Asia Pac. J. Math. 2026 **13**:50

NONLINEAR ANISOTROPIC ELLIPTIC PROBLEM WITH NON HOMOGENOUS BOUNDARY CONDITION INVOLVING DIFFUSE RADON MEASURE DATA AND VARIABLE EXPONENTS

MALICK ZOUNGRANA*, ABDOUL AZIZ KALIFA DIANDA, IBRAHIME KONATE

LAboratoire Sciences et Technologies (LAST), Université Thomas Sankara, 12 BP 417 Ouagadougou 12, Burkina Faso

*Corresponding author: zoungmalick@yahoo.fr

Received Jan. 10, 2026

Abstract. This article concerns the existence and uniqueness result of entropy or renormalized solution for a non linear elliptic problem under non homogenous boundary condition governed by a general anisotropic operator with variable exponents. We use a diffuse Radon measure data which does not charge the sets of zero $p(.)$- capacity.

2020 Mathematics Subject Classification. 47A75, 35B38, 35P30, 34L05, 34L30.

Key words and phrases. non homogenous boundary; generalized Lebesgue-Sobolev spaces; anisotropic Sobolev spaces; weak solution; entropy solution; maximal monotone graph; bounded Radon diffuse measure; Marcinkiewicz spaces.

1. Introduction

In this paper we are interested in the following nonlinear multivalued elliptic anisotropic problem.

$$P(\mu,g)\begin{cases} -\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\big(x,\frac{\partial u}{\partial x_i}\big)+\beta(u)\ni\mu & \text{in } \Omega \\ \\ \displaystyle\sum_{i=1}^{N}a_i\big(x,\frac{\partial u}{\partial x_i}\big).\eta_i+\gamma(u)\ \ni g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N (N\geq 3)$ with smooth boundary $\partial\Omega$, β and γ are maximal monotone graphs on $\mathbb{R}$ such that $0\in\beta(0)$ and $0\in\gamma(0)$, μ a bounded Radon diffuse measure, g a bounded Radon diffuse measure on $\partial\Omega$, $\overrightarrow{\eta}=(\eta_1,...,\eta_N)$ the outward unit normal to $\partial\Omega$. More precisely, we are interested to the existence and uniqueness of renormalized or entropy solution of the problem (1.1).

DOI: 10.28924/APJM/13-50

We set $\overline{dom(\beta)} = [m, M]$ with $m \leq 0 \leq M$ and denote by

$$p_M(x) := max(p_1(x), ..., p_N(x)) \;\; and \;\; p_m(x) := min(p_1(x), ..., p_N(x)).$$

Let $\overrightarrow{p(.)} = (p_1(.), ..., p_N(.))$ be such that for any $i = 1, ..., N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x \in \overline{\Omega}} p_i(x) \leq p_i^+ := \sup_{x \in \overline{\Omega}} p_i(x) < \infty. \tag{1.2}$$

The operator $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function (i.e $a_i(x, \xi)$ is continuous in ξ for a.e $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}$) satisfying:

• There exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1\Big(j_i(x) + |\xi|^{p_i(x)-1}\Big) \tag{1.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a nonnegative function in $L^{p_i'(.)}(\Omega)$, with $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1.$

• For $\xi, \eta \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \mathbb{R}$, there exists a positive constant C_2 such that

$$\Big(a_i(x, \xi) - a_i(x, \eta)\Big)\Big(\xi - \eta\Big) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} \;\; if \;\; |\xi - \eta| \geq 1 \\ \\ C_2|\xi - \eta|^{p_i^-} \;\; if \;\; |\xi - \eta| < 1 \end{cases} \tag{1.4}$$

and,

• There exists a positive constant C_3 such that

$$a_i(x, \xi).\xi \geq C_3|\xi|^{p_i(x)}, \;\; \text{for} \;\; \xi \in \mathbb{R} \;\; \text{and almost every } x \in \Omega. \tag{1.5}$$

The assumptions on a_i are classical in the study of nonlinear PDEs.

Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \;\; \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p} - N}{\overline{p}(N-1)} \tag{1.6}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.7}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-}$, and for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} \;\; \text{if } p(x) < N \\ \\ +\infty \qquad\quad \text{if } p(x) \geq N. \end{cases}$$

We make the following assumption.

$$r \in C(\overline{\Omega}) \text{ with } 1 < r^- \leq r^+ < \min\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}. \tag{1.8}$$

For any $l_0 > 0$, we consider a function h_0 such that.

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,

(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \ \gamma(s) \ \text{ if } \gamma(s) \ \neq \emptyset \\ +\infty & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset \\ -\infty & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

Various mathematical problems with variable exponent has received considerable attention in recent years. The motivation of their study is due to their varied application. These problems arise in many applications as the modeling of electro-rheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electro-magnetic field [12,14,29–31], reaction-diffusion systems, modeling of propagation of epidemic disease [2]. These problems are also used in the field of image processing [1,10].

Many authors have studied problems with a maximal monotone graph and measure data in classical Lebesgue and Sobolev spaces (see [3–5,11,13,19]).

In [27] and [28] the authors extended these problems to the Sobolev spaces with variable exponent in the context of isotropic operators.

This paper is concerned with the anisotropic elliptic strongly non linear equation with variable exponent in which the $\overrightarrow{p}(.)$-Laplacian is more general. In the papers (see [6,7,18,21]), this kind of operator was considered.

We denote by $M_b(\Omega)$ the space of bounded Radon measures in Ω, equiped with its standard norm $\|.\|_{M_b(\Omega)}$. Note that, if μ belongs to $M_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω.

Given $\mu \in M_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $Cap_{p(.)}(A, \Omega) = 0$.

For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \Big\{ u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \geq 0 \ \text{ on } \ \Omega \Big\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$Cap_{p(.)}(A, \Omega) = \inf_{u \in S_{p(.)}(A)} \left\{ \int_\Omega |\nabla u|^{p(x)} dx \right\}.$$

In the case $S_{p(.)}(A) = \emptyset$, we set $Cap_{p(.)}(A, \Omega) = +\infty$

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$.

Note that, since we are dealing with the nonhomogenous boundary conditions, we cannot work with the common space $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. However, the common space is $W^{1,\overrightarrow{p}(.)}(\Omega)$, so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'_m(.)}(\Omega)$ (the dual of $W_0^{1,p_m(.)}(\Omega)$).

To overcome this difficulty, we use the same ideas as authors in [28]. We consider a smooth domain Ω in order to work with the space $W_0^{1,\tilde{p}_m(.)}(U_\Omega)$, where $\tilde{p}_m(.) : U_\Omega \longrightarrow (1,\infty)$ is a continuous function such that $\tilde{p}_m(x) = p_m(x)$ for all $x \in \overline{\Omega}$, and return after to the space $W^{1,p_m(.)}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, Ω is an extension domain (see [8]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E : W^{1,p_m(.)}(\Omega) \to W_0^{1,\tilde{p}_m(.)}(U_\Omega),$$

for which

(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p_m(.)}(\Omega)$,

(ii) $||E(u)||_{W_0^{1,\tilde{p}_m(.)}(U_\Omega)} \leq C||u||_{W^{1,p_m(.)}(\Omega)}$, where C is a constant depending only on Ω.

We define

$$\mathfrak{M}_b^{p_m(.)}(\Omega) := \{\mu \in \mathcal{M}_b^{\tilde{p}_m(.)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p_m(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p_m(.)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u d\mu.$$

On the other hand, as μ is diffuse, there exist (see [27,28]) $f \in L^1(U_\Omega)$ and $F \in (L^{\tilde{p}'_m(.)}(U_\Omega))^N$ such that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} fE(u)dx + \int_{U_\Omega} F.\nabla E(u)dx.$$

We give a useful convergence result (see [27]).

Lemma 1.1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$.*

We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then,

$$z \in dom(\beta).$$

The rest of the paper is organized as follows. In Section 2, we introduce some fundamental preliminary results which are useful in this work. Then, we study the existence and uniqueness of entropy or renormalized solution in Section 3.

