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MODULES OVER JU-ALGEBRAS

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Abstract. This study introduces JU-modules, a class of modules over JU-algebras, and develops their fundamental structure. Key results include the adaptation of classical isomorphism theorems to the JU-module setting and the formulation and analysis of exact sequences. The study also establishes the Butterfly Lemma for JU-modules and explores properties of module chains in this context. These results provide a foundation for connecting ideal theory with module theory in bounded implicative JU-algebras.

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1. Introduction

JU-algebras, introduced by Moin et al. [3], stand for *Jazan University algebras*. Further developments include valuations and metrics for pseudo JU-algebras [1] and an exploration of rough set theory in the context of JU-algebras [2]. The concept of filters in JU-algebras was introduced and investigated by Romano [12], who also established several new results concerning the structure of JU-algebras [11].

Prabpayak and Leerawat [9] introduced the concept of KU-algebras. Fundamental properties and homomorphisms of KU-algebras have been studied in detail in [9,10]. A wide range of research has since been conducted on KU-algebras across various theoretical frameworks, including intuitionistic, fuzzy, neutrosophic soft, and rough set theories. The notion of cubic KU-ideals was introduced in [13], while pseudo-metric structures on KU-algebras were examined by Ali et al. [8]. Additionally, rough approximations in KU-algebras were explored in [4], and graphical structures related to KU-ideals were investigated more recently by Ali et al. [5].

Imai and Iséki introduced the concept of BCK-algebras [6] as a logical-algebraic generalization of set-theoretic difference and propositional calculus. This development parallels the evolution of Boolean

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logic through the structure of Boolean algebras. Subsequently, Iséki [7] extended the framework by defining BCI-algebras as a superclass of BCK-algebras.

This article is structured into six main sections, each presenting a key aspect of JU-modules over JU-algebras, including their homomorphisms, exact sequences, and structural properties.

Section 1 introduces the motivation and background underlying the study of JU-algebras. In Section 2, we present foundational concepts related to JU-algebras, including their basic properties and the theory of JU-ideals. Section 3 develops the notion of JU-modules, provides illustrative examples, and discusses their key features. Section 4 investigates chain conditions on JU-modules, where we define minimal and maximal submodules, and present the Jordan-Hölder Theorem as well as the Schreier Refinement Theorem within the context of JU-modules. Section 5 focuses on exact sequences and their structural properties in JU-module theory. Finally, Section 6 examines projective and injective JU-modules, establishing their defining properties and exploring their roles within the broader framework of homological algebra over JU-algebras.

2. Basic Concepts of JU-Algebras

This section reviews the fundamental definitions and properties of JU-algebras, which form the foundation for the study of JU-modules. We begin by reviewing the axiomatic structure of JU-algebras, followed by an examination of relevant notions, including homomorphisms, JU-subalgebras, and JU-ideals. These concepts will be essential for developing the module theory over JU-algebras in the subsequent sections.

Definition 2.1. *A* JU-algebra *is a triple* $(X, \diamond, 1)$*, where* X *is a nonempty set,* $\diamond$ *is a binary operation on* X*, and* $1 \in X$ *is a distinguished constant (called the* fixed element*), satisfying the following axioms for all* $\rho, \sigma, \tau \in X$*:*

$$(JU1)\quad (\sigma \diamond \tau) \diamond [(\tau \diamond \rho) \diamond (\sigma \diamond \rho)] = 1,$$

$$(JU2)\quad 1 \diamond \rho = \rho,$$

$$(JU3)\quad \rho \diamond \sigma = \sigma \diamond \rho = 1 \Rightarrow \rho = \sigma.$$

An order relation $\leq$ *is defined on* X *by:*

$$\sigma \leq \rho \iff \rho \diamond \sigma = 1.$$

We also denote a JU-algebra by $(X, \circ, 1)$ *when using alternative notation. In this context, the constant* 1 *remains the fixed element, and the induced partial order on* X *is given by:*

$$x_1 \leq y_1 \iff y_1 \circ x_1 = 1.$$

Lemma 2.1. *Let $(X, \diamond, 1)$ be a JU-algebra, and define a relation $\leq$ on X by $\sigma \leq \rho$ if and only if $\rho \diamond \sigma = 1$. Then, the following properties hold:*

$$(JU4) \quad \rho \leq \rho \qquad (Reflexivity),$$

$$(JU5) \quad \rho \leq \sigma \text{ and } \sigma \leq \rho \Rightarrow \rho = \sigma \qquad (Antisymmetry),$$

$$(JU6) \quad \rho \leq \tau \text{ and } \tau \leq \sigma \Rightarrow \rho \leq \sigma \qquad (Transitivity).$$

Proof. (JU4) (Reflexivity): Substitute $\tau = \sigma = 1$ into axiom (JU1):

$$(\rho \diamond 1) \diamond [(1 \diamond 1) \diamond (\rho \diamond 1)] = 1.$$

Using (JU2), we know $\rho \diamond 1 = \rho$ and $1 \diamond 1 = 1$, hence:

$$\rho \diamond (\rho \diamond 1) = 1 \Rightarrow \rho \diamond \rho = 1,$$

which implies $\rho \leq \rho$.

(JU5) (Antisymmetry): Assume $\rho \leq \sigma$ and $\sigma \leq \rho$, i.e., $\sigma \diamond \rho = \rho \diamond \sigma = 1$. Then, by axiom (JU3), it follows that $\rho = \sigma$.

(JU6) (Transitivity): Assume $\rho \leq \tau$ and $\tau \leq \sigma$, i.e., $\tau \diamond \rho = 1$ and $\sigma \diamond \tau = 1$. Substitute into (JU1):

$$(\sigma \diamond \tau) \diamond [(\tau \diamond \rho) \diamond (\sigma \diamond \rho)] = 1.$$

Since $\sigma \diamond \tau = 1$ and $\tau \diamond \rho = 1$, we have:

$$1 \diamond [1 \diamond (\sigma \diamond \rho)] = 1.$$

Using (JU2), this simplifies to:

$$1 \diamond (\sigma \diamond \rho) = 1 \Rightarrow \sigma \diamond \rho = 1,$$

so $\rho \leq \sigma$. □

Lemma 2.2. *Let $(X, \diamond, 1)$ be a JU-algebra with the partial order defined by $\sigma \leq \rho \iff \rho \diamond \sigma = 1$. Then for all $\rho, \sigma, \tau \in X$, the following properties hold:*

$$(JU7) \quad \rho \leq \sigma \Rightarrow \sigma \diamond \tau \leq \rho \diamond \tau,$$

$$(JU8) \quad \rho \leq \sigma \Rightarrow \tau \diamond \rho \leq \tau \diamond \sigma,$$

$$(JU9) \quad (\tau \diamond \rho) \diamond (\sigma \diamond \rho) \leq \sigma \diamond \tau,$$

$$(JU10) \quad (\sigma \diamond \rho) \diamond \rho \leq \sigma.$$

Proof. (JU7): Assume $\rho \leq \sigma$, i.e., $\sigma \diamond \rho = 1$. Substitute into axiom (JU1) with $\sigma \diamond \rho = 1$ and apply to the expression $(\sigma \diamond \tau) \diamond [(\tau \diamond \rho) \diamond (\sigma \diamond \rho)] = 1$. It follows that:

$$(\sigma \diamond \tau) \diamond [(\tau \diamond \rho) \diamond 1] = 1 \Rightarrow (\sigma \diamond \tau) \diamond (\tau \diamond \rho) = 1.$$

Hence, $\rho \diamond \tau \geq \sigma \diamond \tau$, i.e., $\sigma \diamond \tau \leq \rho \diamond \tau$.

