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# INTUITIONISTIC FUZZY HYPER $B$-IDEALS OF HYPER $B$-ALGEBRAS

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Abstract. The study introduces and analyzes intuitionistic fuzzy hyper $B$-ideals in hyper $B$-algebras. The study defines intuitionistic fuzzy subhyper $B$-algebras, hyper $B$-ideals, strong hyper $B$-ideals, $s$-weak hyper $B$-ideals, and weak hyper $B$-ideals, establishing their complete hierarchical relationships. Moreover, the paper examines these structure under some operations.

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## 1. Introduction

The development of algebraic systems has long been a central theme in mathematics, with new structures being introduced to extend existing concepts. In 2002, Neggers and Kim [3] proposed $B$-algebras to explore algebraic operations and their relationship with group theory. On the other hand, the theory of hyperstructures, introduced earlier by Marty [4], generalized classical operations by allowing the result of an operation to be a set rather than a single element. Endam [2] merged these two directions by introducing hyper $B$-algebras, which were later expanded by Vicedo and Vilela [9] through the study of ideals and homomorphisms.

Parallel to these algebraic developments, Zadeh [10] introduced fuzzy sets as a framework for handling uncertainty, later extended by Atanassov [1] into intuitionistic fuzzy sets (IFS) that incorporate both membership and non-membership values. These ideas have been successfully applied in areas such as decision-making. Their integration with algebraic systems has given rise to new structures, including fuzzy hyper $B$-algebras. In fact, Tabaranza and Vilela [7] applied the fuzzy set framework to hyper $B$-algebra and to hyper $B$-ideals of hyper $B$-algebra.

Despite these advances, the intersection of intuitionistic fuzziness and hyper $B$-algebra theory remains to be explored, in particular, the study of intuitionistic fuzzy hyper $B$-ideals has not yet been

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formally developed. Motivated by this gap, this study introduces the notion of intuitionistic fuzzy hyper $B$-ideals of hyper $B$-algebras.

## 2. Preliminaries

If $A$ and $B$ are nonempty subsets of $H$, then $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$. For simplicity, we write $x \circledast y$ instead of $x \circledast \{y\}$, $\{x\} \circledast y$, or $\{x\} \circledast \{y\}$. When $A \subseteq H$ and $x \in H$, we write $A \circledast x$ instead of $A \circledast \{x\}$. Similarly, we write $x \circledast A$ for $\{x\} \circledast A$. Thus, $A \circledast x = \bigcup_{a \in A} a \circledast x$ and $x \circledast A = \bigcup_{a \in A} x \circledast a$.

**Definition 2.1.** [2] A *hyper $B$-algebra* is a set $H$ with constant $0$ and hyperoperation $\circledast$, denoted by $(H; \circledast, 0)$, satisfy the following axioms: for all $x, y, z \in H$,

(H1) $x \ll x$;

(H2) $x \circledast H = H = H \circledast x$; and

(H3) $(x \circledast y) \circledast z = x \circledast (z \circledast (0 \circledast y))$,

where $x \ll y$ if and only if $0 \in x \circledast y$, and for every $A, B \subseteq H$, $A \ll B$ if and only if for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyper order*.

**Example 2.2.** [2] Let $H = \{0, a, b\}$ be a set with hyperoperation $\circledast$ defined by the Cayley below.

| $\circledast$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{a\}$ | $\{0, b\}$ | $\{0, a\}$ |
| $b$ | $\{b\}$ | $\{0, a\}$ | $\{0, a\}$ |

Then $(H; \circledast, 0)$ is a hyper $B$-algebra.

**Example 2.3.** [8] Let $H = [0, 1]$ be a set with hyperoperation $\circledast$ defined on $H$ as follows: For all $x, y \in H$,

$$x \circledast y = \begin{cases} [0, x], & \text{if } x > y, \\ \{0, x\}, & \text{if } x = y, \\ [0, y], & \text{if } x < y. \end{cases}$$

Then $(H; \circledast, 0)$ is a hyper $B$-algebra.

**Definition 2.4.** [9] Let $I$ be a subset of a hyper $B$-algebra $H$ such that $0 \in I$. Then

(i) $I$ is a *hyper $B$-ideal* of $H$ if for all $x, y \in H$, $x \circledast y \ll I$ and $y \in I$ imply that $x \in I$,

(ii) $I$ is a *weak hyper $B$-ideal* of $H$ if for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$, and

(iii) $I$ is a *strong hyper $B$-ideal* of $H$ if for all $x, y \in H$, $x \circledast y \cap I \neq \varnothing$ and $y \in I$ imply that $x \in I$.

**Definition 2.5.** [8] Let $H$ be a hyper $B$-algebra and $\mu$ be a fuzzy subset of $H$. Then $\mu$ is called:

(i) a *fuzzy hyper $B$-ideal* of $H$ if for all $x, y \in H$, $x \ll y$ implies $\mu(x) \geq \mu(y)$ and

$$\mu(0) \geq \mu(x) \geq \min\Big\{\inf_{z \in x \circledast y} \mu(z),\ \mu(y)\Big\};$$

(ii) a *fuzzy weak hyper $B$-ideal* of $H$ if for all $x, y \in H$,

$$\mu(0) \geq \mu(x) \geq \min\Big\{\inf_{z \in x \circledast y} \mu(z),\ \mu(y)\Big\};$$

(iii) a *fuzzy strong hyper $B$-ideal* of $H$ if for all $x, y \in H$,

$$\inf_{a \in x \circledast x} \mu(a) \geq \mu(x) \geq \min\Big\{\sup_{z \in x \circledast y} \mu(z),\ \mu(y)\Big\}.$$

**Definition 2.6.** [1,6] Let $X$ be a nonempty set. An *intuitionistic fuzzy set* $A$ of $X$ is expressed in the form $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ define the degree of membership and nonmembership of $x \in X$ to the set $A$, respectively. Moreover, for every $x \in X$, $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. When $\mu_A(x) + \lambda_A(x) = 1$, $A$ is called a *fuzzy set*.

## 3. Intuitionistic Fuzzy Hyper $B$-Ideals of Hyper $B$-Algebras

In what follows, a hyper $B$-algebra $(H, \circledast, 0)$ shall be denoted by $H$, and an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in H\}$ shall be denoted by $A$, and the term "intuitionistic fuzzy set" will be abbreviated as $IFS$.