2. Preliminary

We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponents. Set

$$C_+(\overline{\Omega}) = \Big\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \quad a.e. \quad x \in \Omega \Big\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \Big\{ u : \Omega \longrightarrow \mathbb{R} \ \text{ a measurable real valued function such that} \int_\Omega |u|^{p(x)} dx < \infty \Big\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(.)} := inf\Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\}.$$

The $p(.)$ modular from $L^{p(.)}(\Omega)$ into $\mathbb{R}$ is defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [15], [16]) will be used later.

$$min\Bigg\{ |u|^{p^-}_{p(.)}; |u|^{p^+}_{p(.)} \Bigg\} \leq \rho_{p(.)}(u) \leq max\Bigg\{ |u|^{p^-}_{p(.)}; |u|^{p^+}_{p(.)} \Bigg\}. \tag{2.1}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1$ for any $x \in \Omega$, we have the Holder type inequality

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) |u|_{p(.)} |v|_{q(.)}. \tag{2.2}$$

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous (see [25], Theorem 2.8).

Herein, we need the following anisotropic Sobolev space with variable exponent

$$W^{1,\overrightarrow{p}(.)}(\Omega) := \Bigg\{ u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial xi} \in L^{p_i(.)}(\Omega), i = 1, ..., N \Bigg\},$$

which is a separable and reflexive Banach space (see [26]) under the norm

$$\|u\|_{\overrightarrow{p}(.)} = |u|_{p_M(.)} + \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|_{p_i(.)}$$

We need the following embedding and trace results.

Theorem 2.1. ([15], *Corollary 2.1) Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set and for all $i = 1, ..., N$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e $x \in \Omega$. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$, a.e $x \in \Omega$ such that*

$$ess \inf_{x\in\Omega} (p_M(x) - q(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega). \tag{2.3}$$

Theorem 2.2. *([7], Theorem 6) Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set with smooth boundary and let $\overrightarrow{p}(.) \in (C_+(\overline{\Omega}))^N$, $r \in C(\overline{\Omega})$ satisfying the condition $1 \leq r(x) < min\Big\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\Big\}$; $\forall x \in \partial\Omega$. Then, there exists a compact boundary trace embedding*

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

In particular,

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{1}(\partial\Omega).$$

We introduce now the numbers

$$q = \frac{N(\overline{p}-1)}{N-1} \;\; and \;\; q^\star = \frac{N(\overline{p}-1)}{N-\overline{p}} = \frac{Nq}{N-q}$$

The following result is due to Troisi (see [32]).

Theorem 2.3. *Let $p_1, ..., p_N \in [1, \infty)$; $g \in W^{1,(p_1,...,p_N)}(\Omega)$ and*

$$\begin{cases} q = (\overline{p})^* \;\; if(\overline{p})^* < N, \\ q \in [1, \infty) \;\; if \;\; (\overline{p})^* \geq N. \end{cases} \tag{2.4}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, ..., p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$\|g\|_{L^q(\Omega)} \leq C_4 \Pi_{i=1}^N \left[\|g\|_{L^{p_M}(\Omega)} + \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}. \tag{2.5}$$

In this paper we will use the Marcinkiewicz space $\mathbb{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of all measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) := meas(\{x \in \Omega : |g(x)| > k\}), k \geq 0 \tag{2.6}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \;\; \text{for some finite constant} \;\; C > 0. \tag{2.7}$$

We will use the following pseudo norm in $\mathbb{M}^q(\Omega)$.

$$\|g\|_{\mathbb{M}^q} := inf\Big\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \forall k > 0\Big\}. \tag{2.8}$$

Finally, we will use throughout the paper, the truncation function T_k, $(k > 0)$ defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.9}$$

It is clear that $\lim_{k\to+\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

In order to simplify the notation, for any $v \in W^{1,\overrightarrow{p}(.)}(\Omega)$ we use v instead of $v|_{\partial\Omega}$ for the trace of v on $\partial\Omega$.

Set $\mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for any $k > 0$, $T_k(u) \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

We define the space $\mathcal{T}^{1,\overrightarrow{p}(.)}_{tr}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\overrightarrow{p}(.)}(\Omega)$ satisfying:

i) $u_n \longrightarrow u$ a.e. in Ω as $n \to +\infty$.

ii) $\dfrac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \dfrac{\partial T_k(u)}{\partial x_i} \in L^1(\Omega), \forall k > 0$.

iii) There exists a mesurable function v on $\partial\Omega$ such that $u_n \longrightarrow v$ a.e. on $\partial\Omega$ as $n \to +\infty$.

