy sets [36] and neutrosophic sets [33], where necessary and sufficient conditions for each type of subset within IUP-algebras were carefully characterized. Suayngam et al. [35] then proposed the application of Pythagorean fuzzy sets to define corresponding IUP-substructures, highlighting structural and level-wise characteristics. More recently, Suayngam et al. [32] introduced a robust hybrid framework by fusing Pythagorean neutrosophic sets with IUP-algebras, leading to new theoretical extensions such as Pythagorean neutrosophic IUP-subalgebras, ideals, filters, and strong ideals. In 2025, Inthachot et al. [10] introduced bipolar fuzzy sets into the IUP-algebraic setting, offering an approach that captures both positive and negative tendencies in fuzzy membership. These dual-valued structures are especially relevant for applications in bipolar decision models and logic programming. Lastly, Suayngam et al. [31] synthesized prior developments by formulating intuitionistic neutrosophic IUP-algebras, thereby unifying multiple forms of uncertainty into a coherent algebraic system and deepening structural insights into IUP-subset relationships. Collectively, these contributions trace a compelling evolution of IUP-algebraic theory—from foundational constructs to a robust framework that seamlessly integrates fuzzy logic, uncertainty theory, and algebraic reasoning, reinforcing its relevance in both theoretical and applied domains.

From another perspective, a researcher introduced a novel concept, the Sheffer stroke operation, which was initially defined by Sheffer [30]. The first application of this operation demonstrated that every Boolean axiom or operation can be expressed in terms of the Sheffer stroke. This discovery has attracted significant interest from the research community, as it enables the reduction of axioms and the simplification of formula structures. Moreover, it has inspired numerous researchers to apply the Sheffer stroke operation to various other algebraic structures, such as ortholattices [4], orthoimplication algebras [1], Sheffer stroke basic algebras [24], filters of strong Sheffer stroke non-associative MV-algebras [23], Sheffer stroke Hilbert algebras [21], (fuzzy) filters of Sheffer stroke BL-algebras [22], Sheffer stroke BG-algebras [17], Sheffer stroke UP-algebras [15], Sheffer stroke BE-algebras [11], Sheffer stroke BCK-algebras [20], Sheffer stoke BCH-algebras [16], Sheffer stroke BM-algebras [18], Sheffer stroke INK-algebras [19], and others.

This paper is organized into several sections. The Introduction presents formal definitions of IUP-algebras and systematically develops their fundamental structural properties. The Main Results section introduces the concept of Sheffer stroke IUP-algebras and outlines their key characteristics. In this section, eight special subsets are defined: derived Sheffer stroke IUP-subalgebra, derived Sheffer stroke IUP-filter, derived Sheffer stroke IUP-ideal, derived Sheffer stroke strong IUP-ideal, transformed Sheffer stroke IUP-subalgebra, transformed Sheffer stroke IUP-filter, transformed Sheffer stroke IUP-ideal, and transformed Sheffer stroke strong IUP-ideal. The relationships among these subsets are then summarized. The subsequent section applies the concepts of Cartesian product and homomorphism to the framework of Sheffer stroke IUP-algebras. Finally, the paper concludes by summarizing the findings and discussing potential directions for future research and applications.

2. PRELIMINARIES

To establish a rigorous foundation for the study of Sheffer stroke IUP-algebras and their related substructures, it is essential to revisit key algebraic concepts and logical frameworks that support their development. This section outlines the fundamental definitions and properties of IUP-algebras, subalgebras, filters, and related notions, which form the basis for the algebraic constructions introduced in the later section.

By consolidating these core preliminaries, we aim to provide the necessary tools and terminology for understanding the structure and behavior of the proposed Sheffer stroke-based algebraic system. This foundational layer ensures clarity and consistency as we build toward the main results of this work.

Definition 2.1. [9] *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $\star$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 \star x = x) \tag{IUP-1}$$

$$(\forall x \in X)(x \star x = 0) \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x \star y) \star (x \star z) = y \star z) \tag{IUP-3}$$

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	3	0	5	1	2	4
2	5	2	0	4	1	3
3	1	3	4	0	5	2
4	4	5	3	2	0	1
5	2	4	1	5	3	0

Then $X = (X, \star, 0)$ is an IUP-algebra.

Example 2.2. [9] *Let $(G, \bullet, e)$ be a group with the identity element e in which each element is its own inverse. Under this condition, $(G, \bullet, e)$ inherently satisfies the axioms of an IUP-algebra.*

Example 2.3. [9] *Let X be a set, and let $\mathcal{P}(X)$ denote its power set. As shown in Example 2.2, $(\mathcal{P}(X), \triangle, \emptyset)$ forms an IUP-algebra, where $\triangle$ represents the symmetric difference between sets.*

Example 2.4. [9] *Let $(G, \cdot, e)$ be a group with the identity element e. Define a binary operation $\bullet$ on G by:*

$$(\forall x, y \in G)(x \bullet y = y \cdot x^{-1}) \tag{2.1}$$

Then $(G, \bullet, e)$ is an IUP-algebra.