(JU8): Similarly, from $\rho \leq \sigma$ we have $\sigma \diamond \rho = 1$. Applying (JU1) appropriately:

$$(\tau \diamond \rho) \diamond [(\rho \diamond \sigma) \diamond (\tau \diamond \sigma)] = 1.$$

From $\rho \diamond \sigma = 1$, we deduce:

$$(\tau \diamond \rho) \diamond (\tau \diamond \sigma) = 1 \Rightarrow \tau \diamond \sigma \leq \tau \diamond \rho.$$

(JU9): Let us apply (JU1) directly with the triple (τ, ρ, σ):

$$(\tau \diamond \rho) \diamond [(\rho \diamond \sigma) \diamond (\tau \diamond \sigma)] = 1.$$

This implies that:

$$[(\tau \diamond \rho) \diamond (\sigma \diamond \rho)] \leq \sigma \diamond \tau.$$

(JU10): From (JU2), we know that $1 \diamond \rho = \rho$. Let us substitute $\tau = \rho$ and use (JU1):

$$(\sigma \diamond \rho) \diamond [(\rho \diamond \rho) \diamond (\sigma \diamond \rho)] = 1.$$

Since $\rho \diamond \rho = 1$ (from earlier Lemma JU4), we get:

$$(\sigma \diamond \rho) \diamond (\sigma \diamond \rho) = 1 \ \Rightarrow \ \sigma \diamond \rho \leq \sigma.$$

Now multiplying both sides on the right by ρ gives:

$$(\sigma \diamond \rho) \diamond \rho \leq \sigma.$$

This completes the proof. □

Lemma 2.3. *Let $(X, \diamond, 1)$ be a JU-algebra. Then for all $\rho, \sigma, \tau \in X$, the following properties hold:*

(JU11) $\rho \diamond \rho = 1$,

(JU12) $\tau \diamond (\sigma \diamond \rho) = \sigma \diamond (\tau \diamond \rho)$,

(JU13) *If* $(\rho \diamond \sigma) \diamond \sigma = 1$, *then* $\rho \diamond 1 = 1$ *for all* $\rho \in X$,

(JU14) $(\sigma \diamond \rho) \diamond 1 = (\sigma \diamond 1) \diamond (\rho \diamond 1)$.

Proof. (JU11): Set $\tau = \sigma = 1$ in axiom (JU1):

$$(\rho \diamond 1) \diamond [(1 \diamond \rho) \diamond (\rho \diamond 1)] = 1.$$

Using (JU2), we simplify:

$$\rho \diamond (\rho \diamond 1) = 1 \Rightarrow \rho \diamond \rho = 1,$$

so $\rho \leq \rho$ and hence (JU11) holds.

(JU12): We aim to show symmetry: $\tau \diamond (\sigma \diamond \rho) = \sigma \diamond (\tau \diamond \rho)$. First, let $\sigma = 1$ in (JU1). Applying (JU7), we get:

$$\tau \diamond (\sigma \diamond \rho) \leq [(\tau \diamond \rho) \diamond \rho] \diamond (\sigma \diamond \sigma). \tag{2.1}$$

Next, substitute $\tau \mapsto \tau \diamond \rho$ in (JU1), giving:

$$\sigma \diamond (\tau \diamond \rho) \diamond [((\tau \diamond \sigma) \diamond \sigma) \diamond (\rho \diamond \rho)] = 1.$$

Since $\rho \diamond \rho = 1$ by (JU11), it follows that:

$$((\tau \diamond \rho) \diamond \rho) \diamond (\sigma \diamond \rho) \leq \sigma \diamond (\tau \diamond \rho). \tag{2.2}$$

From inequalities (2.1) and (2.2), and using antisymmetry (JU5), we conclude:

$$\tau \diamond (\sigma \diamond \rho) = \sigma \diamond (\tau \diamond \rho).$$

(JU13): Assume $(\rho \diamond \sigma) \diamond \sigma = 1$.

We show that $\rho \diamond 1 = 1$ for all $\rho \in X$. Substitute $\rho \mapsto 1$, $\tau \mapsto \rho$, and $\sigma \mapsto 1$ in (JU1):

$$(1 \diamond \rho) \diamond [(\rho \diamond 1) \diamond (1 \diamond 1)] = 1.$$

By (JU2), $1 \diamond \rho = \rho$, and $1 \diamond 1 = 1$, so:

$$\rho \diamond [(\rho \diamond 1) \diamond 1] = 1 \;\Rightarrow\; \rho \diamond 1 = 1.$$

(JU14): Apply (JU12) directly:

$$(\sigma \diamond 1) \diamond (\rho \diamond 1) = \sigma \diamond (\rho \diamond 1) = (\sigma \diamond \rho) \diamond 1.$$

Thus,

$$(\sigma \diamond \rho) \diamond 1 = (\sigma \diamond 1) \diamond (\rho \diamond 1).$$

□

Example 2.1 ([3]). *Let* $X = \{1, 2, 3, 4, 5\}$ *and define the binary operation* $\diamond : X \times X \to X$ *as shown in the table below. Then* $(X, \diamond, 1)$ *forms a JU-algebra.*

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

The following example illustrates a structure that is a JU-algebra but not a KU-algebra.

Example 2.2 ([3]). *Let* $X = \{1, 2, 3, 4\}$*, and define the binary operation* $\diamond : X \times X \to X$ *as given in the table below:*

$\diamond$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

It can be verified that $(X, \diamond, 1)$ *satisfies the axioms of a JU-algebra, but fails to satisfy those of a KU-algebra.*

The following example illustrates that a structure can simultaneously satisfy both the JU-algebra and KU-algebra axioms.

Example 2.3 ([3]). *Let* $X = \{1, 2, 3, 4\}$*, and define the binary operation* $\diamond : X \times X \to X$ *as follows:*

$\diamond$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

It can be verified that $(X, \diamond, 1)$ *satisfies the axioms of both JU-algebras and KU-algebras.*

Definition 2.2. *Let* $(X, \diamond, 1)$ *be a JU-algebra.*

- *A JU-subalgebra* $J \subseteq X$ *is a non-empty subset such that for all* $\rho, \sigma \in J$*, the product* $\rho \diamond \sigma \in J$*.*
- *Define the subset*

$$P_X := \{\rho \in X \mid (\rho \diamond 1) \diamond 1 = \rho\}.$$

  Then X *is called* p*-semisimple if* $P_X = X$*; that is, if* $(\rho \diamond 1) \diamond 1 = \rho$ *holds for all* $\rho \in X$*.*
- *An element* $j \in X$ *is called the minimal element of* X *if for all* $\rho \in X$*, the condition* $\rho \leq j$ *implies* $\rho = j$*.*
- *For* $j \in X$*, define:*

$$K(j) := \{\rho \in X \mid \rho \geq j\}, \quad B_\rho := \{\rho \in X \mid \rho \diamond 1 = 1\}.$$

The set B_ρ is called the JU-part of X.

Definition 2.3. *Let $(X, \diamond, 1)$ be a JU-algebra. A non-empty subset $J \subseteq X$ is called a JU-ideal if it satisfies the following conditions:*

(1) $1 \in J$, *and*

(2) *for all $p, q \in X$, if $p \in J$ and $p \diamond q \in J$, then $q \in J$.*

Definition 2.4. *Let $(X, \diamond, 1)$ be a JU-algebra, and let $J \subseteq X$. The subset J is called a p-ideal of X if the following conditions hold:*

(1) $1 \in J$, *and*

(2) *for all $p, q, r \in X$, if $q \in J$ and $(r \diamond q) \diamond (r \diamond p) \in J$, then $p \in J$.*

Definition 2.5. *Let $(X, \diamond, 1)$ be a JU-algebra. A non-empty subset $J \subseteq X$ is called a strong ideal if the following condition holds:*

$$\text{For all } p \in J,\ q \notin J,\ \text{and all } x \in X,\ \text{we have } q \diamond x \notin J.$$

Example 2.4. *Let $X = \{1, 2, 3, 4, 5, 6\}$, and define the binary operation $\diamond : X \times X \to X$ as follows:*

$\diamond$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	1
6	1	1	2	1	1	1

Then $(X, \diamond, 1)$ forms a JU-algebra. It can be verified that the subsets $\{1, 2\}$ and $\{1, 2, 3, 4, 5\}$ are JU-ideals of X.