We also define the reflexive space

$$E := W^{1,\overrightarrow{p}(.)}(\Omega) \times L^{p_M(.)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u, \varphi) \in E : v = \tau(u)\},$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}^{1,\overrightarrow{p}(.)}_{tr}(\Omega)$ in the usual sense, since $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$. In the sequel, we will identify an element $(u, \varphi) \in X_0$ with its representative $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

We need the following Lemma proved in [6].

Lemma 2.4. *Let g be a nonnegative function in $W^{1,\overrightarrow{p}(.)}(\Omega)$. Assume $\overline{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p_M^-}\,dx + \sum_{i=1}^N \int_{\{|g|\le k\}} \left|\frac{\partial g}{\partial x_i}\right|^{p_i^-} dx \le C(k+1), \tag{2.10}$$

for every $k > 0$. Then, there exists a constant D, depending on C such that $\|g\|_{\mathbb{M}^{q^*}(\Omega)} \le D$, where $q^* = \dfrac{N(\overline{p}-1)}{N-\overline{p}}$.

Definition 2.5. *For any $\mu \in \mathfrak{M}^{p_m(.)}_b(\Omega)$ and $g \in \mathfrak{M}^{p_m(.)}_b(\partial\Omega)$, a renormalized solution of problem $P(\mu, g)$ is a triple $(u,\ b, w) \in \mathcal{T}^{1,\overrightarrow{p}(.)}_{tr}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$,*

$u \in dom(\beta)$ $\mathcal{L}^N$- a.e. in Ω, $b \in\ \beta(u)$ $\mathcal{L}^N$- a.e. in Ω, $tr(u) \in L^1(\partial\Omega)$, $w \in \gamma(u)\mathcal{L}^N$- a.e. on $\partial\Omega$ such that there exists $\nu \in \mathcal{M}^{p_m(.)}_b(\Omega)$ with $\nu \perp \mathcal{L}^N$,

$$\nu^+ \textit{ is concentred on } [u = M], \nu^- \textit{ is concentred on } [u = m],$$

there exists $\lambda \in \mathcal{M}^{p_m(.)}_b(\partial\Omega)$ such that $\lambda \perp \mathcal{L}^N$,

$$\lambda^+ \textit{ is concentred on } [u = M],\ \lambda^- \textit{ is concentred on } [u = m]$$

and

$$\sum_{i=1}^{N}\int_{\Omega}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial \varphi}{\partial x_i}dx+\int_{\Omega}b\varphi dx+\int_{\Omega}\varphi d\nu+\int_{\partial\Omega}w\varphi d\sigma+\int_{\partial\Omega}\varphi d\lambda=\int_{\Omega}\varphi d\mu+\int_{\partial\Omega}\varphi dg, \quad (2.11)$$

for any $\varphi\in W^{1,\overrightarrow{p}(.)}(\Omega)\cap L^{\infty}(\Omega)$. *Moreover,*

$$\lim_{n\to+\infty}\int_{[n\le|u|\le n+1]}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx=0,\ \textit{for}\ \ i=1,...,N. \quad (2.12)$$

Theorem 2.6. *Assume that (1.2)-(1.8) hold,* $\mu\in\mathfrak{M}_b^{p_m(.)}(\Omega)$ *and* $g\in\mathfrak{M}_b^{p_m(.)}(\partial\Omega)$. *Then, problem* $P(\mu,g)$ *admits a unique renormalized solution.*

3. Existence of solutions to Problem (1.1) with data in L^∞

In this section we consider the following problem.

$$P(T_k(b),\Upsilon)\begin{cases}-\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial u}{\partial x_i})+T_k(b(u))+\epsilon|u|^{P_M(x)-2}u=\ \Upsilon\ \text{ in }\ \Omega\\[2ex] \displaystyle\sum_{i=1}^{N}a_i(x,\frac{\partial u}{\partial x_i}).\eta_i+T_k(y(u))=\psi \qquad\qquad \text{on }\ \partial\Omega,\end{cases}$$

where b and y are surjective, continuous and nondecreasing function such that $b(0)=0$ and $y(0)=0$, $\Upsilon\in L^{\infty}(\Omega)$, $\psi\in L^{\infty}(\partial\Omega)$ for $k>0$ and $\epsilon>0$.