Proposition 2.1. [9] *In an IUP-algebra $X = (X, \star, 0)$, the following assertions are valid:*

$$(\forall x, y \in X)((x \star 0) \star (x \star y) = y) \tag{2.2}$$

$$(\forall x \in X)((x \star 0) \star (x \star 0) = 0) \tag{2.3}$$

$$(\forall x, y \in X)((x \star y) \star 0 = y \star x) \tag{2.4}$$

$$(\forall x \in X)((x \star 0) \star 0 = x) \tag{2.5}$$

$$(\forall x, y \in X)(x \star ((x \star 0) \star y) = y) \tag{2.6}$$

$$(\forall x, y \in X)(((x \star 0) \star y) \star x = y \star 0) \tag{2.7}$$

$$(\forall x, y, z \in X)(x \star y = x \star z \Leftrightarrow y = z) \tag{2.8}$$

$$(\forall x, y \in X)(x \star y = 0 \Leftrightarrow x = y) \tag{2.9}$$

$$(\forall x \in X)(x \star 0 = 0 \Leftrightarrow x = 0) \tag{2.10}$$

$$(\forall x, y, z \in X)(y \star x = z \star x \Leftrightarrow y = z) \tag{2.11}$$

$$(\forall x, y \in X)(x \star y = y \Rightarrow x = 0) \tag{2.12}$$

$$(\forall x, y, z \in X)((x \star y) \star 0 = (z \star y) \star (z \star x)) \tag{2.13}$$

$$(\forall x, y, z \in X)(x \star y = 0 \Leftrightarrow (z \star x) \star (z \star y) = 0) \tag{2.14}$$

$$(\forall x, y, z \in X)(x \star y = 0 \Leftrightarrow (x \star z) \star (y \star z) = 0) \tag{2.15}$$

$$\text{the right and the left cancellation laws hold} \tag{2.16}$$

Within IUP-algebras, four fundamental subsets stand out: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets form a critical framework that deepens our understanding and facilitates the application of IUP-algebras across different mathematical contexts.

Definition 2.2. [9] *A nonempty subset S of an IUP-algebra $X = (X, \star, 0)$ is called*

(i) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \star y \in S) \tag{2.17}$$

(ii) *an IUP-filter of X if it satisfies the following conditions:*

$$\textit{the constant } 0 \textit{ of } X \textit{ is in } S \tag{2.18}$$

$$(\forall x, y \in X)(x \star y \in S, x \in S \Rightarrow y \in S) \tag{2.19}$$

(iii) *an IUP-ideal of X if it satisfies the condition (2.18) and the following condition:*

$$(\forall x, y, z \in X)(x \star (y \star z) \in S, y \in S \Rightarrow x \star z \in S) \tag{2.20}$$

(iv) *a strong IUP-ideal of X if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \star y \in S) \tag{2.21}$$

According to [9], IUP-filters constitute a unifying concept that encompasses both IUP-ideals and IUP-subalgebras. These two subsets, IUP-ideals and IUP-subalgebras, are generalizations of strong IUP-ideals. Particularly, in an IUP-algebra X, strong IUP-ideals are equivalent to the entire algebra X itself. This hierarchical relationship among these subsets is visually represented in Figure 1, illustrating the structure of special subsets within an IUP-algebra X.

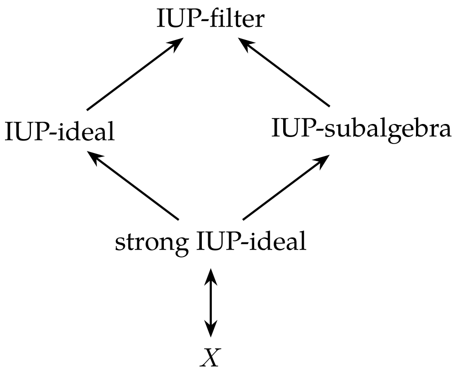

FIGURE 1. Structure of the four concepts of subsets in IUP-algebras

3. STRUCTURAL PROPERTIES OF SHEFFER STROKE IUP-ALGEBRAS

Building upon the foundational definitions and algebraic tools presented in the previous section, this part introduces the core contributions of this paper. Specifically, we define a new class of algebraic structures called Sheffer stroke IUP-algebras, formulated by integrating the Sheffer stroke operation into the IUP-algebraic framework. This novel system encapsulates logical and algebraic properties in a unified structure, offering a fresh perspective on implication-based algebras.

The main results presented here focus on the internal structure of Sheffer stroke IUP-algebras, including characterizations of subalgebras, filters, ideals, and strong ideals. Several theorems and propositions are provided to investigate the algebraic behavior and interactions of these substructures under the Sheffer operation. These findings not only establish fundamental properties but also lay the groundwork for further theoretical exploration and potential extensions to fuzzy and neutrosophic settings, as discussed in future research.

For clarity and consistency throughout this paper, we summarize in Table 1 the abbreviations and technical terms that will be used to represent the algebraic structures and related concepts introduced in the subsequent sections.

TABLE 1. List of abbreviations and specialized terms used in this paper

Abbreviation	**Definition**
SIUP-algebra	Sheffer stroke IUP-algebra
dSIUP-subalgebra	Derived Sheffer stroke IUP-subalgebra
dSIUP-filter	Derived Sheffer stroke IUP-filter
dSIUP-ideal	Derived Sheffer stroke IUP-ideal
dSsIUP-ideal	Derived Sheffer stroke strong IUP-ideal
tSIUP-subalgebra	Transformed Sheffer stroke IUP-subalgebra
tSIUP-filter	Transformed Sheffer stroke IUP-filter
tSIUP-ideal	Transformed Sheffer stroke IUP-ideal
tSsIUP-ideal	Transformed Sheffer stroke strong IUP-ideal

Definition 3.1. [30] *Let $X = (X, \mid)$ be a groupoid. The operation $\mid$ is said to be a Sheffer stroke if it satisfies the following conditions:*

$$(\forall x, y \in X)(x \mid y = y \mid x) \tag{S1}$$

$$(\forall x, y \in X)((x \mid x) \mid (x \mid y) = x) \tag{S2}$$

$$(\forall x, y, z \in X)(x \mid ((y \mid z) \mid (y \mid z)) = ((x \mid y) \mid (x \mid y)) \mid z) \tag{S3}$$

$$(\forall x, y \in X)((x \mid ((x \mid x) \mid (y \mid y))) \mid (x \mid ((x \mid x) \mid (y \mid y))) = x) \tag{S4}$$