**Definition 3.1.** Let $H$ be a hyper $B$-algebra and $A$ be an $IFS$ of $H$. Then $A$ is called an *intuitionistic fuzzy sub-hyper $B$-algebra* of $H$ if the following conditions hold.

(i) $\inf_{a \in x \circledast y}\{\mu_A(a)\} \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{a \in x \circledast y}\{\lambda_A(a)\} \leq \max\{\lambda_A(x), \lambda_A(y)\}$, for all $x, y \in H$.

(ii) For all $b, x \in H$, there exists $y \in H$ such that $x \in b \circledast y$, $\mu_A(y) \geq \min\{\mu_A(b), \mu_A(x)\}$, and $\lambda_A(y) \leq \max\{\lambda_A(b), \lambda_A(x)\}$.

(iii) For all $b, x \in H$, there exists $z \in H$ such that $x \in z \circledast b$, $\mu_A(z) \geq \min\{\mu_A(b), \mu_A(x)\}$, and $\lambda_A(z) \leq \max\{\lambda_A(b), \lambda_A(x)\}$.

Conditions $(ii)$ and $(iii)$ are called the *left intuitionistic fuzzy reproduction axiom* and the *right intuitionistic fuzzy reproduction axiom*, respectively.

**Example 3.2.** Consider the hyper $B$-algebra $H = [0,1]$ in Example 2.3. Define an $IFS$ $A$ on $H$ by $\mu_A(x) = 1 - x$, and $\lambda_A(x) = x$. Thus, by routine calculations, $A$ is an intuitionistic fuzzy subhyper $B$-algebra.

**Definition 3.3.** Let $H$ be a hyper $B$-algebra and $A$ be an $IFS$ of $H$. Then $A$ is said to be an *intuitionistic fuzzy hyper $B$-ideal* of $H$ if for all $x, y \in H$:

$(i)$ $x \ll y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$;

(ii) $\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf\limits_{a\in x\circledast y}\{\mu_A(a)\},\ \mu_A(y)\right\}$; and

(iii) $\lambda_A(0) \leq \lambda_A(x) \leq \max\left\{\sup\limits_{b\in x\circledast y}\{\lambda_A(b)\},\ \lambda_A(y)\right\}$.

**Example 3.4.** Consider the hyper $B$-algebra $H = \{0, a, b\}$ in Example 2.2. Define an IFS $A$ on $H$ by $\mu_A(0) = 0.7$, $\mu_A(a) = 0.4$, $\mu_A(b) = 0.4$, $\lambda_A(0) = 0.1$, and $\lambda_A(a) = \lambda_A(b) = 0.3$. Then by routine calculations, $A$ is an intuitionistic fuzzy hyper $B$-ideals of $H$.

**Definition 3.5.** Let $H$ be a hyper $B$-algebra and $A$ be an $IFS$ of $H$. Then $A$ is said to be an *intuitionistic fuzzy strong hyper $B$-ideal* of $H$ if for all $x, y \in H$:

(i) $\inf\limits_{a\in x\circledast y}\{\mu_A(a)\} \geq \mu_A(x) \geq \min\left\{\sup\limits_{b\in x\circledast y}\{\mu_A(b)\},\ \mu_A(y)\right\}$, and

(ii) $\sup\limits_{c\in x\circledast y}\{\lambda_A(c)\} \leq \lambda_A(x) \leq \max\left\{\inf\limits_{d\in x\circledast y}\{\lambda_A(d)\},\ \lambda_A(y)\right\}$.

**Example 3.6.** Consider the hyper $B$-algebra $H = \{0, a, b\}$ in Example 2.2. Define an $IFS$ $A$ by $\mu_A(0) = \mu_A(a) = \mu_A(b) = 0.5$, and also $\lambda_A(0) = 0.4$ and $\lambda_A(a) = \lambda_A(b) = 0.4$. Thus, by routine calculations, $A$ is an intuitionistic fuzzy strong hyper $B$-ideal of $H$.

**Definition 3.7.** Let $H$ be a hyper $B$-algebra and $A$ be an $IFS$ of $H$. Then $A$ is said to be an *intuitionistic fuzzy s-weak hyper $B$-ideal* of $H$ if the following conditions hold:

($S_1$) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in H$; and

($S_2$) for every $x, y \in H$, there exist $a, b \in x \circledast y$ such that $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(y)\}$, and $\lambda_A(x) \leq \max\{\lambda_A(b), \lambda_A(y)\}$.

**Definition 3.8.** Let $H$ be a hyper $B$-algebra and $A$ be an intuitionistic fuzzy set of $H$. Then $A$ is said to be an *intuitionistic fuzzy weak hyper $B$-ideal* of $H$ if for all $x, y \in H$

(i) $\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf\limits_{a\in x\circledast y}\{\mu_A(a)\},\ \mu_A(y)\right\}$, and

(ii) $\lambda_A(0) \leq \lambda_A(x) \leq \max\left\{\sup\limits_{b\in x\circledast y}\{\lambda_A(b)\},\ \lambda_A(y)\right\}$.

**Example 3.9.** The intuitionistic fuzzy hyper $B$-ideal $A$ of $H$ given in Example 3.4 is also both an intuitionistic fuzzy s-weak hyper $B$-ideal and an intuitionistic fuzzy weak hyper $B$-ideal of $H$.

**Lemma 3.10.** Let $A$ be an intuitionistic fuzzy strong hyper $B$-ideal of $H$, and let $x, y \in H$. Then for all $a, b \in x \circledast y$, the following statements hold:

(i) $\mu_A(0) \geq \inf\limits_{a\in x\circledast y}\mu_A(a) \geq \mu_A(x)$ and $\lambda_A(0) \leq \sup_{a\in x\circledast y}\mu_A(a) \leq \lambda_A(x)$;

(ii) If $x \ll y$, then $\mu_A(x) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$;

(iii) $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(y)\} \geq \min\{\inf\limits_{a\in x\circledast y}\{\mu_A(a), \mu_A(y)\}\}$; and

(iv) $\lambda_A(x) \leq \max\left\{\lambda_A(b), \lambda_A(y)\right\} \leq \max\left\{\sup_{b\in x\circledast y}\{\lambda_A(b), \lambda_A(y)\}\right\}$.

*Proof.* Let $x, y \in H$.