Theorem 3.1. *Assume that (1.2)-(1.8) hold, then* $P(T_k(b),\Upsilon)$ *admits at least one weak solution* $u_k\in W^{1,\overrightarrow{p}(.)}(\Omega)$ *such that* $\forall\,\varphi\ \in\ W^{1,\overrightarrow{p}(.)}(\Omega)\cap L^{\infty}(\Omega)$,

$$\sum_{i=1}^{N}\int_{\Omega}a_i\Big(x,\frac{\partial u_k}{\partial x_i}\Big)\frac{\partial\varphi}{\partial x_i}dx+\int_{\Omega}T_k(b(u_k))\varphi dx+\epsilon\int_{\Omega}|u_k|^{P_M(x)-2}u_k\varphi dx+\int_{\partial\Omega}T_k(y(u_k))\varphi d\sigma$$

$$=\int_{\Omega}\Upsilon\varphi dx+\int_{\partial\Omega}\psi\varphi d\sigma. \quad (3.1)$$

Furthermore,

$$\forall k>||\Upsilon||_{\infty},\ \ |b(u_k)|\le||\Upsilon||_{\infty}\ \text{a.e. in}\ \Omega, \quad (3.2)$$

$$\forall k>||\psi||_{\infty},\ \ |\psi(u_k)|\le||\Upsilon||_{\infty}\ \text{a.e. on}\ \partial\Omega. \quad (3.3)$$

Indeed, we define an operator $A_k:X_0\longrightarrow X_0'$ by

$$\langle A_k(u),\varphi\rangle\ =\ \langle A(u),\varphi\rangle+\int_{\partial\Omega}T_k(b(u))\varphi dx+\int_{\partial\Omega}T_k(y(u))\varphi d\sigma,\ \forall\,u,\varphi\ \in X_0,$$

where

$$\langle A(u),\varphi\rangle\ =\ \sum_{i=1}^{N}\int_{\Omega}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial\varphi}{\partial x_i}dx+\epsilon\int_{\Omega}|u|^{p_M(x)-2}u\varphi dx.$$

We show with similar arguments as in [18] that the operator A_k is of type (M), bounded and coercive. Therefore, by setting
$\langle F, \varphi\rangle = \int_\Omega \Upsilon \varphi dx + \int_{\partial\Omega} \psi\varphi dx$, it follows that $F \in E' \subset X_0'$. Then, we can deduce the existence of a function $u_k \in X_0$ such that

$$\langle A_k(u_k), \varphi\rangle \ = \ \langle F, \varphi\rangle, \ \text{ for all } \varphi \in X_0.$$

4. Proof of Theorem 2.6

The poof is done in three steps.

Step 1. The approximate problem

For every $\epsilon > 0$, we consider the Yosida regularization β_ϵ of β and γ_ϵ of γ see [32], given by

$$\beta_\epsilon = \frac{1}{\epsilon}(I - (I + \epsilon\beta)^{-1})$$

and

$$\gamma_\epsilon = \frac{1}{\epsilon}(I - (I + \epsilon\gamma)^{-1}).$$

Thanks to [32], there exist nonnegative, convex and l.s.c. functions j and z defined on $\mathbb{R}$ such that

$$\beta = \partial j$$

and

$$\gamma = \partial z.$$

To regularise j and z, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2\epsilon}|s-r|^2 + j(r) \right\}, \ \forall\, s \in \mathbb{R}, \ \forall \epsilon \ > 0$$

and

$$z_\epsilon(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2\epsilon}|s-r|^2 + z(r) \right\}, \ \forall\, s \in \mathbb{R}, \ \forall \epsilon \ > 0.$$