Definition 3.2. *A Sheffer stroke IUP-algebra (briefly, SIUP-algebra) is an algebra $X = (X, \mid, \uparrow, 0)$ of type $(2, 2, 0)$, where X is a nonempty set, $\mid$ and $\uparrow$ are binary operations on X, and 0 is a constant in X. The binary operation $\uparrow$ is defined by*

$$x \uparrow y = x \mid (y \mid y)$$

for all $x, y \in X$*. The following axioms must also be satisfied:*

$$(\forall x \in X)(x \uparrow x = 0 \mid 0) \tag{SIUP-1}$$

$$\begin{aligned}(\forall x, y, z \in X)((((x \uparrow y) \mid (y \uparrow x)) \uparrow ((x \uparrow z) \mid (z \uparrow x))) \mid (((x \uparrow z) \mid (z \uparrow x)) \uparrow ((x \uparrow y) \mid (y \uparrow x)))\\ = (y \uparrow z) \mid (z \uparrow y))\end{aligned} \tag{SIUP-2}$$

Lemma 3.1. *Let* $X = (X, \mid, \uparrow, 0)$ *be an SIUP-algebra. Then the following properties hold for all* $x, y, z \in X$*:*

(1) $x \mid (x \mid (x \mid x)) = x \mid x$, *that is,* $x \mid (x \uparrow x) = x \mid x$

(2) $(x \mid (0 \mid 0)) \mid (x \mid (0 \mid 0)) = x$, *that is,* $(x \uparrow 0) \mid (x \uparrow 0) = x$

(3) $0 \mid x = 0 \mid 0$

(4) $(x \mid (x \mid x)) \mid (x \mid x) = x$, *that is,* $(x \uparrow x) \uparrow x = x$

(5) $(0 \mid 0) \mid (x \mid x) = x$, *that is,* $(0 \mid 0) \uparrow x = x$

(6) $x \mid (((x \mid (y \mid y)) \mid (y \mid y)) \mid ((x \mid (y \mid y)) \mid (y \mid y))) = 0 \mid 0$, *that is,* $x \uparrow ((x \uparrow y) \uparrow y) = 0 \mid 0$

(7) $(x \mid (0 \mid 0)) \mid (0 \mid (x \mid x)) = x$, *that is,* $(x \uparrow 0) \mid (0 \uparrow x) = x$

Proof. Let $x, y, z \in X$.

(1)

$$\begin{aligned} x \mid (x \mid (x \mid x)) &= ((x \mid x) \mid (x \mid x)) \mid (x \mid (x \mid x)) && \text{(by (S2))}\\ &= x \mid x. && \text{(by (S2))}\end{aligned}$$

(2)

$$\begin{aligned} (x \mid (0 \mid 0)) \mid (x \mid (0 \mid 0)) &= (x \mid (x \mid (x \mid x))) \mid (x \mid (x \mid (x \mid x))) && \text{(by (SIUP-1))}\\ &= (x \mid x) \mid (x \mid x) && \text{(by (1))}\\ &= x. && \text{(by (S2))}\end{aligned}$$

(3)

$$\begin{aligned} 0 \mid x &= ((0 \mid x) \mid (0 \mid 0)) \mid ((0 \mid x) \mid (0 \mid 0)) && \text{(by (2))}\\ &= 0 \mid 0. && \text{(by (S2))}\end{aligned}$$

(4)

$$\begin{aligned} (x \mid (x \mid x)) \mid (x \mid x) &= (x \mid x) \mid (x \mid x) && \text{(by (1))}\\ &= x. && \text{(by (S2))}\end{aligned}$$

(5)

$$\begin{aligned}(0\,|\,0)\,|\,(x\,|\,x) &= (x\,|\,(x\,|\,x))\,|\,(x\,|\,x) && \text{(by (SIUP-1))}\\ &= x. && \text{(by (1))}\end{aligned}$$

(6)

$$\begin{aligned}x\,|\,(((x\,|\,(y\,|\,y))\,|\,(y\,|\,y))\,|\,((x\,|\,(y\,|\,y))\,|\,(y\,|\,y))) &= ((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))\,|\,(x\,|\,(y\,|\,y)) && \text{(by (S3))}\\ &= 0\,|\,0. && \text{(by (SIUP-1))}\end{aligned}$$

(7)

$$\begin{aligned}(x\,|\,(0\,|\,0))\,|\,(0\,|\,(x\,|\,x)) &= ((0\,|\,0)\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|\,(0\,|\,0) && \text{(by (S2) and (3))}\\ &= (x\,|\,x)\,|\,(x\,|\,0) && \text{(by (5) and (3))}\\ &= x. && \text{(by (S2))}\end{aligned}$$

□

Definition 3.3. *A nonempty subset S of an SIUP-algebra $X = (X, |, \uparrow, 0)$ is called a derived Sheffer stroke IUP-subalgebra (briefly, dSIUP-subalgebra) of X if it satisfies the following condition:*

$$(\forall x, y \in S)((x \uparrow y)\,|\,(y \uparrow x) \in S) \tag{3.1}$$

Definition 3.4. *A nonempty subset S of an SIUP-algebra $X = (X, |, \uparrow, 0)$ is called a derived Sheffer stroke IUP-filter (briefly, dSIUP-filter) of X if it satisfies the following condition:*

$$\textit{the constant } 0 \textit{ of } X \textit{ is in } S \tag{3.2}$$

$$(\forall x, y \in X)((x \uparrow y)\,|\,(y \uparrow x) \in S, x \in S \Rightarrow y \in S) \tag{3.3}$$

Definition 3.5. *A nonempty subset S of an SIUP-algebra $X = (X, |, \uparrow, 0)$ is called a derived Sheffer stroke IUP-ideal (briefly, dSIUP-ideal) of X if it satisfies the condition (3.2) and the following condition:*

$$(\forall x, y, z \in X)((x \uparrow ((y \uparrow z)\,|\,(z \uparrow y)))\,|\,(((y \uparrow z)\,|\,(z \uparrow y)) \uparrow x) \in S, y \in S \Rightarrow (x \uparrow z)\,|\,(z \uparrow x) \in S) \tag{3.4}$$