(i) Since $H$ is a hyper $B$-algebra, by axiom (H1), $x \ll x$ for all $x \in H$, so $0 \in x \circledast x$. Then $\mu_A(0) \in \{\mu_A(a) \mid a \in x \circledast x\}$ and $\lambda_A(0) \in \{\lambda_A(a) \mid a \in x \circledast x\}$. Hence, it follows that $\mu_A(0) \geq \inf_{a\in x\circledast x} \mu_A(a)$ and $\lambda_A(0) \leq \sup_{b\in x\circledast x} \lambda_A(b)$. Because $A$ is a strong hyper $B$-ideal,

$$\inf_{a\in x\circledast x} \mu_A(a) \geq \mu_A(x) \text{ and } \sup_{b\in x\circledast x} \lambda_A(b) \leq \lambda_A(x).$$

So,

$$\mu_A(0) \geq \inf_{a\in x\circledast x} \mu_A(a) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \sup_{b\in x\circledast x} \lambda_A(b) \leq \lambda_A(x).$$

(ii) Let $x \ll y$, then $0 \in x \circledast y$. By (i), it follows that $\sup_{c\in x\circledast y} \mu_A(c) \geq \mu_A(0)$ and $\inf_{d\in x\circledast y} \lambda_A(d) \leq \lambda_A(0)$. From (i), $\mu_A(0) \geq \mu_A(y)$ and $\lambda_A(0) \leq \lambda_A(y)$, and so,

$$\mu_A(x) \geq \min\left\{\sup_{c\in x\circledast y} \mu_A(c),\, \mu_A(y)\right\} \geq \min\{\mu_A(0),\, \mu_A(y)\} = \mu_A(y),$$

and also

$$\lambda_A(x) \leq \max\left\{\inf_{d\in x\circledast y} \lambda_A(d),\, \lambda_A(y)\right\} \leq \max\{\lambda_A(0),\, \lambda_A(y)\} = \lambda_A(y).$$

(iii) Let $a \in x \circledast y$. From (i), $\mu_A(0) \geq \mu_A(y)$. Also, $\sup_{c\in x\circledast y} \mu_A(c) \geq \mu_A(0) \geq \mu_A(y)$. So, $\min\left\{\sup_{c\in x\circledast y} \mu_A(c),\, \mu_A(y)\right\} = \mu_A(y)$. Since $a \in x \circledast y$, thus $\mu_A(a) \leq \sup_{c\in x\circledast y} \mu_A(c)$, and $\mu_A(y) \geq \min\{\mu_A(a),\, \mu_A(y)\}$. By Definition 3.5(i),

$$\begin{aligned}
\mu_A(x) &\geq \min\left\{\sup_{c\in x\circledast y} \mu_A(c),\, \mu_A(y)\right\} \\
&\geq \min\{\mu_A(a),\, \mu_A(y)\} \\
&\geq \min\left\{\inf_{a\in x\circledast y} \mu_A(a),\, \mu_A(y)\right\}.
\end{aligned}$$

(iv) Let $b \in x \circledast y$. From (i), $\lambda_A(0) \leq \lambda_A(y)$. Also, $\inf_{d\in x\circledast y} \lambda_A(d) \leq \lambda_A(0) \leq \lambda_A(y)$. And so, $\max\{\inf_{d\in x\circledast y} \lambda_A(d),\, \lambda_A(y)\} = \lambda_A(y)$. Since $b \in x \circledast y$, it follows that

$$\lambda_A(b) \geq \inf_{d\in x\circledast y} \lambda_A(d) \text{ and } \lambda_A(y) \leq \max\{\lambda_A(b),\, \lambda_A(y)\}.$$

By Definition 3.5$(ii)$,

$$\begin{aligned}
\lambda_A(x) &\leq \max\left\{\inf_{d\in x\circledast y} \lambda_A(d),\, \lambda_A(y)\right\} \\
&\leq \max\{\lambda_A(b),\, \lambda_A(y)\} \\
&\leq \max\left\{\sup_{b\in x\circledast y} \lambda_A(b),\, \lambda_A(y)\right\}.
\end{aligned}$$

Therefore, Lemma 3.10 holds. □

The following results established the hierarchical relationships among the various types of intuitionistic fuzzy hyper $B$-ideals introduced previously.

**Theorem 3.11.** Let $H$ be a hyper $B$-algebra. Then

(i) every intuitionistic fuzzy strong hyper $B$-ideal of $H$ is an intuitionistic fuzzy hyper $B$-ideal of $H$,

(ii) every intuitionistic fuzzy hyper $B$-ideal of $H$ is an intuitionistic fuzzy weak hyper $B$-ideal of $H$, and

(iii) every intuitionistic fuzzy strong hyper $B$-ideal of $H$ is an intuitionistic fuzzy weak hyper $B$-ideal of $H$.

*Proof.* Let $x, y \in H$.

(i) Let $x \ll y$. By Lemma 3.10(ii), $\mu_A(x) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$. Next, by Lemma 3.10(i), $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in H$. Now, by Definition 3.5, for any $x, y \in H$, $\inf_{a\in x\circledast y} \mu_A(a) \geq \mu_A(x)$ and $\sup_{b\in x\circledast y} \lambda_A(b) \leq \lambda_A(x)$. Also, $\mu_A(x) \geq \min\left\{\sup_{c\in x\circledast y} \mu_A(c),\, \mu_A(y)\right\}$ and $\lambda_A(x) \leq \max\left\{\inf_{d\in x\circledast y} \lambda_A(d),\, \lambda_A(y)\right\}$. By (H1), $\mu_A(0) \geq \inf_{a\in x\circledast x} \mu_A(a) \geq \mu_A(x)$ and $\lambda_A(0) \leq \sup_{b\in x\circledast x} \lambda_A(b) \leq \lambda_A(x)$. Note that $\mu_A(x) \geq \min\left\{\sup_{c\in x\circledast y} \mu_A(c),\, \mu_A(y)\right\} \geq \min\left\{\inf_{a\in x\circledast y} \mu_A(a),\, \mu_A(y)\right\}$, where the last inequality holds because $\sup_{c\in x\circledast y} \mu_A(c) \geq \inf_{a\in x\circledast y} \mu_A(a)$. Also, for $\lambda_A$,

$$\lambda_A(x) \leq \max\left\{\inf_{d\in x\circledast y} \lambda_A(d),\, \lambda_A(y)\right\} \leq \max\left\{\sup_{b\in x\circledast y} \lambda_A(b),\, \lambda_A(y)\right\},$$

where the last inequality holds because $\inf_{d\in x\circledast y} \lambda_A(d) \leq \sup_{b\in x\circledast y} \lambda_A(b)$.