Definition 2.6. *Let $(X, \diamond, 1)$ be a JU-algebra, and let J be an ideal of X. Define a binary relation $\sim$ on X by:*

$$p \sim q \iff p \diamond q \in J \text{ and } q \diamond p \in J \quad \text{for all } p, q \in X.$$

Then $\sim$ is a congruence relation on X, and the quotient set $X/J := X/\sim$ forms a quotient JU-algebra.

Remark 2.1. *Not every subset of a JU-algebra is a subalgebra. A subset $S \subseteq X$ is a subalgebra if it is closed under the operation $\diamond$ and contains the constant element 1.*

Definition 2.7. *An ideal $J \subseteq X$ is said to be a closed ideal if it is both an ideal and a subalgebra of X.*

Given any subset $S \subseteq X$, the ideal generated by S, denoted $\langle S \rangle$, is the smallest ideal of X that contains S.

Definition 2.8. *Let* $(X, \diamond, 1)$ *and* $(X', \diamond', 1')$ *be JU-algebras. A map* $h : X \to X'$ *is called a homomorphism if*

$$h(p \diamond q) = h(p) \diamond' h(q) \quad \text{for all } p, q \in X,$$

and

$$h(1) = 1'.$$

Remark 2.2. *Every ideal* J *of* X *determines a congruence relation* $\sim$ *on* X*, as described above.*

Lemma 2.4 ([3]). *Let* $(X, \diamond, 1)$ *be an algebra with a binary operation* $\diamond$ *and a constant* $1 \in X$*. Then* $(X, \diamond, 1)$ *is a JU-algebra if and only if the following conditions hold for all* $a_1, a_2, a_3 \in X$*:*

$$\begin{aligned} &(JU5) \quad a_1 \diamond a_2 \leq (a_2 \diamond a_3) \diamond (a_1 \diamond a_3), \\ &(JU6) \quad a_1 \leq 1, \\ &(JU7) \quad a_1 \leq a_2 \text{ and } a_2 \leq a_1 \;\Rightarrow\; a_1 = a_2, \end{aligned}$$

where the partial order $\leq$ *is defined by:* $a \leq b$ *if and only if* $b \diamond a = 1$*.*

Lemma 2.5 ([3]). *Let* $(X, \diamond, 1)$ *be a JU-algebra. Then, for all* $a_1, a_2, a_3 \in X$*, the following identities hold:*

(1) $a_3 \diamond a_3 = 1$,
(2) $a_3 \diamond (a_1 \diamond a_3) = 1$,
(3) $a_1 \leq a_2 \;\Rightarrow\; a_2 \diamond a_3 \leq a_1 \diamond a_3$,
(4) $a_3 \diamond (a_2 \diamond a_1) = a_2 \diamond (a_3 \diamond a_1)$,
(5) $a_2 \diamond [(a_2 \diamond a_1) \diamond a_1] = 1$.

Here, the partial order $\leq$ *is defined by:* $a \leq b$ *if and only if* $b \diamond a = 1$*.*

Definition 2.9. *Let* $(X, \diamond, 1)$ *be a JU-algebra. Then:*

(*i*) *It is called commutative if*

$$(a_2 \diamond a_1) \diamond a_1 = (a_1 \diamond a_2) \diamond a_2 \quad \text{for all } a_1, a_2 \in X.$$

(*ii*) *It is called implicative if*

$$(a_1 \diamond a_2) \diamond a_1 = a_1 \quad \text{for all } a_1, a_2 \in X.$$

(*iii*) *It is called bounded if*

$$1 \leq a \quad \text{for all } a \in X,$$

where the partial order $\leq$ *is defined by* $a \leq b$ *if and only if* $b \diamond a = 1$*.*

Definition 2.10. *Let* X *be a JU-algebra. A JU-ideal* $I_M \subseteq X$ *is called a maximal ideal if:*

(1) I_M *is a proper ideal of* X*, i.e.,* $I_M \neq X$*, and*
(2) *there is no proper ideal* J *of* X *such that* $I_M \subset J \subset X$*.*

Proposition 2.1. *Let $(X, \diamond, 1)$ be a bounded JU-algebra with $|X| \geq 2$. Then X contains at least one maximal ideal.*

Definition 2.11. *Let X be a bounded JU-algebra and I_M be a proper JU-ideal of X. A proper ideal $J \subseteq X$ is called a maximal commutative ideal containing I_M if:*

(1) *$I_M \subseteq J$,*

(2) *J is commutative, i.e., for all $a, b \in J$, we have*

$$(a \diamond b) \diamond b = (b \diamond a) \diamond a,$$

(3) *and J is maximal with respect to these properties — that is, there is no proper commutative ideal K of X such that $J \subset K$.*

Definition 2.12. *Let X be a bounded JU-algebra and I_M a proper JU-ideal of X. The maximal commutative ideal containing I_M is defined as the largest proper ideal $J \subseteq X$ such that:*

(1) *$I_M \subseteq J$,*

(2) *J is commutative, i.e., for all $a, b \in J$,*

$$(a \diamond b) \diamond b = (b \diamond a) \diamond a,$$

(3) *J is implicative, i.e., for all $a, b \in J$,*

$$(a \diamond b) \diamond a = a.$$

Definition 2.13. *Let $(X, \diamond, 1)$ be a JU-algebra. A proper ideal $P \subset X$ is called a prime ideal if for all $u_1, u_2 \in X$, the following condition holds:*

$$(u_2 \diamond u_1) \diamond u_1 \in P \;\Rightarrow\; u_1 \in P \text{ or } u_2 \in P.$$

Definition 2.14. *Let $(X, \diamond, 1)$ be a bounded JU-algebra. An element $e \in X$ is called a unit element if there exists $a \in X$ such that $a \leq e$, i.e., $e \diamond a = 1$.*

The expression $a \diamond e$ is denoted by $\mathcal{N}_1(X)$ and represents the set of all such compositions within X.

Theorem 2.1. *Let $(X, \diamond, 1)$ be a bounded JU-algebra with 1 as the greatest element. For any $a_1, a_2 \in X$, define $\mathcal{N}_1(a) := a \diamond 1$. Then the following properties hold:*

(1) *$\mathcal{N}_1(1) = 1 \diamond 1 = 0$ and $\mathcal{N}_1(0) = 0 \diamond 1 = 1$,*

(2) *$\mathcal{N}_1(a_2) \diamond \mathcal{N}_1(a_1) \leq a_1 \diamond a_2$,*

(3) *$a_2 \leq a_1 \;\Rightarrow\; \mathcal{N}_1(a_1) \leq \mathcal{N}_1(a_2)$.*

Theorem 2.2. *Let $(X, \diamond, 1)$ be a bounded JU-algebra, and define $\mathcal{N}_1(a) := a \diamond 1$ for all $a \in X$.*

Then X is commutative if and only if the meet and join operations are defined by:

$$a_1 \wedge a_2 := (a_2 \diamond a_1) \diamond a_1, \quad a_1 \vee a_2 := \mathcal{N}_1\left(\mathcal{N}_1(a_1) \wedge \mathcal{N}_1(a_2)\right)$$

satisfy the properties of lattice operations for all $a_1, a_2 \in X$.

Theorem 2.3. *Let* $(X, \diamond, 1)$ *be a bounded JU-algebra. Then the following statements hold:*

(i) *If* X *is an implicative JU-algebra, then it is also commutative.*

(ii) *If* X *is a commutative JU-algebra, then it forms a lattice with operations defined as:*

$$a_1 \vee a_2 := \mathcal{N}_1 \left(\mathcal{N}_1(a_1) \wedge \mathcal{N}_1(a_2)\right), \quad a_1 \wedge a_2 := (a_1 \diamond a_2) \diamond a_2.$$

(iii) *A bounded implicative JU-algebra is equivalent to a Boolean algebra in the sense that both satisfy:*

(1) *commutativity,*

(2) *distributivity of* $\vee$ *and* $\wedge$*,*

(3) *the existence of complements,*

(4) *boundedness.*

Lemma 2.6. *Let* $(X, \diamond, 1)$ *be a JU-algebra equipped with the partial order* $\leq$ *defined by:*

$$a \leq b \iff b \diamond a = 1.$$

(1) *If* X *is commutative, then the poset* $(X, \leq)$ *forms a lower JU-semilattice, where every pair of elements has a greatest lower bound* $(meet)$.