Definition 3.6. *A nonempty subset S of an SIUP-algebra $X = (X, |, \uparrow, 0)$ is called a derived Sheffer stroke strong IUP-ideal (briefly, dSsIUP-ideal) of X if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow (x \uparrow y)\,|\,(y \uparrow x) \in S) \tag{3.5}$$

Theorem 3.1. *If S is a dSsIUP-ideal of an SIUP-algebra $X = (X, |, \uparrow, 0)$, then it satisfies the condition (3.2).*

Proof. Assume that S is a dSsIUP-ideal of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$. Let $x \in S$. Then

$$\begin{aligned} 0 &= (0 \mid 0) \mid (0 \mid 0) && \text{(by (S2))} \\ &= (x \uparrow x) \mid (x \uparrow x). && \text{(by (SIUP-1))} \end{aligned}$$

By the assumption, $0 \in S$. Hence, S satisfies the condition (3.2). □

Theorem 3.2. *If S is a dSIUP-subalgebra of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$, then it satisfies the condition* (3.2).

Proof. Assume that S is a dSIUP-subalgebra of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$. Let $x \in S$. Then

$$\begin{aligned} 0 &= (0 \mid 0) \mid (0 \mid 0) && \text{(by (S2))} \\ &= (x \uparrow x) \mid (x \uparrow x). && \text{(by (SIUP-1))} \end{aligned}$$

By the assumption, $0 \in S$. Hence, S satisfies the condition (3.2). □

Theorem 3.3. *A nonempty subset S of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ is a dSsIUP-ideal of X if and only if $S = X$.*

Proof. Assume that a nonempty subset S of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ is a dSsIUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned} x &= (x \mid x) \mid (x \mid 0) && \text{(by (S2))} \\ &= (x \mid (0 \mid 0)) \mid (0 \mid (x \mid x)) && \text{(by (5) and (3))} \\ &= (x \uparrow 0) \mid (0 \uparrow x). \end{aligned}$$

By the assumption and Theorem 3.1, $x \in S$. Hence, $S = X$.

The converse is obvious. □

Theorem 3.4. *Every dSsIUP-ideal of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ is a dSIUP-subalgebra of X.*

Proof. It is straightforward by Theorem 3.3. □

Example 3.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\mid$ defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	3	2	5	4	7	6
2	1	3	3	1	6	4	4	6
3	1	2	1	2	2	1	2	1
4	1	5	6	2	5	1	2	6
5	1	4	4	1	1	4	4	1
6	1	7	4	2	2	4	7	1
7	1	6	6	1	6	1	1	6

Then X is an SIUP-algebra. Observe that $S = \{0, 3, 4, 7\}$ is a dSIUP-subalgebra of X. Since $7 \in S$ but $(1 \uparrow 7) \,|\, (7 \uparrow 1) = (1 \,|\, (7 \,|\, 7)) \,|\, (7 \,|\, (1 \,|\, 1)) = (1 \,|\, 6) \,|\, (7 \,|\, 0) = 7 \,|\, 1 = 6 \notin S$, we have S is not a dSsIUP-ideal of X.

Theorem 3.5. *Every dSIUP-subalgebra of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$ is a dSIUP-filter of X.*

Proof. Assume that S is a dSIUP-subalgebra of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$. By Theorem 3.2. Then $0 \in S$. Let $x, y \in X$ be such that $(x \uparrow y) \,|\, (y \uparrow x), x \in S$. Then

$$\begin{aligned}
y &= (y \uparrow 0) \,|\, (0 \uparrow y) && \text{(by (7))}\\
&= (((x \uparrow y) \,|\, (y \uparrow x)) \uparrow ((x \uparrow 0) \,|\, (0 \uparrow x))) \,|\, \\
&\quad (((x \uparrow 0) \,|\, (0 \uparrow x)) \uparrow ((x \uparrow y) \,|\, (y \uparrow x))) && \text{(by (SIUP-2))}\\
&= (((x \uparrow y) \,|\, (y \uparrow x)) \uparrow x) \,|\, \\
&\quad (x \uparrow ((x \uparrow y) \,|\, (y \uparrow x))). && \text{(by (7))}
\end{aligned}$$

By the assumption, $y \in S$. Hence, S is a dSIUP-filter of X. □

Theorem 3.6. *Every dSIUP-filter of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$ is a dSIUP-subalgebra of X.*

Proof. Assume that S is a dSIUP-filter of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$. Let $x, y \in S$. Then

$$\begin{aligned}
(((x \uparrow y) \,|\, (y \uparrow x)) \uparrow x) \,|\, (x \uparrow ((x \uparrow y) \,|\, (y \uparrow x))) &= (((x \uparrow y) \,|\, (y \uparrow x)) \uparrow ((x \uparrow 0) \,|\, (0 \uparrow x))) \,|\, \\
&\quad (((x \uparrow 0) \,|\, (0 \uparrow x)) \uparrow ((x \uparrow y) \,|\, (y \uparrow x))) && \text{(by (7))}\\
&= (y \uparrow 0) \,|\, (0 \uparrow y) && \text{(by (SIUP-2))}\\
&= y. && \text{(by (7))}
\end{aligned}$$

By the assumption, $(x \uparrow y) \,|\, (y \uparrow x) \in S$. Hence, S is a dSIUP-subalgebra of X. □

Theorem 3.7. *Every dSIUP-filter of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$ is a dSIUP-ideal of X.*