(ii) By Definition 3.3(ii) and (iii), $\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf_{a\in x\circledast y} \mu_A(a),\, \mu_A(y)\right\}$ and

$$\lambda_A(0) \leq \lambda_A(x) \leq \max\left\{\sup_{b\in x\circledast y} \lambda_A(b),\, \lambda_A(y)\right\}.$$

(iii) This follows directly from (i) and (ii). □

The converse of Theorem 3.11 need not be true as shown in the following examples.

**Example 3.12.**

(i) Consider the intuitionistic fuzzy hyper $B$-ideal defined in Example 3.4. Observe that if $x = 0$ and $y = b$, then $0 \circledast b = \{b\}$, and $\inf_{a\in 0\circledast b} \mu_A(a) = \mu_A(b) = 0.4$. But $\mu_A(0) = 0.7$, and so, $0.4 \not\geq 0.7$. Thus, $A$ fails to be an intuitionistic fuzzy strong hyper $B$-ideal.

(ii) Consider the hyper $B$-algebra $H = \{0, a, b\}$ defined in Example 3.4. Define an $IFS$ $A$ on $H$ by $\mu_A(0) = 0.8$, $\mu_A(a) = 0.5$, $\mu_A(b) = 0.6$, $\lambda_A(0) = 0.1$, $\lambda_A(a) = 0.4$, and $\lambda_A(b) = 0.3$. By routine calculations, $A$ is an intuitionistic fuzzy weak hyper $B$-ideal of $H$. However, $A$ is not an intuitionistic fuzzy hyper $B$-ideal because it fails Definition 3.3$(i)$. For instance, $b \ll a$ since $0 \in b \circledast a = \{0, a\}$, but notice that $\mu_A(b) = 0.6 < 0.5 = \mu_A(a)$ which violates Definition 3.3$(i)$ because $\mu_A(b) \geq \mu_A(a)$.

(iii) From $(ii)$ above, $A$ is an intuitionistic fuzzy weak hyper $B$-ideal but not intuitionistic fuzzy strong hyper $B$-ideal of $H$. For instance, take $x = b$ and $y = a$, $b \circledast a = \{0, a\}$, and $\inf_{a \in b \circledast a} \mu_A(a) = \min\{\mu_A(0), \mu_A(a)\} = \min\{0.8, 0.5\} = 0.5$. But $\mu_A(b) = 0.6$, and Definition 3.5(i) requires $0.5 \geq 0.6$, which is false.

**Remark 3.13.** Every intuitionistic fuzzy weak hyper $B$-ideal of $H$ is not necessarily an intuitionistic fuzzy strong and hyper $B$-ideal of $H$. Moreover, every intuitionistic hyper $B$-ideal is not necessarily an intuitionistic fuzzy strong hyper $B$-ideal.

**Theorem 3.14.** Let $H$ be a hyper $B$-algebras. Then

(i) every intuitionistic fuzzy strong hyper $B$-ideal of $H$ is an intuitionistic fuzzy s-weak hyper $B$-ideal of $H$, and

(ii) every intuitionistic fuzzy hyper $B$-ideal of $H$ is an intuitionistic fuzzy s-weak hyper $B$-ideal of $H$.

*Proof.* Let $x, y \in H$.

(i) From Lemma 3.10 $(iii)$ and $(iv)$, for all $a, b \in x \circledast y$, $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(y)\}$ and $\lambda_A(x) \leq \max\{\lambda_A(b), \lambda_A(y)\}$.

(ii) Follows directly from Theorem 3.11(ii). □

The converse of Theorem 3.14 need not be true as shown in the following examples.

**Example 3.15.**

(i) Consider Example 3.9. The intuitionistic fuzzy hyper set $A$ of $H$ is an intuitionistic fuzzy s-weak hyper $B$-ideal of $H$. However, $A$ fails to be an intuitionistic fuzzy strong hyper $B$-ideal. Let $x = 0$ and $y = b$. Then $0 \circledast b = \{b\}$, so $\inf_{a \in 0 \circledast b} \mu_A(a) = \mu_A(b) = 0.4$. But $\mu_A(0) = 0.7$, and Definition 3.5(i) requires that $\inf_{a \in 0 \circledast b} \mu_A(a) \geq \mu_A(0)$. Since $0.4 \ngeq 0.7$, the condition is violated.

(ii) It is important to note that an intuitionistic fuzzy s-weak hyper $B$-ideal need not be an intuitionistic fuzzy hyper $B$-ideal. The s-weak condition does not include the order-reversing property, in particular, $x \ll y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$). From Example 3.12(ii), the intuitionistic fuzzy set $A$ is an intuitionistic fuzzy weak hyper $B$-ideal and hence an intuitionistic fuzzy s-weak hyper $B$-ideal by the same verification, but not a hyper $B$-ideal as it fails Definition 3.3(i). Hence, an intuitionistic fuzzy s-weak hyper $B$-ideal is not necessarily an intuitionistic fuzzy hyper $B$-ideal.

**Remark 3.16.** Every intuitionistic fuzzy s-weak hyper $B$-ideal of $H$ need not to be an intuitionistic fuzzy strong and hyper $B$-ideal of $H$.

**Theorem 3.17.** Let $A$ be an IFS in a hyper $B$-algebra $H$. If $A$ is an intuitionistic fuzzy s-weak hyper $B$-ideal of $H$, then $A$ is also an intuitionistic fuzzy weak hyper $B$-ideal of $H$.

*Proof.* Since $x, y \in H$. Then $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$. By Definition 3.7, there exist $a, b \in x \circledast y$ such that $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(y)\}$ and $\lambda_A(x) \leq \max\{\lambda_A(b), \lambda_A(y)\}$. Since we know that $\mu_A(a) \geq \inf_{c\in x\circledast y} \mu_A(c)$ and $\lambda_A(b) \leq \sup_{d\in x\circledast y} \lambda_A(d)$, it follows directly that $\mu_A(x) \geq \min\{\inf_{c\in x\circledast y} \mu_A(c),\ \mu_A(y)\}$ and $\lambda_A(x) \leq \max\{\sup_{d\in x\circledast y} \lambda_A(d),\ \lambda_A(y)\}$. □

To prove the converse of Theorem 3.17, we need the following notion. An $IFS$ $A$ in $H$ is said to satisfy the *inf-sup property* if for any subset $T \subseteq H$ there exist $x_0, y_0 \in T$ such that $\mu_A(x_0) = \inf_{x\in T} \mu_A(x)$ and $\lambda_A(y_0) = \sup_{y\in T} \lambda_A(y)$.