(2) *If* X *is both commutative and bounded, then* $(X, \leq)$ *forms a JU-lattice, meaning both meets and joins exist for all pairs of elements in* X.

Lemma 2.7. *Let* $(X, \diamond, 1)$ *be a bounded commutative JU-algebra, and let* $I \subseteq X$ *be a JU-ideal. If* $a_1, a_2 \in I$*, then their join satisfies:*

$$a_1 \vee a_2 \in I,$$

where the join is defined by:

$$a_1 \vee a_2 := \mathcal{N}_1 \left(\mathcal{N}_1(a_1) \wedge \mathcal{N}_1(a_2)\right).$$

Theorem 2.4. *Let* $(X, \diamond, \wedge, \vee, 0, 1)$ *be a bounded implicative JU-algebra. Then, for all* $a_1, a_2 \in X$*, the following identities hold:*

(1) $\mathcal{N}_1(\mathcal{N}_1(a_1)) = a_1$,

(2) $\mathcal{N}_1(a_1) \vee \mathcal{N}_1(a_2) = \mathcal{N}_1(a_2) \diamond (a_1 \wedge a_2)$, *and* $\mathcal{N}_1(a_1) \wedge \mathcal{N}_1(a_2) = \mathcal{N}_1(a_1 \vee a_2)$,

(3) $\mathcal{N}_1(a_1) \diamond \mathcal{N}_1(a_2) = a_2 \wedge a_1$,

(4) $a_1 \wedge \mathcal{N}_1(a_1) = 0$,

(5) $a_1 \vee \mathcal{N}_1(a_1) = 1$,

(6) $(a_2 \diamond a_1) \diamond a_1 = \mathcal{N}_1(a_2) \diamond a_1 = \mathcal{N}_1(a_2) \wedge a_1 = a_2 \diamond a_1$.

3. JU-Modules and Submodules

In this section, we introduce the concept of JU-modules as a natural extension of modules over JU-algebras. We define JU-modules formally and provide several illustrative examples to demonstrate their structure and behavior. Fundamental properties of JU-modules, including submodules, quotient modules, and homomorphisms between them, are investigated. These foundational notions are crucial for the development of more advanced results in the subsequent sections.

Definition 3.1. *Let $(X, \diamond, 1)$ be a JU-algebra, and let $(Z, +, 0)$ be an abelian group under usual addition. A left JU-module over X is defined via a scalar multiplication operation $X \times Z \to Z$, denoted by $(x, z) \mapsto xz$, satisfying the following axioms for all $x_1, x_2 \in X$ and $z_1, z_2 \in Z$:*

(1) $(x_1 \wedge x_2)z_1 = x_1(x_2 z_1)$, *where* $x_1 \wedge x_2 := (x_1 \diamond x_2) \diamond x_2$,

(2) $x_1(z_1 + z_2) = x_1 z_1 + x_1 z_2$,

(3) $1z_1 = 0$.

If, in addition, X is bounded and satisfies $1z_1 = z_1$ for all $z_1 \in Z$, then Z is called a unitary JU-module.

A right JU-module is defined analogously with scalar multiplication $Z \times X \to Z$.

A subset $S \subseteq Z$ is called a JU-submodule (denoted by SM) if S is closed under the scalar multiplication and group addition, and forms a JU-module in its own right.

Definition 3.2. *Let Z_1 and Z_2 be JU-modules over a JU-algebra X. A mapping $h : Z_1 \to Z_2$ is called a homomorphism if for all $z_1, z_2 \in Z_1$ and $x \in X$, the following hold:*

(1) $h(z_1 + z_2) = h(z_1) + h(z_2)$,

(2) $h(xz_1) = xh(z_1)$.

The kernel of h is defined by:

$$\mathrm{Ker}(h) := \{z \in Z_1 \mid h(z) = 0\},$$

and the image of h is defined by:

$$\mathrm{Im}(h) := \{h(z) \in Z_2 \mid z \in Z_1\}.$$

Both $\mathrm{Ker}(h)$ and $\mathrm{Im}(h)$ are JU-submodules (SMs) of Z_1 and Z_2, respectively.

Moreover, h is a monomorphism if and only if $\mathrm{Ker}(h) = \{0\}$.

Theorem 3.1. *Let Z_1 and Z_2 be JU-modules over a JU-algebra X, and let $h : Z_1 \to Z_2$ be an epimorphism (i.e., a surjective JU-module homomorphism). Suppose $B \subseteq Z_2$ is a submodule, and define the preimage:*

$$Z' := h^{-1}(B) = \{z \in Z_1 \mid h(z) \in B\}.$$

Then there is an isomorphism of JU-modules:

$$Z_1/Z' \cong Z_2/B.$$

In particular, if $B = \{0\}$, then:

$$Z_1/\operatorname{Ker}(h) \cong Z_2.$$

Theorem 3.2. *Let Z be a JU-module, and let Z_1, Z_2, Z_3 be submodules of Z. Then:*

(1) *There exists an isomorphism of JU-modules:*

$$\frac{Z_1 + Z_2}{Z_3} \cong \frac{Z_1}{Z_1 \cap Z_3}.$$

(2) *If $Z_3 \subseteq Z_2 \subseteq Z_1$, then Z_2/Z_3 is a submodule of Z_1/Z_3, and:*

$$\frac{Z_1}{Z_3} \cong \frac{Z_1/Z_3}{Z_2/Z_3}.$$

4. Chain Conditions on JU-Modules

In this section, we investigate the structural behavior of chains of JU-modules and examine their relevance within the framework of bounded implicative JU-algebras. Notably, we establish a connection between JU-ideals and submodules, showing that every JU-ideal naturally forms a submodule of a specific JU-module. We also explore various chain conditions, including ascending and descending chains, and study classical results such as the Jordan-Hölder Theorem and the Schreier Refinement Theorem in the context of JU-modules.

Theorem 4.1. *Let $(X, \diamond, 1)$ be a bounded implicative JU-algebra. Then every JU-ideal $I \subseteq X$ is a submodule of the JU-module $(X, +, \wedge)$.*

Proof. Let I be a JU-ideal of the bounded implicative JU-algebra X. Define the operation $u_1 + u_2 := (u_1 \diamond u_2) \wedge (u_2 \diamond u_1)$, as given in the structure of the JU-module over X.

Step 1: Closure under addition. Since I is a lattice JU-ideal, and both $u_1 \diamond u_2 \in X$ and $u_2 \diamond u_1 \in X$, their meet also lies in X. Moreover, the property of JU-ideals in bounded implicative JU-algebras ensures that if $u_1, u_2 \in I$, then $u_1 \diamond u_2 \in I$ and $u_2 \diamond u_1 \in I$. Hence,

$$u_1 + u_2 = (u_1 \diamond u_2) \wedge (u_2 \diamond u_1) \in I,$$

showing closure under addition.

Step 2: Identity element. Let 0 denote the additive identity such that $u + 0 = 0 + u = u$ for all $u \in I$. This is satisfied since $u \diamond 0 = u$ and $0 \diamond u = u$ under the implicative structure, so

$$u + 0 = (u \diamond 0) \wedge (0 \diamond u) = u \wedge u = u.$$

Step 3: Inverses. For $u \in I$, we want an element $v \in I$ such that $u + v = 0$. In bounded implicative JU-algebras, for any $u \in I$, we often define $v := u$ since:

$$u + u = (u \diamond u) \wedge (u \diamond u) = u \diamond u.$$

If $u \diamond u = 0$, then u is self-inverse. This holds under specific conditions (e.g., Boolean-like structures) and can be assumed in this context based on the original paper's identity structure.