Proof. Assume that S is a dSIUP-filter of an SIUP-algebra $X = (X, \,|\, , \uparrow, 0)$. Then $0 \in S$. Let $x, y, z \in X$ be such that $(x \uparrow ((y \uparrow z) \,|\, (z \uparrow y))) \,|\, (((y \uparrow z) \,|\, (z \uparrow y)) \uparrow x), y \in S$. Then

$$\begin{aligned}
&(((x \uparrow ((y \uparrow z) \,|\, (z \uparrow y))) \,|\, (((y \uparrow z) \,|\, (z \uparrow y)) \uparrow x)) \uparrow ((x \uparrow z) \,|\, (z \uparrow x))) \,|\, \\
&(((x \uparrow z) \,|\, (z \uparrow x)) \uparrow ((x \uparrow ((y \uparrow z) \,|\, (z \uparrow y))) \,|\, (((y \uparrow z) \,|\, (z \uparrow y)) \uparrow x)))\\
&= (((y \uparrow z) \,|\, (z \uparrow y)) \uparrow z) \,|\, \\
&\quad (z \uparrow ((y \uparrow z) \,|\, (z \uparrow y))) && \text{(by (SIUP-2))}\\
&= (((z \uparrow y) \,|\, (y \uparrow z)) \uparrow ((z \uparrow 0) \,|\, (0 \uparrow z))) \,|\,
\end{aligned}$$

$$\begin{aligned} &(((z \uparrow 0) \,|\, (0 \uparrow z)) \uparrow ((z \uparrow y) \,|\, (y \uparrow z))) && \text{(by (S1) and (7))} \\ &= (y \uparrow 0) \,|\, (0 \uparrow y) && \text{(by (SIUP-2))} \\ &= y. && \text{(by (7))} \end{aligned}$$

By the assumption, $(x \uparrow z) \,|\, (z \uparrow x) \in S$. Hence, S is a dSIUP-ideal of X. □

Theorem 3.8. *Every dSIUP-ideal of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is a dSIUP-filter of* X.

Proof. Assume that S is a dSIUP-ideal of an SIUP-algebra $X = (X, \,|\,, \uparrow, 0)$. Then $0 \in S$. Let $x, y \in X$ be such that $(x \uparrow y) \,|\, (y \uparrow x), x \in S$. Then

$$\begin{aligned} (y \uparrow ((x \uparrow 0) \,|\, (0 \uparrow x))) \,|\, (((x \uparrow 0) \,|\, (0 \uparrow x)) \uparrow y) &= (y \uparrow x) \,|\, (x \uparrow y) \\ &= (x \uparrow y) \,|\, (y \uparrow x). \end{aligned}$$

By the assumption, $y = (y \uparrow 0) \,|\, (0 \uparrow y) \in S$. Hence, S is a dSIUP-filter of X. □

Definition 3.7. *A nonempty subset* S *of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is called a transformed Sheffer stroke IUP-subalgebra (briefly, tSIUP-subalgebra) of* X *if it satisfies the following condition:*

$$(\forall x, y \in S)(x \,|\, y \in S)$$

Definition 3.8. *A nonempty subset* S *of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is called a transformed Sheffer stroke IUP-filter (briefly, tSIUP-filter) of* X *if it satisfies the condition (3.2) and the following condition:*

$$(\forall x, y \in X)(x \,|\, y \in S, x \in S \Rightarrow y \in S)$$

Definition 3.9. *A nonempty subset* S *of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is called a transformed Sheffer stroke IUP-ideal (briefly, tSIUP-ideal) of* X *if it satisfies the condition (3.2) and the following condition:*

$$(\forall x, y, z \in X)(x \,|\, (y \,|\, z) \in S, y \in S \Rightarrow x \,|\, z \in S)$$

Definition 3.10. *A nonempty subset* S *of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is called a transformed Sheffer stroke strong IUP-ideal (briefly, tSsIUP-ideal) of* X *if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \,|\, y \in S)$$

Theorem 3.9. *A nonempty subset* S *of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$ *is a tSsIUP-ideal of* X *if and only if* $S = X$.

Proof. Assume that a nonempty subset S of an SIUP-algebra $X = (X, \,|\,, \uparrow, 0)$ is a tSsIUP-ideal of X. Let $x \in X$. By Theorem 3.10, $0 \in S$. Then $x \,|\, 0 \in S$. Thus, $x = (x \,|\, x) \,|\, (x \,|\, 0) \in S$. Hence, $S = X$.

The converse is obvious. □

Theorem 3.10. *If* S *is a tSsIUP-ideal of an SIUP-algebra* $X = (X, \,|\,, \uparrow, 0)$*, then it satisfies the condition (3.2).*

Proof. It is straightforward by Theorem 3.9. □

Theorem 3.11. *Every tSsIUP-ideal of an SIUP-algebra* $X = (X, \mid, \uparrow, 0)$ *is a tSIUP-ideal of* X*.*

Proof. It is straightforward by Theorem 3.9. □

Theorem 3.12. *Every tSIUP-ideal of an SIUP-algebra* $X = (X, \mid, \uparrow, 0)$ *is a tSsIUP-ideal of* X*.*

Proof. Assume that S is a tSIUP-ideal of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$. Let $x \in X$ and $y \in S$. Since $(0 \mid 0) \mid (0 \mid y) = 0 \in S$. By the assumption, $y \mid (0 \mid x) = y \mid (0 \mid 0) = (0 \mid 0) \mid y \in S$. Thus, $x \mid y = y \mid x \in S$. Hence, S is a tSsIUP-ideal of X. □

Theorem 3.13. *Every tSsIUP-ideal of an SIUP-algebra* $X = (X, \mid, \uparrow, 0)$ *is a tSIUP-subalgebra of* X*.*