By Proposition 2.11 in [32] we have

$$\begin{cases} dom(\beta) \subset dom(j) \subset \overline{dom(j)} = \overline{dom(\beta)}, \\ j_s(s) = \dfrac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon) \text{ where } J_\epsilon := (I + \epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

and

$$\begin{cases} dom(\gamma) \subset dom(z) \subset \overline{dom(z)} = \overline{dom(\gamma)}, \\ z_s(s) = \dfrac{\epsilon}{2}|\gamma_\epsilon(s)|^2 + z(Z_\epsilon) \text{ where } Z_\epsilon := (I + \epsilon\gamma)^{-1}, \\ z_\epsilon \text{ is a convex, Frechet-differentiable function and } \gamma_\epsilon = \partial z_\epsilon, \\ z_\epsilon \uparrow z \text{ as } \epsilon \downarrow 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, β_ϵ and γ_ϵ are nondecreasing and Lipschitz-continuous function (see [28]).
Since μ and $g \in \mathcal{M}_b^{\tilde{p}_m(.)}(U_\Omega)$, recall that $\mu = f - div(F)$ and $g = h - div(H)$ in $\mathcal{D}'(U_\Omega)$ with f and $h \in L^1(U_\Omega)$ and F and $H \in (L^{\tilde{p}'_m(.)}(U_\Omega))^N$ where U_Ω is the open subset of $\mathbb{R}^N$ which extends Ω via the operator E.
We regularize f,h, g and μ respectively as follows.
For any $\epsilon > 0$ and $x \in U_\Omega$, we define the functions

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x).$$

$$h_\epsilon(x) = T_{\frac{1}{\epsilon}}(h(x))\chi_{\partial\Omega(x)}.$$

Let $(F_\epsilon)_{\epsilon>1} \subset C_0^\infty(U_\Omega)$ and $(H_\epsilon)_{\epsilon>1} \subset C_0^\infty(U_\Omega)$ are sequences such that $F_\epsilon \longrightarrow F$ strongly in $(L^{\tilde{p}'_m(.)}(U_\Omega))^N$ and $H_\epsilon \longrightarrow H$ strongly in $(L^{\tilde{p}'_m(.)}(U_\Omega))^N$. For any $\epsilon > 0$, we set $\tilde{F}_\epsilon = \chi_\Omega F_\epsilon$, $\tilde{H}_\epsilon = \chi_{\partial\Omega} H_\epsilon$, $\mu_\epsilon = f_\epsilon - div(\tilde{F}_\epsilon)$ and $g_\epsilon = h_\epsilon - div(\tilde{H}_\epsilon)$. For any $\epsilon > 0$, one has

- $\mu_\epsilon \in \mathfrak{M}_b^{p_m(.)}(\Omega),\ \mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_\epsilon \in L^\infty(\Omega)$.
- $g_\epsilon \in \mathfrak{M}_b^{p_m(.)}(\partial\Omega)$, $g_\epsilon \rightharpoonup g$ in $\mathcal{M}_b(U_\Omega)$ and $g_\epsilon \in L^\infty(\partial\Omega)$.
- $(f_\epsilon)_{\epsilon>0}$, $(h_\epsilon)_{\epsilon>0}$ are sequences of bounded functions which converges to $f \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$ respectively.

Moreover,

$$||f_\epsilon||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)},\ \forall\, \epsilon > 0, \tag{4.1}$$

$$||h_\epsilon||_{L^1(\partial\Omega)} \leq ||h||_{L^1(\partial\Omega)},\ \forall\, \epsilon > 0, \tag{4.2}$$

$$\left| \int_\Omega T_k(\varphi) d\mu_\epsilon \right| \leq kC(\mu, \Omega),\ \forall\, k > 0,\ \forall\, \varphi \in \mathcal{T}^{1,p_m(.)}(\Omega) \tag{4.3}$$

and

$$\left| \int_{\partial\Omega} T_k(\varphi) dg_\epsilon \right| \leq kC(g, \partial\Omega),\ \forall\, k > 0,\ \forall\, \varphi \in \mathcal{T}^{1,p_m(.)}(\partial\Omega). \tag{4.4}$$

We have the following lemma (see [28], Lemma 4.1).