**Theorem 3.18.** Let $A$ be an intuitionistic fuzzy weak hyper $B$-ideal of $H$. If $A$ satisfies the inf-sup property, then $A$ is an intuitionistic fuzzy s-weak hyper $B$-ideal of $H$.

*Proof.* By Definition 3.8, $(S_1)$ is satisfied. Now, since $A$ satisfies the inf-sup property, for every $x, y \in H$, there exist $a_0, b_0 \in x \circledast y$ such that

$$\mu_A(a_0) = \inf_{a\in x\circledast y} \mu_A(a) \text{ and } \lambda_A(b_0) = \sup_{b\in x\circledast y} \lambda_A(b).$$

So, for all $x, y \in H$,

$$\mu_A(x) \geq \min\left\{\inf_{a\in x\circledast y} \mu_A(a),\ \mu_A(y)\right\} = \min\left\{\mu_A(a_0),\ \mu_A(y)\right\},$$

and

$$\lambda_A(x) \leq \max\left\{\sup_{b\in x\circledast y} \lambda_A(b),\ \lambda_A(y)\right\} = \max\left\{\lambda_A(b_0),\ \lambda_A(y)\right\}.$$

Thus, $(S_2)$ is satisfied. □

Note that, in a finite hyper $B$-algebra, every intuitionistic fuzzy set satisfies the inf-sup property. Consequently, by Theorem 3.18, the concepts of intuitionistic fuzzy weak and s-weak hyper $B$-ideals coincide.

The following results examines the relationship between intuitionistic fuzzy hyper $B$-ideals and intuitionistic fuzzy subhyper $B$-algebras.

**Remark 3.19.** An intuitionistic fuzzy hyper $B$-ideal and an intuitionistic fuzzy weak hyper $B$-ideal of $H$ are not necessarily intuitionistic fuzzy sub-hyper $B$-algebras of $H$.

**Example 3.20.** Consider the hyper $B$-algebra $H = \{0, 1, 2, 3\}$ in [7]. Define an intuitionistic fuzzy set $A$ on $H$ by $\mu_A(0) = 0.7$, $\mu_A(1) = 0.6$, $\mu_A(2) = 0.3$, and $\mu_A(3) = 0.3$, and $\lambda_A(0) = 0.2$, $\lambda_A(1) = 0.3$, and $\lambda_A(2) = \lambda_A(3) = 0.6$. By routine calculations, $A$ is an intuitionistic fuzzy hyper $B$-ideal of $H$. By Theorem 3.11(ii), $A$ is also an intutionistic fuzzy weak hyper $B$-ideal of $H$. However, for $x = 1$ and $y = 1$, $1 \circledast 1 = \{0, 2\}$, and

$$\inf_{a \in 1 \circledast 1} \mu_A(a) = \min\{\mu_A(0), \mu_A(2)\} = \min\{0.7, 0.3\} = 0.3.$$

Also, it follows that $\min\{\mu_A(1), \mu_A(1)\} = \min\{0.6, 0.6\} = 0.6$. But $0.3 \not\geq 0.6$, violating Definition 3.1(i).

**Remark 3.21.** An intuitionistic fuzzy sub-hyper $B$-algebra of $H$ is not necessarily an intuitionistic fuzzy hyper $B$-ideal, an intuitionistic fuzzy weak hyper $B$-ideal, nor an intuitionistic fuzzy strong hyper $B$-ideal of $H$.

**Example 3.22.** The intuitionistic fuzzy sub-hyper $B$-algebra $A$ on $H = [0, 1]$ in Example 3.2 is not an intuitionistic fuzzy hyper $B$-ideal. Consider $x = 0.5$ and $y = 0.3$. Then $x \ll y$ since $0 \in x \circledast y = [0, 0.5]$, but $\mu_A(x) = 0.5 < 0.7 = \mu_A(y)$. Hence, it is not an intuitionistic fuzzy hyper $B$-ideal of $H$. Now, consider the hyper $B$-algebra $H = \{0, a, b, c\}$ from [7] with the following Cayley table.

| $\circledast$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0, a, b\}$ | $\{0, a, c\}$ |
| $a$ | $\{0\}$ | $\{0\}$ | $\{0, a, b\}$ | $\{0, a, c\}$ |
| $b$ | $\{0, a, b\}$ | $\{0, a, b\}$ | $\{0, a, b\}$ | $\{b, c\}$ |
| $c$ | $\{0, a, c\}$ | $\{0, a, c\}$ | $\{b, c\}$ | $\{0, a, c\}$ |

Define an IFS $A$ on $H$ by $\mu_A(0) = 0.8,\ \mu_A(a) = 0.7,\ \mu_A(b) = 0.7,\ \mu_A(c) = 0.6,\ \lambda_A(0) = 0.1,$ $\lambda_A(a) = 0.1, \lambda_A(b) = 0.1,$ and $\lambda_A(c) = 0.1$. By routine calculations, $A$ is an intuitionistic fuzzy subhyper $B$-algebra of $H$. However, $A$ is not an intuitionistic fuzzy weak hyper $B$-ideal. Take $x = a$ and $y = 0$. From the Cayley table, $a \circledast 0 = \{0\}$. Then $\min\{0.8, 0.8\} = 0.8$. Observe that $\inf_{z \in a \circledast 0} \mu(z) = \mu(0) = 0.8,$ and $\mu(y) = \mu(0) = 0.8$. The weak hyper $B$-ideal condition requires $\mu(a) = 0.7 \geq 0.8$, which is false. Therefore, $A$ is an intuitionistic fuzzy subhyper $B$-algebra but not an intuitionistic fuzzy weak hyper $B$-ideal of $H$. Also, the intuitionistic fuzzy set $A$ in Example 3.2 is not an intuitionistic fuzzy strong hyper $B$-ideal. Consider $x = 0.8$ and $y = 0.4$. Then $\sup_{b \in [0,0.8]} \mu_A(b) = \sup_{b \in [0,0.8]}(1 - b) = 1 - 0 = 1$, and $\mu_A(y) = \mu_A(0.4) = 0.6$, so $\min\{1, 0.6\} = 0.6$. But $\mu_A(x) = 0.2$, and $0.2 \not\geq 0.6$.