Proof. It is straightforward by Theorem 3.9. □

Example 3.2. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $\mid$ *defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	2	2	2	2	2	2	2	2
1	2	3	3	2	2	3	3	2
2	2	3	0	1	5	4	7	6
3	2	2	1	1	5	6	5	6
4	2	2	5	5	5	2	5	2
5	2	3	4	6	2	4	3	6
6	2	3	7	5	5	3	7	2
7	2	2	6	6	2	6	2	6

Then X *is an SIUP-algebra. Observe that* $S = \{0, 2\}$ *is a tSIUP-subalgebra of* X*. Since* $1 \in X$ *and* $2 \in S$ *but* $1 \mid 2 = 3 \notin S$*, we have* S *is not a tSsIUP-ideal of* X*.*

Theorem 3.14. *Every tSsIUP-ideal of an SIUP-algebra* $X = (X, \mid, \uparrow, 0)$ *is a tSIUP-filter of* X*.*

Proof. It is straightforward by Theorem 3.9. □

Example 3.3. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $\mid$ *defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	3	2	5	4	7	6
2	1	3	3	1	1	3	3	1
3	1	2	1	2	5	6	5	6
4	1	5	1	5	5	1	5	1
5	1	4	3	6	1	4	3	6
6	1	7	3	5	5	3	7	1
7	1	6	1	6	1	6	1	6

Then X is an SIUP-algebra. Observe that $S = \{0, 7\}$ is a tSIUP-filter of X. Since $3 \in X$ and $7 \in S$ but $3 \mid 7 = 6 \notin S$, we have S is not a tSsIUP-ideal of X.

Example 3.4. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\mid$ defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	2	2	2	2	2	2	2	2
1	2	3	3	2	2	3	3	2
2	2	3	0	1	5	4	7	6
3	2	2	1	1	5	6	5	6
4	2	2	5	5	5	2	5	2
5	2	3	4	6	2	4	3	6
6	2	3	7	5	5	3	7	2
7	2	2	6	6	2	6	2	6

Then X is an SIUP-algebra. Observe that $S = \{0, 2, 6, 7\}$ is a tSIUP-subalgebra of X. Since $7 \mid 3 = 6 \in X$ and $7 \in S$ but $3 \notin S$, we have S is not a tSIUP-filter of X.

Example 3.5. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\mid$ defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	3	2	5	4	7	6
2	1	3	3	1	3	1	3	1
3	1	2	1	2	6	4	4	6
4	1	5	3	6	5	1	3	6
5	1	4	1	4	1	4	4	1
6	1	7	3	4	3	4	7	1
7	1	6	1	6	6	1	1	6

Then X is an SIUP-algebra. Observe that $S = \{0, 3, 7\}$ is a tSIUP-filter of X. Since $0, 7 \in S$ but $0 \mid 7 = 1 \notin S$, we have S is not a tSIUP-subalgebra of X.

Theorem 3.15. *A nonempty subset S is dSsIUP-ideal of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ if and only if S is a tSsIUP-ideal of X.*

Proof. It is straightforward by Theorems 3.3 and 3.9. □

Theorem 3.16. *Every tSIUP-subalgebra of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ is a dSIUP-subalgebra of X.*

Proof. Assume that S is a tSIUP-subalgebra of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$. Let $x, y \in S$. Then $x \mid x, y \mid y \in S$, that is, $x \mid (y \mid y), y \mid (x \mid x) \in S$. Thus, $(x \mid (y \mid y)) \mid (y \mid (x \mid x)) \in S$. Hence, S is dSIUP-subalgebra of X. □

Example 3.6. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $\mid$ *defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	3	2	5	4	7	6
2	1	3	3	1	3	1	3	1
3	1	2	1	2	6	4	4	6
4	1	5	3	6	5	1	3	6
5	1	4	1	4	1	4	4	1
6	1	7	3	4	3	4	7	1
7	1	6	1	6	6	1	1	6

Then X *is an SIUP-algebra. Observe that* $S = \{0, 3\}$ *is a dSIUP-subalgebra of* X*. Since* $0 \in S$ *but* $0 \mid 0 = 1 \notin S$*, we have* S *is not a tSIUP-subalgebra of* X*.*

Example 3.7. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $\mid$ *defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	3	3	3	3	3	3	3	3
1	3	4	3	4	3	4	4	3
2	3	3	5	5	5	3	5	3
3	3	4	5	0	1	2	7	6
4	3	3	5	1	1	6	5	6
5	3	4	3	2	6	2	4	6
6	3	4	5	7	5	4	7	3
7	3	3	3	6	6	6	3	6

Then X *is an SIUP-algebra. Observe that* $S = \{0, 1, 3, 4\}$ *is a dSIUP-subalgebra of* X*. Since* $1 \mid 2 = 3 \in S$ *and* $1 \in S$ *but* $2 \notin S$*, we have* S *is not a tSIUP-filter of* X*.*

Example 3.8. *Let* $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *be a set with a binary operation* $\mid$ *defined by the following Cayley table:*

$\mid$	0	1	2	3	4	5	6	7
0	3	3	3	3	3	3	3	3
1	3	4	5	4	3	6	5	6
2	3	5	5	5	3	3	5	3
3	3	4	5	0	1	2	7	6
4	3	3	3	1	1	1	1	3
5	3	6	3	2	1	2	1	6
6	3	5	5	7	1	1	7	3
7	3	6	3	6	3	6	3	6

Then X is an SIUP-algebra. Observe that $S = \{0, 1, 2\}$ is a tSIUP-filter of X. Since $1 \in S$ and $2 \in S$ but $(1 \uparrow 2) \,|\, (2 \uparrow 1) = (1 \,|\, (2 \,|\, 2)) \,|\, (2 \,|\, (1 \,|\, 1)) = (1 \,|\, 5) \,|\, (2 \,|\, 4) = 6 \,|\, 3 = 7 \notin S$, we have S is not a dSIUP-subalgebra of X.