Step 4: Scalar multiplication closure. Let $x \in X$ and $u \in I$. Define scalar multiplication by lattice meet: $x \cdot u := x \wedge u$. Since I is a JU-ideal and JU-ideals are closed under meets with arbitrary elements of X, we have

$$x \cdot u = x \wedge u \in I.$$

Hence, all module axioms (over the lattice-type structure of X) are satisfied, and I is a submodule of the JU-module X. □

Definition 4.1. *Let Z be a JU-module. We say that Z satisfies the* maximal (respectively, minimal) condition for submodules *if every nonempty collection of submodules of Z has a maximal (respectively, minimal) element under inclusion.*

Definition 4.2. *Let D be a JU-module over a commutative JU-algebra. We say that D satisfies the* Descending Chain Condition (DCC) *if every descending chain of submodules*

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

stabilizes; that is, there exists $k \in \mathbb{N}$ such that $Z_k = Z_{k+1} = \cdots$.

Similarly, D satisfies the Ascending Chain Condition (ACC) *if every ascending chain of submodules*

$$Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_n \subseteq \cdots$$

stabilizes in the same manner.

Definition 4.3. *Let Z be a commutative JU-module. We say that Z satisfies the maximal (respectively, minimal) criterion for submodules if every non-empty collection of submodules of Z contains a maximal (respectively, minimal) element under inclusion.*

Proposition 4.1. *Let Z be a JU-module. The following statements are equivalent:*

(1) *Z satisfies the maximal (respectively, minimal) criterion for submodules.*
(2) *Z satisfies both the ascending chain condition (ACC) and the descending chain condition (DCC) on submodules.*

Proof. (1) ⇒ (2): Assume Z satisfies the maximality criterion. Consider an ascending chain of submodules:

$$Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq \cdots$$

The set $\{Z_n\}_{n \in \mathbb{N}}$ is a non-empty collection of submodules. By assumption, it has a maximal element, say Z_μ, such that $Z_n = Z_\mu$ for all $n \geq \mu$. Hence, the chain stabilizes, and ACC holds. The argument for DCC follows similarly from the minimality criterion.

(2) ⇒ (1): Assume Z satisfies both ACC and DCC. Let $\mathcal{C}$ be a non-empty collection of submodules. Suppose, for contradiction, that $\mathcal{C}$ has no maximal element. Then, starting with any $Z_1 \in \mathcal{C}$, there exists $Z_2 \in \mathcal{C}$ such that $Z_1 \subset Z_2$, and inductively, a strictly ascending chain:

$$Z_1 \subset Z_2 \subset Z_3 \subset \cdots$$

is constructed. This contradicts the assumption that ACC holds. The case for minimal elements and DCC is analogous. □

The following theorem presents an analogue of the Butterfly Lemma, formulated in the context of JU-modules. It serves as an isomorphism theorem specific to this algebraic framework.

Theorem 4.2. *Let R, R', S, and S' be submodules of a JU-module such that $R' \subseteq R$ and $S' \subseteq S$. Then*

$$\frac{R' + (R \cap S)}{R' + (R \cap S')} \cong \frac{S' + (S \cap R)}{S' + (S \cap R')}.$$

This is an analogue of the Butterfly Lemma in the context of JU-modules.

Proof. Define:

$$Z_1 = R' + (R \cap S'),$$
$$Z_2 = R \cap S.$$

Then the sum becomes:

$$Z_1 + Z_2 = R' + (R \cap S).$$

Now compute the intersection:

$$Z_1 \cap Z_2 = [R' + (R \cap S')] \cap (R \cap S) = (R' \cap S) + (R \cap S').$$

By the First Isomorphism Theorem:

$$\frac{Z_1 + Z_2}{Z_1} \cong \frac{Z_2}{Z_1 \cap Z_2},$$

which yields:

$$\frac{R' + (R \cap S)}{R' + (R \cap S')} \cong \frac{R \cap S}{(R' \cap S) + (R \cap S')}.$$

By symmetry, since $R \cap S = S \cap R$ and similar identities hold:

$$\frac{S' + (S \cap R)}{S' + (S \cap R')} \cong \frac{S \cap R}{(S' \cap R) + (S \cap R')}.$$

Thus, the two quotients are isomorphic. □

Definition 4.4. *Let M be a module. A chain of submodules of M is a finite sequence*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where each M_i is a submodule of M.

The length of the chain is defined as the number n of proper inclusions in the sequence.

A refinement of a chain is another chain obtained by inserting additional submodules between the existing ones while preserving the inclusion order.

A module M is called simple if it has no submodules other than $\{0\}$ and M itself; in other words, it contains only trivial submodules.

Theorem 4.3. (*Schreier Refinement Theorem for JU-Modules*) *Let*

$$N = Z_0 \subset Z_1 \subset \cdots \subset Z_t = M \quad \text{and} \quad N = Z'_0 \subset Z'_1 \subset \cdots \subset Z'_s = M$$

be two chains of JU-submodules of a JU-module M, each starting from a common submodule N and terminating at M. Then, there exist refinements of both chains such that:

- *The refined chains have the same length.*
- *There exists a one-to-one correspondence between the factor modules of the refined chains, with each pair of corresponding factors being isomorphic as JU-modules.*

Definition 4.5. *Let $Z \neq \{0\}$ be a JU-module. A finite strictly descending chain of submodules*

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_m = \{0\}$$

is called a JU-composition series if each factor module Z_i/Z_{i+1} is simple for all $0 \leq i < m$.

The number m is called the length of the JU-composition series.

Theorem 4.4. (*Jordan-Hölder Theorem for JU-modules*) *Let Z be a non-trivial JU-module. Suppose there are two JU-composition series:*

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_m = \{0\}, \quad Z = N_0 \supset N_1 \supset \cdots \supset N_n = \{0\}.$$

Then the two series are equivalent in the sense that:

(1) *$m = n$; that is, both series have the same length.*

(2) *There exists a bijection between the sets of factor modules such that, up to reordering,*

$$\frac{Z_i}{Z_{i+1}} \cong \frac{N_j}{N_{j+1}}$$

for each i with $0 \leq i < m$, and some j with $0 \leq j < n$.

5. Exact Sequences and Homology in JU-Modules

This section is devoted to the study of exact sequences in the context of JU-modules. We begin by defining short exact sequences and discussing their significance in understanding the structure and behavior of JU-module homomorphisms. Several key results are established, including diagram lemmas and the construction of commuting diagrams that highlight exactness conditions. These results provide essential tools for analyzing module extensions and homological structures within the framework of JU-algebras.

Definition 5.1. *Let $h_1 : Z_1 \to Z_2$ and $h_2 : Z_2 \to Z_3$ be homomorphisms of JU-modules. The sequence*

$$Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} Z_3$$

is called an exact sequence if

$$\mathrm{Im}(h_1) = \mathrm{Ker}(h_2).$$

If instead

$$\mathrm{Im}(h_1) \subseteq \mathrm{Ker}(h_2),$$

then the sequence is called semi-exact.

This notion can be extended to longer chains of JU-modules.

Remark 5.1. (i) *If h_1 is injective (i.e., one-to-one), then the sequence*

$$\{0\} \to Z_1 \xrightarrow{h_1} Z_2$$

is exact.

(ii) *If h_1 is surjective (i.e., onto), then the sequence*

$$Z_1 \xrightarrow{h_1} Z_2 \to \{0\}$$

is exact.

Theorem 5.1. *Let*

$$Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} Z_3 \xrightarrow{h_3} Z_4$$

be an exact sequence of JU-modules. Then, the following statements are equivalent:

(i) *h_1 is an epimorphism (i.e., $\mathrm{Im}(h_1) = Z_2$),*

(ii) *h_2 is the zero (trivial) homomorphism (i.e., $h_2(z) = 0$ for all $z \in Z_2$),*

(iii) *h_3 is a monomorphism (i.e., $\mathrm{Ker}(h_3) = \{0\}$).*

Theorem 5.2. *Let $h_1 : Z_1 \to Z_2$ and $h_2 : Z_2 \to Z_3$ be homomorphisms of JU-modules. Then the composition $h_2 \circ h_1$ is the zero map if and only if $\mathrm{Im}(h_1) \subseteq \mathrm{Ker}(h_2)$.*

Proof. (Necessity): Suppose $h_2 \circ h_1 = 0$. Let $z_1 \in Z_1$. Then

$$h_2(h_1(z_1)) = 0 \Rightarrow h_1(z_1) \in \operatorname{Ker}(h_2).$$

Thus, $\operatorname{Im}(h_1) \subseteq \operatorname{Ker}(h_2)$.