The next result establishes closure properties under arbitrary intersections. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFS in $X$, where $A_i = \{\langle x, \mu_{A_i}(x), \lambda_{A_i}(x)\rangle \mid x \in X\}$ for each $i \in J$. The arbitrary intersection is defined as $\bigwedge_{i \in J} A_i = \{\langle x, \inf_{i \in J} \mu_{A_i}(x), \sup_{i \in J} \lambda_{A_i}(x)\rangle \mid x \in X\}$, as introduced in [5].

**Theorem 3.23.** Let $\{A_\alpha \mid \alpha \in H\}$ where $\{A_\alpha = \{\langle x, \mu_{A_\alpha}(x), \lambda_{A_\alpha}(x)\rangle \mid x \in H\}$ be a nonempty family of intuitionistic fuzzy sets in $H$.

(i) If $A_\alpha$ is an intuitionistic fuzzy hyper $B$-ideal of $H$ for all $\alpha \in \Lambda$, then so is $\bigwedge_{\alpha \in \Lambda} A_\alpha$.

(ii) If $A_\alpha$ is an intuitionistic fuzzy weak hyper $B$-ideal of $H$ for all $\alpha \in \Lambda$, then so is $\bigwedge_{\alpha\in\Lambda} A_\alpha$.

(iii) If $A_\alpha$ is an intuitionistic fuzzy strong hyper $B$-ideal of $H$ for all $\alpha \in \Lambda$, then so is $\bigwedge_{\alpha\in\Lambda} A_\alpha$.

*Proof.* Let $A = \bigwedge_{\alpha\in\Lambda} A_\alpha = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in H\}$, where $\mu_A(x) = \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x)$ and $\lambda_A(x) = \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x)$ for all $x \in H$.

(i) Let $x, y \in H$. If $x \ll y$, then for each $\alpha \in \Lambda$, $\mu_{A_\alpha}(x) \geq \mu_{A_\alpha}(y)$ and $\lambda_{A_\alpha}(x) \leq \lambda_{A_\alpha}(y)$. Therefore, $\mu_A(x) = \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x) \geq \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(y) = \mu_A(y)$, and also for $\lambda_A(x)$,

$$\lambda_A(x) = \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x) \leq \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(y) = \lambda_A(y).$$

For each $\alpha \in \Lambda$, note that $\mu_{A_\alpha}(0) \geq \mu_{A_\alpha}(x)$ and $\lambda_{A_\alpha}(0) \leq \lambda_{A_\alpha}(x)$. Taking infimum and supremum,

$$\mu_A(0) = \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(0) \geq \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x) = \mu_A(x),$$

and

$$\lambda_A(0) = \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(0) \leq \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x) = \lambda_A(x).$$

Taking infimum over $\alpha \in \Lambda$,

$$\begin{aligned}
\mu_A(x) &= \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x) \\
&\geq \inf_{\alpha\in\Lambda} \left\{ \min\left\{ \inf_{a\in x\circledast y} \mu_{A_\alpha}(a),\, \mu_{A_\alpha}(y) \right\} \right\} \\
&= \min\left\{ \inf_{\alpha\in\Lambda} \left\{ \inf_{a\in x\circledast y} \mu_{A_\alpha}(a) \right\},\, \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(y) \right\} \\
&= \min\left\{ \inf_{a\in x\circledast y} \left\{ \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(a) \right\},\, \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(y) \right\} \\
&= \min\left\{ \inf_{a\in x\circledast y} \mu_A(a),\, \mu_A(y) \right\}.
\end{aligned}$$

For the $\lambda_A$, for each $\alpha \in \Lambda$, $\lambda_{A_\alpha}(x) \leq \max\left\{ \sup_{a\in x\circledast y} \lambda_{A_\alpha}(a),\, \lambda_{A_\alpha}(y) \right\}$. Taking supremum over $\alpha \in \Lambda$,

$$\begin{aligned}
\lambda_A(x) &= \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x) \\
&\leq \sup_{\alpha\in\Lambda} \left\{ \max\left\{ \sup_{a\in x\circledast y} \lambda_{A_\alpha}(a),\, \lambda_{A_\alpha}(y) \right\} \right\} \\
&= \max\left\{ \sup_{\alpha\in\Lambda} \left\{ \sup_{a\in x\circledast y} \lambda_{A_\alpha}(a) \right\},\, \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(y) \right\} \\
&= \max\left\{ \sup_{a\in x\circledast y} \left\{ \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(a) \right\},\, \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(y) \right\} \\
&= \max\left\{ \sup_{a\in x\circledast y} \lambda_A(a),\, \lambda_A(y) \right\}.
\end{aligned}$$

(ii) The proof follows exactly in (i).

(iii) Assume each $A_\alpha = \{\langle x, \mu_{A_\alpha}(x), \lambda_{A_\alpha}(x)\rangle \mid x \in H\}$ is an intuitionistic fuzzy strong hyper $B$-ideal of $H$. Let $x, y \in H$. For each $\alpha \in \Lambda$ and for each $a \in x \circledast y$, we have $\mu_{A_\alpha}(a) \geq \mu_{A_\alpha}(x)$. Hence, for all $a \in x \circledast y$. $\inf_{\alpha\in\Lambda} \mu_{A_\alpha}(a) \geq \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x)$. Therefore, $\inf_{a\in x\circledast y} \mu_A(a) = \inf_{a\in x\circledast y} \{\inf_{\alpha\in\Lambda} \mu_{A_\alpha}(a)\} \geq \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x) = \mu_A(x)$. Similarly, for each $\alpha \in \Lambda$ and for each $a \in x \circledast y$, we have $\lambda_{A_\alpha}(a) \leq \lambda_{A_\alpha}(x)$. Taking supremum over $\alpha \in \Lambda$, for all $a \in x \circledast y$, $\sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(a) \leq \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x)$. Therefore,

$$\sup_{a\in x\circledast y} \lambda_A(a) = \sup_{a\in x\circledast y} \left\{\sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(a)\right\} \leq \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x) = \lambda_A(x).$$