Lemma 4.1. *The Yosida regularization β_ϵ and γ_ϵ are surjective operators.*

Now, we consider the following approximating scheme problem

$$P(\beta_\epsilon, \mu_\epsilon, g_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u_\epsilon}{\partial x_i}) + \beta_\epsilon(u_\epsilon) + \epsilon |u_\epsilon|^{P_M(x)-2} u_\epsilon = \mu_\epsilon \text{ in } \Omega \\ \\ \displaystyle\sum_{i=1}^N a_i(x, \frac{\partial u_\epsilon}{\partial x_i}).\eta_i + \gamma_\epsilon(u_\epsilon) = g_\epsilon \text{ on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\epsilon > 0$.

Theorem 4.2. *The problem (4.5) admits at least one weak solution u_ϵ in the sense that $u_\epsilon \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^1(\partial\Omega)$, $\beta_\epsilon(u_\epsilon) \in L^\infty(\Omega)$, $\gamma_\epsilon(u_\epsilon) \in L^\infty(\partial\Omega)$ and $\forall\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,*

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i}dx + \int_\Omega \beta_\epsilon(u_\epsilon)\varphi dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon \varphi dx + \int_{\partial\Omega}\gamma_\epsilon(u_\epsilon)\varphi d\sigma = \int_\Omega \varphi d\mu_\epsilon + \int_{\partial\Omega} g_\epsilon \varphi d\sigma. \quad (4.6)$$

For the proof of Theorem 4.2 we fix $k > max\Big(||\Upsilon||_\infty, ||\psi||_\infty\Big)$ in $P(T_k(b), \Upsilon)$ and set $\Upsilon = \mu_\epsilon$, $b = \beta_\epsilon$, $y = \gamma_\epsilon$ and $\psi = g_\epsilon$ to end the proof of the Theorem 4.2.

Step 2. A priori estimate

Lemma 4.3. *Let u_ϵ be weak a solution of $P(\beta_\epsilon, \mu_\epsilon, g_\epsilon)$. Then, there exist positive constants $C(\mu, \Omega)$ and $C(g, \partial\Omega)$ such that for any $k > 0$,*

$$\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx \le k\left(\frac{C(\mu,\Omega) + C(g,\partial\Omega)}{C_3}\right), \quad (4.7)$$

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le k(C(\mu,\Omega) + C(g,\partial\Omega)) \quad (4.8)$$

and

$$\int_{\partial\Omega} \gamma_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le k(C(\mu,\Omega) + C(g,\partial\Omega)). \quad (4.9)$$

Proof. We begin by Proving (4.7) and (4.8).

By Taking $\varphi = T_k(u_\epsilon)$ as test function in (4.6), we get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial T_k(u_\epsilon)}{\partial x_i}dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx + \int_{\partial\Omega}\gamma_\epsilon(u_\epsilon)T_k(u_\epsilon)d\sigma$$
$$= \int_\Omega T_k(u_\epsilon)d\mu_\epsilon + \int_{\partial\Omega} g_\epsilon T_k(u_\epsilon)d\sigma.$$

Then, taking into account that $\int_{\partial\Omega} g_\epsilon T_k(u_\epsilon)d\sigma \le kC(g, \partial\Omega)$, we use (1.5) and (4.3) in the last equality to get

$$C_3\sum_{i=1}^{N}\int_{|u_\epsilon|\le k}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx + \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx + \int_{\partial\Omega}\gamma_\epsilon(u_\epsilon)T_k(u_\epsilon)d\sigma + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)$$
$$\le k(C(\mu,\Omega) + C(g,\partial\Omega)). \quad (4.10)$$

Since T_k, $\beta_\epsilon, s \mapsto s^{P_M(.)-2}s$ are nondecreasing and $\beta_\epsilon(0) = T_k(0) = 0$, all the integrals in (4.10) are nonnegative. Therefore, we deduce from (4.10) that

$$\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx \le k\left(\frac{C(\mu,\Omega) + C(g,\partial\Omega)}{C_3}\right)$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le k(C(\mu,\Omega) + C(g,\partial\Omega)).$$

Let us prove (4.9). We use the fact that all integrals in (4.10) are nonnegative to obtain

$$\int_{\partial\Omega} \gamma_\epsilon(u_\epsilon)T_k(u_\epsilon)d\sigma \le k(C(\mu,\Omega) + C(g,\partial\Omega)). \quad (4.11)$$

□

We have the following lemma (we refer to [28], Proposition 4.2).