In what follows, we present the relationships among the dSsIUP-ideal, dSIUP-subalgebra, tSsIUP-ideal, tSIUP-subalgebra, and tSIUP-filter of X, as illustrated below.

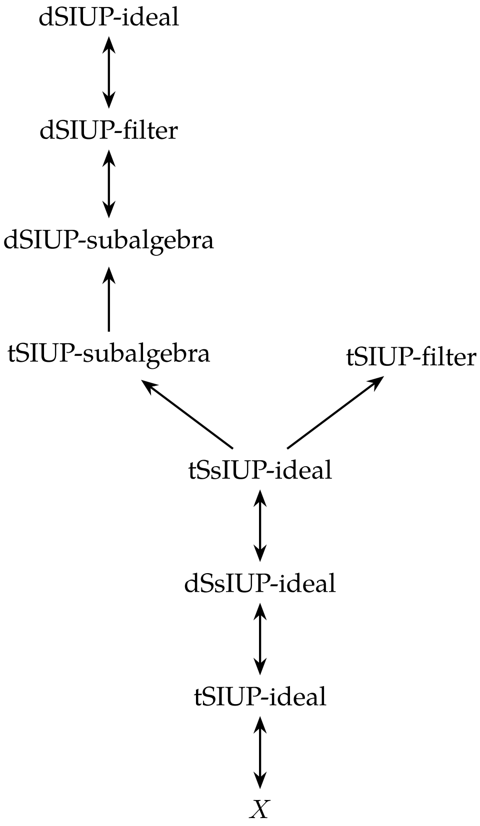

FIGURE 2. Structure of the eight new concepts of subsets in SIUP-algebras

Theorem 3.17. *Let $(X, \,|\,, \uparrow, 0)$ be an SIUP-algebra. If we define the binary operation $\star$ on X by $x \star y := (x \uparrow y) \,|\, (y \uparrow x)$, then $(X, \star, 0)$ is an IUP-algebra.*

Proof. Let x, y, z be any elements in X.

(IUP-1):

$$\begin{aligned} 0 \star x &= (0 \uparrow x) \,|\, (x \uparrow 0) \\ &= x. && \text{(by (7))} \end{aligned}$$

(IUP-2):

$$x \star x = (x \uparrow x) \,|\, (x \uparrow x)$$

$$\begin{aligned} &= (0 \mid 0) \mid (0 \mid 0) && \text{(by (SIUP-1))} \\ &= 0. && \text{(by (S2))} \end{aligned}$$

(IUP-3):

$$\begin{aligned} (x \star y) \star (x \star z) &= (((x \uparrow y) \mid (y \uparrow x)) \uparrow ((x \uparrow z) \mid (z \uparrow x))) \mid \\ &\quad (((x \uparrow z) \mid (z \uparrow x)) \uparrow ((x \uparrow y) \mid (y \uparrow x))) \\ &= (y \uparrow z) \mid (y \uparrow z) && \text{(by (SIUP-2))} \\ &= y \star z. \end{aligned}$$

Hence, $(X, \star, 0)$ is an IUP-algebra. □

Theorem 3.18. *Let* $(X, \mid_X, \uparrow_X, 0_X)$ *and* $(Y, \mid_Y, \uparrow_Y, 0_Y)$ *be SIUP-algebras. Then,* $(X \times Y, \mid_{X\times Y}, \uparrow_{X\times Y}, 0_{X\times Y})$ *is an SIUP-algebra where the set* $X \times Y$ *is the Cartesian product of* X *and* Y *and the operation* $\mid_{X\times B}$ *on this set is defined by* $(x_1, y_1) \mid_{X\times Y} (x_2, y_2) := (x_1 \mid_X x_2, y_1 \mid_Y y_2)$*, the operation* $\uparrow_{X\times B}$ *on this set is defined by* $(x_1, y_1) \uparrow_{X\times Y} (x_2, y_2) := (x_1 \mid_X (x_2 \mid_X x_2), y_1 \mid_Y (y_2 \mid_Y y_2))$*, and the fixed element is* $0_{X\times Y} := (0_X, 0_Y)$*.*

Proof. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be arbitrary elements in $X \times Y$.

(SIUP-1):

$$\begin{aligned} (x_1, y_1) \uparrow_{X\times Y} (x_1, y_1) &= (x_1 \mid_X (x_1 \mid_X x_1), y_1 \mid_Y (y_1 \mid_Y y_1)) \\ &= (0_X, 0_Y) \\ &= 0_{X\times Y}. \end{aligned}$$

(SIUP-2):

$$\begin{aligned} &((((x_1, y_1) \uparrow_{X\times Y} (x_2, y_2)) \mid_{X\times Y} ((x_2, y_2) \uparrow_{X\times Y} (x_1, y_1))) \uparrow_{X\times Y} \\ &(((x_1, y_1) \uparrow_{X\times Y} (x_3, y_3)) \mid_{X\times Y} ((x_3, y_3) \uparrow_{X\times Y} (x_1, y_1)))) \mid_{X\times Y} \\ &((((x_1, y_1) \uparrow_{X\times Y} (x_3, y_3)) \mid_{X\times Y} ((x_3, y_3) \uparrow_{X\times Y} (x_1, y_1))) \uparrow_{X\times Y} \\ &(((x_1, y_1) \uparrow_{X\times Y} (x_2, y_2)) \mid_{X\times Y} ((x_2, y_2) \uparrow_{X\times Y} (x_1, y_1)))) \\ &= ((((x_1 \uparrow_X x_2) \mid_X (x_2 \uparrow_X x_1)) \uparrow_X ((x_1 \uparrow_X x_3) \mid_X (x_3 \uparrow_X x_1))) \mid_X \\ &\quad (((x_1 \uparrow_X x_3) \mid_X (x_3 \uparrow_X x_1)) \uparrow_X ((x_1 \uparrow_X x_2) \mid_X (x_2 \uparrow_X x_1))), \\ &\quad (((y_1 \uparrow_Y y_2) \mid_Y (y_2 \uparrow_Y y_1)) \uparrow_Y ((y_1 \uparrow_Y y_3) \mid_Y (y_3 \uparrow_Y y_1))) \mid_Y \\ &\quad (((y_1 \uparrow_Y y_3) \mid_Y (y_3 \uparrow_Y y_1)) \uparrow_Y ((y_1 \uparrow_Y y_2) \mid_Y (y_2 \uparrow_Y y_1)))) \\ &= ((x_2 \uparrow_X x_3) \mid_X (x_3 \uparrow_X x_2), (y_2 \uparrow_Y y_3) \mid_Y (y_3 \uparrow_Y y_2)) \\ &= (x_2 \uparrow_X x_3, y_2 \uparrow_Y y_3) \mid_{X\times Y} (x_3 \uparrow_X x_2, y_3 \uparrow_Y y_2) \end{aligned}$$