Moreover,

$$\begin{aligned}\mu_A(x) &= \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(x) \\ &\geq \inf_{\alpha\in\Lambda} \left\{\min\left\{\sup_{a\in x\circledast y} \mu_{A_\alpha}(a),\ \mu_{A_\alpha}(y)\right\}\right\} \\ &= \min\left\{\inf_{\alpha\in\Lambda}\left\{\sup_{a\in x\circledast y} \mu_{A_\alpha}(a)\right\},\ \inf_{\alpha\in\Lambda} \mu_{A_\alpha}(y)\right\}.\end{aligned}$$

Note that $\inf_{\alpha\in\Lambda}\left\{\sup_{a\in x\circledast y} \mu_{A_\alpha}(a)\right\} \geq \sup_{a\in x\circledast y}\left\{\inf_{\alpha\in\Lambda} \mu_{A_\alpha}(a)\right\} = \sup_{a\in x\circledast y} \mu_A(a)$. So, $\mu_A(x) \geq \min\left\{\sup_{a\in x\circledast y} \mu_A(a),\ \mu_A(y)\right\}$. Also, for each $\alpha \in \Lambda$, it follows that

$$\lambda_{A_\alpha}(x) \leq \max\left\{\inf_{a\in x\circledast y} \lambda_{A_\alpha}(a),\ \lambda_{A_\alpha}(y)\right\}.$$

Taking supremum over $\alpha \in \Lambda$,

$$\begin{aligned}\lambda_A(x) &= \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(x) \\ &\leq \sup_{\alpha\in\Lambda}\left\{\max\left\{\inf_{a\in x\circledast y} \lambda_{A_\alpha}(a),\ \lambda_{A_\alpha}(y)\right\}\right\} \\ &= \max\left\{\sup_{\alpha\in\Lambda}\left\{\inf_{a\in x\circledast y} \lambda_{A_\alpha}(a)\right\},\ \sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(y)\right\}.\end{aligned}$$

Note that $\sup_{\alpha\in\Lambda}\{\inf_{a\in x\circledast y} \lambda_{A_\alpha}(a)\} \leq \inf_{a\in x\circledast y}\{\sup_{\alpha\in\Lambda} \lambda_{A_\alpha}(a)\} = \inf_{a\in x\circledast y} \lambda_A(a)$. So, it follows that $\lambda_A(x) \leq \max\{\inf_{a\in x\circledast y} \lambda_A(a),\ \lambda_A(y)\}$. □

The next results are the characterizations of intuitionistic fuzzy hyper $B$-ideals via complement and associated operators. Let $A$ be an IFS of $X$. Then the necessity and possibility operators of $A$ are defined, respectively, by $\square A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in X\}$ and $\lozenge A = \{\langle x, 1-\lambda_A(x), \lambda_A(x)\rangle \mid x \in X\}$, as introduced in [1].

**Lemma 3.24.** An IFS $A$ is an intuitionistic fuzzy hyper $B$-ideal of $H$ if and only if the fuzzy sets $\mu_A$ and $\overline{\lambda}_A$ are fuzzy hyper $B$-ideals of $H$, where $\overline{\lambda}_A(x) = 1 - \lambda_A(x)$ for all $x \in H$.

*Proof.* ( $\Longrightarrow$ ) By Definition 3.3, $\mu_A$ is a fuzzy hyper $B$-ideal of $H$. Now, consider the fuzzy set $\overline{\lambda}_A$, defined by $\overline{\lambda}_A(x) = 1 - \lambda_A(x)$ for all $x \in H$. Let $x, y \in H$ such that $x \ll y$. Since $A$ is an intuitionistic fuzzy hyper $B$-ideal, by Definition 3.3(i), $\lambda_A(x) \leq \lambda_A(y)$. Therefore, $\overline{\lambda}_A(x) = 1 - \lambda_A(x) \geq 1 - \lambda_A(y) = \overline{\lambda}_A(y)$. Next, by Definition 3.3(iii),

$$\lambda_A(x) \leq \max\left\{\sup_{b \in x \circledast y} \lambda_A(b),\ \lambda_A(y)\right\}.$$

So, $1 - \lambda_A(x) \geq 1 - \max\left\{\sup_{b \in x \circledast y} \lambda_A(b),\ \lambda_A(y)\right\}$. Also, $1 - \max\{a, b\} = \min\{1 - a, 1 - b\}$. Therefore,

$$\overline{\lambda}_A(x) \geq \min\left\{1 - \sup_{b \in x \circledast y} \lambda_A(b),\ 1 - \lambda_A(y)\right\}.$$

Now, $1 - \sup_{b \in x \circledast y} \lambda_A(b) = \inf_{b \in x \circledast y}(1 - \lambda_A(b)) = \inf_{b \in x \circledast y} \overline{\lambda}_A(b)$. Also, $1 - \lambda_A(y) = \overline{\lambda}_A(y)$. Thus, $\overline{\lambda}_A(x) \geq \min\left\{\inf_{b \in x \circledast y} \overline{\lambda}_A(b),\ \overline{\lambda}_A(y)\right\}$. Moreover, since $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in H$, by Definition 3.3(ii),

$$\overline{\lambda}_A(0) = 1 - \lambda_A(0) \geq 1 - \lambda_A(x) = \overline{\lambda}_A(x).$$

Therefore, $\overline{\lambda}_A$ satisfies both conditions of a fuzzy hyper $B$-ideal. Hence, $\overline{\lambda}_A$ is a fuzzy hyper $B$-ideal of $H$.

$(\Longleftarrow)$ Conversely, since $\mu_A$ is a fuzzy hyper $B$-ideal, for all $x, y \in H$,

$$\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf_{a \in x \circledast y} \mu_A(a), \mu_A(y)\right\}.$$

Now, since $\overline{\lambda}_A$ is a fuzzy hyper $B$-ideal, $\overline{\lambda}_A(0) \geq \overline{\lambda}_A(x) \geq \min\left\{\inf_{b \in x \circledast y} \overline{\lambda}_A(b), \overline{\lambda}_A(y)\right\}$ for all $x, y \in H$. From $\overline{\lambda}_A(0) \geq \overline{\lambda}_A(x)$, $1 - \lambda_A(0) \geq 1 - \lambda_A(x)$ implies $\lambda_A(0) \leq \lambda_A(x)$.