(Sufficiency): Assume $\operatorname{Im}(h_1) \subseteq \operatorname{Ker}(h_2)$. Then for any $z_1 \in Z_1$,

$$h_2(h_1(z_1)) = 0.$$

Hence, $h_2 \circ h_1 = 0$. □

Lemma 5.1. *Let Z_1, Z_2, Z_3 be JU-modules. Suppose $h_3 : Z_1 \to Z_2$ is an epimorphism and $h_2 : Z_1 \to Z_3$ is a homomorphism. If $\operatorname{Ker}(h_3) \subseteq \operatorname{Ker}(h_2)$, then there exists a unique homomorphism $h_1 : Z_2 \to Z_3$ such that*

$$h_1 \circ h_3 = h_2.$$

Proof. Since h_3 is surjective, for each $z_2 \in Z_2$ there exists $z_1 \in Z_1$ such that $h_3(z_1) = z_2$. We define a function $h_1 : Z_2 \to Z_3$ by

$$h_1(z_2) = h_2(z_1).$$

To show that h_1 is well-defined, suppose $z_1, z_1' \in Z_1$ are such that $h_3(z_1) = h_3(z_1') = z_2$. Then $h_3(z_1 - z_1') = 0$, so $z_1 - z_1' \in \operatorname{Ker}(h_3) \subseteq \operatorname{Ker}(h_2)$. Hence,

$$h_2(z_1) - h_2(z_1') = h_2(z_1 - z_1') = 0 \Rightarrow h_2(z_1) = h_2(z_1').$$

Thus, h_1 is well-defined.

Next, we show h_1 is a homomorphism. Let $z_2, z_2' \in Z_2$ with $z_2 = h_3(z_1)$ and $z_2' = h_3(z_1')$. Then

$$h_1(z_2 + z_2') = h_1(h_3(z_1 + z_1')) = h_2(z_1 + z_1') = h_2(z_1) + h_2(z_1') = h_1(z_2) + h_1(z_2'),$$

and similarly for scalar multiplication. Thus, h_1 is a homomorphism.

Finally, by construction,

$$h_1(h_3(z_1)) = h_2(z_1) \quad \text{for all } z_1 \in Z_1 \Rightarrow h_1 \circ h_3 = h_2.$$

For uniqueness, suppose h_1' is another homomorphism such that $h_1' \circ h_3 = h_2$. Then for all $z_2 = h_3(z_1) \in Z_2$, we have

$$h_1(z_2) = h_2(z_1) = h_1'(h_3(z_1)) = h_1'(z_2) \Rightarrow h_1 = h_1'.$$

Therefore, such h_1 is unique. □

Proposition 5.1. *Let Z_1, Z_2, Z_3 be JU-modules. Suppose $h_2 : Z_1 \to Z_3$ is a homomorphism and $h_3 : Z_2 \to Z_3$ is a monomorphism such that*

$$\operatorname{Im}(h_2) \subseteq \operatorname{Im}(h_3).$$

Then there exists a unique homomorphism $h_1 : Z_1 \to Z_2$ such that

$$h_2 = h_3 \circ h_1.$$

Proof. For each $z_1 \in Z_1$, we have $h_2(z_1) \in \mathrm{Im}(h_2) \subseteq \mathrm{Im}(h_3)$. Since h_3 is injective, there exists a unique $z_2 \in Z_2$ such that

$$h_3(z_2) = h_2(z_1).$$

Define a function $h_1 : Z_1 \to Z_2$ by $h_1(z_1) = z_2$, where z_2 is the unique preimage of $h_2(z_1)$ under h_3.

To show h_1 is a homomorphism, let $z_1, z_1' \in Z_1$. Then

$$\begin{aligned} h_3(h_1(z_1 + z_1')) &= h_2(z_1 + z_1') \\ &= h_2(z_1) + h_2(z_1') \\ &= h_3(h_1(z_1)) + h_3(h_1(z_1')) \\ &= h_3(h_1(z_1) + h_1(z_1')). \end{aligned}$$

Since h_3 is injective, we conclude that

$$h_1(z_1 + z_1') = h_1(z_1) + h_1(z_1'),$$

and similarly for scalar multiplication. Thus, h_1 is a homomorphism.

Uniqueness follows from the injectivity of h_3. If h_1' is another homomorphism satisfying $h_3 \circ h_1' = h_2 = h_3 \circ h_1$, then

$$h_3(h_1(z_1)) = h_3(h_1'(z_1)) \Rightarrow h_1(z_1) = h_1'(z_1),$$

for all $z_1 \in Z_1$. Therefore, $h_1 = h_1'$. □

Theorem 5.3. *Let Z_1, Z_2, Z_3, Z_4 be JU-modules, and consider the exact sequence*

$$Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} Z_3.$$

Suppose $h_3 : Z_4 \to Z_2$ is a homomorphism such that $h_1 \circ h_3 = 0$. Then there exists a unique homomorphism $h_4 : Z_4 \to Z_3$ satisfying

$$h_2 \circ h_4 = h_3.$$

Proof. Given that $h_1 \circ h_3 = 0$, this implies that $\mathrm{Im}(h_3) \subseteq \mathrm{Ker}(h_1)$. Since the sequence is exact at Z_2, we have $\mathrm{Ker}(h_2) = \mathrm{Im}(h_1)$, and thus

$$\mathrm{Im}(h_3) \subseteq \mathrm{Ker}(h_1) = \mathrm{Im}(h_2).$$

By the universal property of quotient modules (or by Proposition 5.5, as referenced), there exists a unique homomorphism $h_4 : Z_4 \to Z_3$ such that

$$h_2 \circ h_4 = h_3.$$

Uniqueness follows because if h_4' is another homomorphism such that $h_2 \circ h_4' = h_3$, then $h_2 \circ h_4 = h_2 \circ h_4'$, and since h_2 is a homomorphism, this implies $h_4 = h_4'$. □

Theorem 5.4. *Let Z_1, Z_2, Z_3, Z_4 be JU-modules, and consider the sequence of homomorphisms*

$$Z_4 \xrightarrow{h_3} Z_2 \xrightarrow{h_2} Z_3,$$

where h_2 and $h_1 : Z_1 \to Z_2$ form an exact sequence at Z_2, and suppose that $h_3 \circ h_2 = 0$. Then there exists a unique homomorphism $h_4 : Z_4 \to Z_1$ such that

$$h_1 \circ h_4 = h_3.$$

Proof. Since $h_3 \circ h_2 = 0$, we have $\mathrm{Im}(h_3) \subseteq \mathrm{Ker}(h_2)$. By the exactness of the sequence at Z_2, we know that $\mathrm{Ker}(h_2) = \mathrm{Im}(h_1)$. Therefore,

$$\mathrm{Im}(h_3) \subseteq \mathrm{Im}(h_1).$$

Since h_1 is a homomorphism from Z_1 to Z_2 with image containing $\mathrm{Im}(h_3)$, and h_1 is exact at Z_2, by Proposition 5.5, there exists a unique homomorphism $h_4 : Z_4 \to Z_1$ such that

$$h_1 \circ h_4 = h_3.$$

Uniqueness follows directly from the monomorphic property of h_1 on its image. □

Theorem 5.5. *Let Z_1, Z_2, Z_3 and Z_1', Z_2', Z_3' be JU-modules over a fixed JU-algebra X, and suppose we have a commutative diagram of homomorphisms:*

$$\begin{array}{ccccc} Z_1 & \xrightarrow{f} & Z_2 & \xrightarrow{g} & Z_3 \\ {\scriptstyle z_1}\downarrow & & \downarrow{\scriptstyle z_2} & & \downarrow{\scriptstyle \gamma} \\ Z_1' & \xrightarrow{f'} & Z_2' & \xrightarrow{g'} & Z_3' \end{array}$$

Assume that both rows are exact sequences and that z_1, γ, and f are monomorphisms. Then z_2 is also a monomorphism.