$$= ((x_2, y_2) \uparrow_{X\times Y} (x_3, y_3)) \,|\,_{X\times Y}((x_3, y_3) \uparrow_{X\times Y} (x_2, y_2)).$$

Hence, $(X \times Y, \,|\,_{X\times Y}, \uparrow_{X\times Y}, 0_{X\times Y})$ is an SIUP-algebra. □

Definition 3.11. *Let $(X, \,|\,_X, \uparrow_X, 0_X)$ and $(Y, \,|\,_Y, \uparrow_Y, 0_Y)$ be SIUP-algebras. A mapping $f : X \to Y$ is called an SIUP-homomorphism if*

$$(\forall x, y \in X)(f(x \,|\,_X y) = f(x) \,|\,_Y f(y)).$$

Theorem 3.19. *Let $(X, \,|\,_X, \uparrow_X, 0_X)$ and $(Y, \,|\,_Y, \uparrow_Y, 0_Y)$ be SIUP-algebras, and let the mapping $f : X \to Y$ be an SIUP-homomorphism. If C is a tSIUP-subalgebra of X, then $f(C)$ is a tSIUP-subalgebra of Y.*

Proof. Assume that $(X, \,|\,_X, \uparrow_X, 0_X)$ and $(Y, \,|\,_Y, \uparrow_Y, 0_Y)$ are SIUP-algebras, and let the mapping $f : X \to Y$ be an SIUP-homomorphism. Let C be a tSIUP-subalgebra of X. Then for arbitrary elements $u, v \in f(C)$, there exist $x, y \in C$ such that $u = f(x)$ and $v = f(y)$. Then $u \,|\,_Y v = f(x) \,|\,_Y f(y) = f(x \,|\,_X y)$. Since $x \,|\,_X y \in C$. Thus, $u \,|\,_Y v \in f(C)$. Hence, $f(C)$ is a tSIUP-subalgebra of Y. □

Theorem 3.20. *Let $(X, \,|\,_X, \uparrow_X, 0_X)$ and $(Y, \,|\,_Y, \uparrow_Y, 0_Y)$ be SIUP-algebras, and let the mapping $f : X \to Y$ be an SIUP-homomorphism. If C is a dSIUP-subalgebra of X, then $f(C)$ is a dSIUP-subalgebra of Y.*

Proof. Assume that $(X, \,|\,_X, \uparrow_X, 0_X)$ and $(Y, \,|\,_Y, \uparrow_Y, 0_Y)$ are SIUP-algebras, and let the mapping $f : X \to Y$ be an SIUP-homomorphism. Let C be a dSIUP-subalgebra of X. Then for arbitrary elements $u, v \in f(C)$, there exist $x, y \in C$ such that $u = f(x)$ and $v = f(y)$. Then $(u \uparrow_Y v) \,|\,_Y (v \uparrow_Y u) = (f(x) \uparrow_Y f(y)) \,|\,_Y (f(y) \uparrow_Y f(x)) = f((x \uparrow_X y) \,|\,_X (y \uparrow_X x))$. Since $(x \uparrow_X y) \,|\,_X (y \uparrow_X x) \in C$. Thus, $(u \uparrow_Y v) \,|\,_Y (v \uparrow_Y u) \in f(C)$. Hence, $f(C)$ is a dSIUP-subalgebra of Y. □

4. Conclusion

In this study, we introduce the concept of Sheffer stroke IUP-algebras and investigate their fundamental properties. Within this framework, we define eight specialized subsets associated with the structure: derived Sheffer stroke IUP-subalgebras, derived Sheffer stroke IUP-filters, derived Sheffer stroke IUP-ideals, derived Sheffer stroke strong IUP-ideals, transformed Sheffer stroke IUP-subalgebras, transformed Sheffer stroke IUP-filters, transformed Sheffer stroke IUP-ideals, and transformed Sheffer stroke strong IUP-ideals. The interrelationships among these eight subsets are rigorously examined to uncover their algebraic dependencies and hierarchical organization.

Moreover, the existence of Sheffer stroke IUP-algebras is established through concrete illustrative examples. We further define the Cartesian product of Sheffer stroke IUP-algebras and investigate the role of homomorphisms within this algebraic framework. These homomorphisms are subsequently extended to the aforementioned specialized subsets, thereby deepening the structural understanding and enhancing the potential theoretical and practical applications of Sheffer stroke IUP-algebras.

In the future, this line of research could be extended by integrating the concept of Sheffer stroke IUP-algebras with various uncertainty-based frameworks, such as fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and related generalizations. Such integrations could provide fertile ground for developing more comprehensive algebraic systems capable of addressing a wider range of mathematical and real-world problems. This potential for cross-framework synthesis highlights the versatility and applicability of Sheffer stroke IUP-algebras in advancing modern algebraic logic and uncertainty modeling.

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