Now, from the inequality for $\overline{\lambda}_A$, it follows that $\inf_{b \in x \circledast y}(1 - \lambda_A(b)) = 1 - \sup_{b \in x \circledast y} \lambda_A(b)$. So,

$$\begin{aligned} 1 - \lambda_A(x) &\geq \min\left\{\inf_{b \in x \circledast y}(1 - \lambda_A(b)),\ 1 - \lambda_A(y)\right\} \\ &\geq \min\left\{1 - \sup_{b \in x \circledast y} \lambda_A(b),\ 1 - \lambda_A(y)\right\}. \end{aligned}$$

Taking complements, $\lambda_A(x) \leq 1 - \min\left\{1 - \sup_{b \in x \circledast y} \lambda_A(b),\ 1 - \lambda_A(y)\right\}$. So, it follows that $1 - \min\{a, b\} = \max\{1 - a, 1 - b\}$, and

$$\lambda_A(x) \leq \max\left\{\sup_{b \in x \circledast y} \lambda_A(b),\ \lambda_A(y)\right\}.$$

Finally, let $x \ll y$. Then $\overline{\lambda}_A(x) \geq \overline{\lambda}_A(y)$. Therefore, $1 - \lambda_A(x) \geq 1 - \lambda_A(y)$ implies that $\lambda_A(x) \leq \lambda_A(y)$. Similarly, since $\mu_A$ is a fuzzy hyper $B$-ideal, $x \ll y$ implies $\mu_A(x) \geq \mu_A(y)$, which is Definition 3.3(i) for $\mu_A$. □

The equivalence in Lemma 3.24 establishes dual relationships for both complements. For the complement of $\overline{\mu}_A$, $x \ll y$ implies $\overline{\mu}_A(x) \leq \overline{\mu}_A(y)$ and

$$\overline{\mu}_A(0) \leq \overline{\mu}_A(x) \leq \max\left\{\sup_{a \in x \circledast y} \overline{\mu}_A(a),\ \overline{\mu}_A(y)\right\},$$

where $\overline{\mu}_A = 1 - \mu_A$. For the complement of $\overline{\lambda}_A$, $x \ll y$ implies $\overline{\lambda}_A(x) \geq \overline{\lambda}_A(y)$ and

$$\overline{\lambda}_A(0) \geq \overline{\lambda}_A(x) \geq \min\left\{ \inf_{a \in x \circledast y} \overline{\lambda}_A(a),\, \overline{\lambda}_A(y) \right\}.$$

**Theorem 3.25.** Let $A$ be an IFS in $H$. Then $A$ is an intuitionistic fuzzy hyper $B$-ideal of $H$ if and only if

(i) $\square A = \{\langle x, \mu_A(x), \overline{\mu}_A(x)\rangle \mid x \in H\}$ is an intuitionistic fuzzy hyper $B$-ideal of $H$, and

(ii) $\Diamond A = \{\langle x, \overline{\lambda}_A(x), \lambda_A(x)\rangle \mid x \in H\}$ is an intuitionistic fuzzy hyper $B$-ideal of $H$,

where $\overline{\mu}_A(x) = 1 - \mu_A(x)$ and $\overline{\lambda}_A(x) = 1 - \lambda_A(x)$ for all $x \in H$.

*Proof.* ( $\Longrightarrow$ ) By Lemma 3.24, the fuzzy sets $\mu_A$ and $\overline{\lambda}_A$ are fuzzy hyper $B$-ideals of $H$. Consider $\square A$, the membership part of $\square A$ satisfies Definition 3.3(i) and (ii). Let $x, y \in H$ with $x \ll y$. Since $\mu_A(x) \geq \mu_A(y)$, $\overline{\mu}_A(x) \leq \overline{\mu}_A(y)$, satisfying Definition 3.3(i). Since $\mu_A(0) \geq \mu_A(x)$ for all $x \in H$, $\overline{\mu}_A(0) \leq \overline{\mu}_A(x)$. Since $\mu_A(x) \geq \min\{\inf_{a \in x \circledast y} \mu_A(a), \mu_A(y)\}$,

$$\overline{\mu}_A(x) \leq 1 - \min\left\{ \inf_{a \in x \circledast y} \mu_A(a), \mu_A(y) \right\} = \max\left\{ \sup_{a \in x \circledast y} \overline{\mu}_A(a), \overline{\mu}_A(y) \right\}.$$

Next, consider $\Diamond A$. Since $\overline{\lambda}_A$ is a fuzzy hyper $B$-ideal by Lemma 3.24, the membership part of $\Diamond A$ satisfies Definition 3.3(i) and (ii). The non-membership part $\lambda_A$ satisfies Definition 3.3 because $A$ is an intuitionistic fuzzy hyper $B$-ideal. Therefore, $\Diamond A$ is an intuitionistic fuzzy hyper $B$-ideal of $H$.
( $\Longleftarrow$ ) Assume that both $\square A$ and $\Diamond A$ are intuitionistic fuzzy hyper $B$-ideals of $H$. Then $\mu_A$ and $\overline{\lambda}_A$ are fuzzy hyper $B$-ideal. By Lemma 3.24, $A$ is an intuitionistic fuzzy hyper $B$-ideal of $H$. □

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## References

[1] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[2] J.C. Endam, On JB-Semigroups and Hyper B-Algebras, Ph.D. Thesis, Mindanao State University - Iligan Institute of Technology, (2014).

[3] J. Neggers, H.S. Kim, On B-Algebras, Mat. Vesnik 54 (2002), 21–29. http://eudml.org/doc/253204.

[4] F. Marty, Sur une Generalization de la Notion de Groupe, in: 8th Congress Math. Scandinaves, Stockholm, Sweden, pp. 45–49, 1934.

[5] L.C. Platil, G.C. Petalcorin, Fuzzy Γ-Semimodules over Γ-Semirings, J. Anal. Appl. 15 (2017), 71–83.

[6] L.C. Platil, T. Tanaka, Multi-Criteria Evaluation for Intuitionistic Fuzzy Sets Based on Set-Relations, Nihonkai Math. J. 34 (2023), 1–18.

[7] G.F. Tabaranza, J.P. Vilela, On Fuzzy Hyper B-Ideals of Hyper B-Algebras, Int. J. Algebr. 12 (2018), 197–209. https://doi.org/10.12988/ija.2018.8621.

[8] G.F. Tabaranza, J.P. Vilela, Fuzzy Hyper B-Algebras, JP J. Algebr. Number Theory Appl. 41 (2019), 205–218. https://doi.org/10.17654/NT041020205.

[9] A.L.O. Vicedo, J.P. Vilela, Hyper B-Ideals in Hyper B-Algebra, J. Ultra Sci. Phys. Sci. 29 (2017), 339–347.

[10] L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.