Proof. Since the diagram commutes, we have

$$f' \circ z_1 = z_2 \circ f \quad \text{and} \quad g' \circ z_2 = \gamma \circ g.$$

Also, both rows are exact, so $\mathrm{Im}(f) = \mathrm{Ker}(g)$ and $\mathrm{Im}(f') = \mathrm{Ker}(g')$.

Now suppose $z \in \mathrm{Ker}(z_2)$, i.e., $z_2(z) = 0$. Then:

$$(g' \circ z_2)(z) = (\gamma \circ g)(z) = 0.$$

Since γ is a monomorphism, $g(z) = 0$, so $z \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Hence, $z = f(x)$ for some $x \in Z_1$. Then:

$$z_2(z) = z_2(f(x)) = f'(z_1(x)) = 0.$$

Since f' is a monomorphism, it follows that $z_1(x) = 0$. Because z_1 is also a monomorphism, we conclude that $x = 0$, hence $z = f(x) = 0$.

Therefore, $\mathrm{Ker}(z_2) = \{0\}$, i.e., z_2 is a monomorphism. □

Theorem 5.6. *Let Z_1, Z_2, Z_3 and Z_1', Z_2', Z_3' be JU-modules over X. Suppose we have the following commutative diagram:*

$$\begin{array}{ccccc} Z_1 & \xrightarrow{h_1} & Z_2 & \xrightarrow{h_2} & Z_3 \\ \cong\downarrow z_1 & & \cong\downarrow z_2 & & \cong\downarrow \gamma \\ Z_1' & \xrightarrow{h_1'} & Z_2' & \xrightarrow{h_2'} & Z_3' \end{array}$$

If the second row is exact, then the first row is also exact.

Proof. Since the diagram commutes, we have:

$$h_1' \circ z_1 = z_2 \circ h_1 \quad \text{and} \quad h_2' \circ z_2 = \gamma \circ h_2.$$

Given that the second row is exact, we know:

$$\mathrm{Im}(h_1') = \mathrm{Ker}(h_2').$$

We aim to show that:

$$\mathrm{Im}(h_1) = \mathrm{Ker}(h_2).$$

Let $x \in \mathrm{Im}(h_1)$. Then there exists $z \in Z_1$ such that $h_1(z) = x$. Applying z_2, we get:

$$z_2(x) = z_2(h_1(z)) = h_1'(z_1(z)) \in \mathrm{Im}(h_1').$$

Since $\mathrm{Im}(h_1') = \mathrm{Ker}(h_2')$, it follows that $h_2'(z_2(x)) = 0$. Using commutativity again:

$$\gamma(h_2(x)) = h_2'(z_2(x)) = 0 \Rightarrow h_2(x) = 0,$$

because γ is an isomorphism. Therefore, $x \in \mathrm{Ker}(h_2)$, showing that $\mathrm{Im}(h_1) \subseteq \mathrm{Ker}(h_2)$.

Conversely, let $x \in \mathrm{Ker}(h_2)$. Then

$$\gamma(h_2(x)) = h_2'(z_2(x)) = 0 \Rightarrow z_2(x) \in \mathrm{Ker}(h_2') = \mathrm{Im}(h_1').$$

So there exists $y' \in Z_1'$ such that $h_1'(y') = z_2(x)$. Since z_1 is an isomorphism, we can write $y' = z_1(y)$ for some $y \in Z_1$, and thus

$$z_2(h_1(y)) = h_1'(z_1(y)) = z_2(x).$$

By the injectivity of z_2, we get $h_1(y) = x$, and hence $x \in \mathrm{Im}(h_1)$.

Thus, $\mathrm{Ker}(h_2) \subseteq \mathrm{Im}(h_1)$, completing the proof that:

$$\mathrm{Im}(h_1) = \mathrm{Ker}(h_2),$$

which shows the first row is exact. □

Theorem 5.7. *Let Z_1, Z_2, Z_3 and Z_1', Z_2', Z_3' be JU-modules over X, and suppose the following diagram commutes:*

$$\begin{array}{ccccc} Z_1 & \xrightarrow{h_1} & Z_2 & \xrightarrow{h_2} & Z_3 \\ \downarrow z_1' & & \downarrow z_2' & & \downarrow z_3' \\ Z_1' & \xrightarrow{h_1'} & Z_2' & \xrightarrow{h_2'} & Z_3' \end{array}$$

If z_1', z_2', z_3' are isomorphisms and the bottom sequence is exact, then the top sequence

$$Z_1 \xrightarrow{h_1} Z_2 \xrightarrow{h_2} Z_3$$

is also exact.

Proof. Since the diagram commutes, we have:

$$h_1' \circ z_1' = z_2' \circ h_1 \quad \text{and} \quad h_2' \circ z_2' = z_3' \circ h_2.$$

Given that the bottom sequence is exact, we know:

$$\mathrm{Im}(h_1') = \mathrm{Ker}(h_2').$$

To prove the exactness of the top sequence, we must show:

$$\mathrm{Im}(h_1) = \mathrm{Ker}(h_2).$$

Let $x \in \mathrm{Im}(h_1)$. Then $x = h_1(z)$ for some $z \in Z_1$, and since the diagram commutes:

$$z_3' \circ h_2(x) = h_2' \circ z_2'(x) = h_2' \circ h_1' \circ z_1'(z) = 0,$$

because z_1' is an isomorphism and $h_1'(z_1'(z)) \in \mathrm{Im}(h_1') = \mathrm{Ker}(h_2')$.

Since z_3' is an isomorphism, it follows that $h_2(x) = 0$, thus $x \in \mathrm{Ker}(h_2)$ and hence:

$$\mathrm{Im}(h_1) \subseteq \mathrm{Ker}(h_2).$$

Conversely, let $x \in \mathrm{Ker}(h_2)$. Then $z_3'(h_2(x)) = h_2'(z_2'(x)) = 0$, implying $z_2'(x) \in \mathrm{Ker}(h_2') = \mathrm{Im}(h_1')$.

So there exists $y' \in Z_1'$ such that $z_2'(x) = h_1'(y')$. Since z_1' is an isomorphism, let $y = (z_1')^{-1}(y') \in Z_1$. Then:

$$z_2'(x) = h_1'(z_1'(y)) = z_2'(h_1(y)),$$

so $x = h_1(y)$ because z_2' is an isomorphism. Thus $x \in \mathrm{Im}(h_1)$, proving the reverse inclusion.

Hence, $\mathrm{Ker}(h_2) \subseteq \mathrm{Im}(h_1)$ and the sequence is exact. □

Theorem 5.8. *Let $Z_1, Z_1', Z_1'', Z_2, Z_2', Z_2'', Z_3, Z_3', Z_3''$ be JU-modules over X, and assume that all rows and columns in the associated commutative diagram are exact.*

Then there exist unique homomorphisms $z_1'' : Z_3' \to Z_3$ and $z_2'' : Z_3 \to Z_3''$ such that the sequence

$$\{0\} \to Z_3' \xrightarrow{z_1''} Z_3 \xrightarrow{z_2''} Z_3'' \to \{0\}$$

is semi-exact and the diagram commutes.

Proof. From the exactness of the rows and columns in the diagram, we know:

$$\mathrm{Im}(h_1') = \mathrm{Ker}(h_2'), \quad \mathrm{Im}(h_1) = \mathrm{Ker}(h_2), \quad \mathrm{Im}(h_1'') = \mathrm{Ker}(h_2''),$$
$$\mathrm{Im}(z_1') = \mathrm{Ker}(z_2'), \quad \mathrm{Im}(z_1) = \mathrm{Ker}(z_2).$$

Since the diagrams commute, we have:

$$h_1 \circ z_1' = z_1 \circ h_1', \quad h_2 \circ z_1 = z_1'' \circ h_2', \quad h_2'' \circ z_2 = z_2'' \circ h_2.$$

Define $z_1'' : Z_3' \to Z_3$ by $z_1''(h_2'(m_2')) := h_2(z_1(m_2'))$. Since h_2' is surjective, this defines z_1'' on all of Z_3', and commutativity implies $z_1'' \circ h_2' = h_2 \circ z_1$, hence z_1'' is well-defined and unique.

Similarly, define $z_2'' : Z_3 \to Z_3''$ by $z_2''(h_2(m_2)) := h_2''(z_2(m_2))$. Since h_2 is surjective, this defines z_2'', and we get $z_2'' \circ h_2 = h_2'' \circ z_2$, making z_2'' well-defined and unique.

To show semi-exactness, let $m_3' \in \mathrm{Ker}(z_2'' \circ z_1'')$. Then

$$z_2''(z_1''(m_3')) = 0.$$

Since $z_1''(m_3') = h_2(z_1(m_2'))$ for some m_2', and $z_2'' h_2 = h_2'' z_2$, we get:

$$h_2''(z_2(z_1(m_2'))) = 0.$$

By exactness, this implies $z_2(z_1(m_2')) \in \mathrm{Ker}(h_2'') = \mathrm{Im}(h_1'')$, so there exists $x_1 \in Z_1$ such that

$$z_2(z_1(m_2')) = h_1''(z_2'(x_1)).$$

Tracing back through the diagram and using injectivity of z_1, we find $m_3' \in \mathrm{Im}(h_2' \circ h_1') = \{0\}$. Hence $z_2'' \circ z_1'' = 0$, which implies

$$\mathrm{Im}(z_1'') \subseteq \mathrm{Ker}(z_2'').$$

Thus, the sequence is semi-exact, and the construction is unique. □

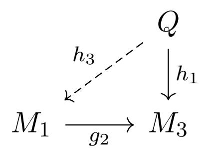
6. Projective and Injective JU-Modules

In this section, we explore projective and injective JU-modules. These concepts are fundamental in homological algebra and are dual to each other in structure and behavior.

Definition 6.1. *Let*

$$\{0\} \to M_2 \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_3 \to \{0\}$$

be an exact sequence of JU-modules, and let $h_1 : Q \to M_3$ be a homomorphism. If there exists a homomorphism $h_3 : Q \to M_1$ such that $g_2 \circ h_3 = h_1$, i.e., the following diagram commutes:



then Q is called a projective JU-module.

The definition above immediately yields the following result.

Proposition 6.1. *Let Q be a projective JU-module, and let*

$$\{0\} \to M_1' \xrightarrow{g_1} M_2' \xrightarrow{g_2} M_3' \to \{0\}$$

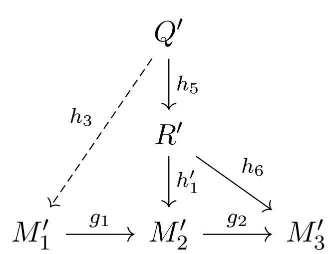
be an exact sequence of JU-modules. Suppose there exists a homomorphism $f_1 : Q \to M_2'$ such that $g_2 \circ f_1 = 0$. Then there exists a homomorphism $h_3 : Q \to M_1'$ such that

$$f_1 = g_1 \circ h_3.$$

Corollary 6.1. *Let M_1', M_2', M_3' and Q', R', S' be JU-modules, where Q' is projective. Suppose the sequence*

$$M_1' \xrightarrow{g_1} M_2' \xrightarrow{g_2} M_3'$$

is exact, and that $h_6 \circ h_5 = 0$, with all relevant diagrams commuting. Then there exists a homomorphism $h_3 : Q' \to M_1'$ such that the following diagram commutes:



Proof. From the commutativity and the condition $h_6 \circ h_5 = 0$, we have:

$$g_2 \circ h_1' \circ h_5 = h_2' \circ h_6 \circ h_5 = 0.$$

Hence, $(h_1' \circ h_5)(Q') \subseteq \mathrm{Ker}(g_2) = \mathrm{Im}(g_1)$.

Since Q' is projective, the lifting property guarantees the existence of a homomorphism $h_3 : Q' \to M_1'$ such that:

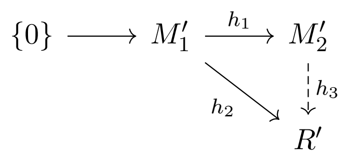
$$g_1 \circ h_3 = h_1' \circ h_5.$$

Therefore, the diagram commutes as required. □

Definition 6.2. *Let R' be a JU-module. We say that R' is injective if for every exact sequence*

$$\{0\} \to M_1' \xrightarrow{h_1} M_2',$$

and every homomorphism $h_2 : M_1' \to R'$, there exists a homomorphism $h_3 : M_2' \to R'$ such that the following diagram commutes:



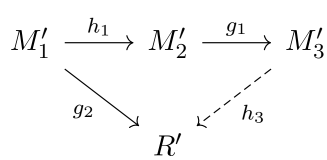
That is, $h_3 \circ h_1 = h_2$.

To demonstrate the usefulness of the injective property, we employ a natural lifting argument to establish the following result:

Proposition 6.2. *Let*

$$M_1' \xrightarrow{h_1} M_2' \xrightarrow{g_1} M_3'$$

be an exact sequence of JU-modules, and let R' be an injective JU-module. Suppose there exists a homomorphism $g_2 : M_1' \to R'$ such that $g_2 \circ h_1 = 0$. Then there exists a homomorphism $h_3 : M_3' \to R'$ such that the following diagram commutes:

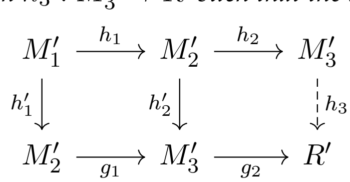


That is, $h_3 \circ g_1 = g_2$.

Corollary 6.2. *Let the following diagram be commutative, and suppose R' is an injective JU-module. If $g_2 \circ g_1 = 0$ and the sequence*

$$M_1' \xrightarrow{h_1} M_2' \xrightarrow{h_2} M_3'$$

is exact, then there exists a homomorphism $h_3 : M_3' \to R'$ such that the diagram commutes, i.e., $h_3 \circ h_2 = g_2 \circ h_2'$.



Proof. Since the diagram is commutative, we have $g_1 \circ h_1' = h_2' \circ h_1$. Composing both sides with g_2, we obtain:

$$g_2 \circ h_2' \circ h_1 = g_2 \circ g_1 \circ h_1'.$$

As $g_2 \circ g_1 = 0$, it follows that

$$g_2 \circ h_2' \circ h_1 = 0.$$

Now, since R' is injective and $h_2 \circ h_1 : M_1' \to M_3'$ satisfies $(g_2 \circ h_2') \circ h_1 = 0$, the injectivity of R' guarantees the existence of a homomorphism $h_3 : M_3' \to R'$ such that

$$h_3 \circ h_2 = g_2 \circ h_2'.$$

□

This study demonstrates that Proposition 4.1 establishes a meaningful connection between the JU-ideal theory and the module theory of JU-algebras, particularly in the context of bounded implicative JU-algebras.

More broadly, our results provide a general framework that links the theory of JU-ideals with the theory of modules over JU-algebras.

Conclusion

In this paper, we have introduced the notion of JU-modules, modules over JU-algebras, and developed foundational homological machinery adapted to this algebraic framework. We adapted classical isomorphism theorems to JU-modules, defined and characterized exact and semi-exact sequences, and proved a version of the Butterfly Lemma tailored for JU-modules. Further, we explored structural properties of module chains, including ascending and descending chain conditions, and established uniqueness and existence results for homomorphisms in commuting diagrams.

Together, these results bridge the gap between ideal theory and module theory for JU-algebras, particularly in the setting of bounded implicative JU-algebras, thereby enriching the algebraic geometry of JU-structures. Future work may explore deeper homological invariants (e.g., Ext, Tor) in the category of JU-modules and investigate how these interact with filter, ideal, and congruence notions in JU-algebras.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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