Lemma 4.4.

(i) *For any $k > 0$ The sequences $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$,$(\beta_\epsilon(T_k(u_\epsilon))_{\epsilon>0}$ are uniformly bounded in $L^1(\Omega)$.*

(ii) *For any $k > 0$, the sequences $(\gamma_\epsilon(u_\epsilon))_{\epsilon>0}$,$(\gamma_\epsilon(T_k(u_\epsilon))_{\epsilon>0}$ are uniformly bounded in $L^1(\Omega)$.*

Proof.

(i) Dividing the terms in (4.8) by $k > 0$ and letting k goes to 0, we get

$$\lim_{k\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)\frac{1}{k}T_k(u_\epsilon)dx = \int_\Omega \beta_\epsilon(u_\epsilon)sign_0(u_\epsilon)dx.$$

As

$$\int_\Omega \beta_\epsilon(u_\epsilon)sign_0(u_\epsilon)dx = \int_\Omega |\beta_\epsilon(u_\epsilon)|dx \leq C(\mu,\Omega) + C(\mu,\partial\Omega), \tag{4.12}$$

Then,$(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$. Since for any $k > 0$,

$$\int_\Omega |\beta_\epsilon(T_k(u_\epsilon))|dx \leq \int_\Omega |\beta_\epsilon(u_\epsilon)|dx,$$

we deduce that $(\beta_\epsilon(T_k(u_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.

(ii) Reasonning as to (i) we obtain (ii). □

Proposition 4.5. *Let u_ϵ be a weak solution of (4.5). Then, for all $k > 0$,*

$$\sum_{i=1}^{N}\int_{|u_\epsilon|\leq k}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx \leq N.meas(\Omega) + k\left(\frac{C(\mu,\Omega)+C(g,\partial\Omega)}{C_3}\right). \tag{4.13}$$

Let us prove (4.13).

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\{|u_\epsilon|\leq k\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-}dx &= \sum_{i=1}^{N}\int_{\{|u_\epsilon|\leq k,|\frac{\partial u_\epsilon}{\partial x_i}|\leq 1\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-}dx + \sum_{i=1}^{N}\int_{\{|u_\epsilon|\leq k,|\frac{\partial u_\epsilon}{\partial x_i}|>1\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i^-}dx\\
&\leq Nmeas(\Omega) + \sum_{i=1}^{N}\int_{\{|u_\epsilon|\leq k,|\frac{\partial u_\epsilon}{\partial x_i}|\geq 1\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\
&\leq Nmeas(\Omega) + \sum_{i=1}^{N}\int_{\{|u_\epsilon|\leq k\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\
&\leq Nmeas(\Omega) + \sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\
&\leq Nmeas(\Omega) + k\left(\frac{C(\mu,\Omega)+C(g,\partial\Omega)}{C_3}\right).
\end{aligned}$$

Lemma 4.6. *If u_ϵ is a weak solution of (4.5), then there exists a constant D which depends on μ and Ω such that*

$$meas\{|u_\epsilon| > k\} \leq \frac{D}{\min(b_\epsilon(k), |b_\epsilon(-k)|)},\ \forall\, k > 0 \tag{4.14}$$

and a constant D' which depends on μ and Ω such that

$$meas\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(p_M^-)'}}},\ \forall k \geq 1. \tag{4.15}$$

Proof. • For the proof of (4.17), we refer to [20,28].

• For the proof of (4.18), we refer to [6]. □

We need the following lemma (see [6], [17], [18], [24]).

Lemma 4.7. *For any $k > 0$, there exists some positive constants C and C' such that*

(i) $||u_\epsilon||_{\mathcal{M}^{q^*}(\Omega)} \leq C$;

(ii) $\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_{\mathcal{M}^{p_i^- \frac{q}{p}}(\Omega)} \leq C',\ \ \forall\, i = 1, ..., N.$

Step 3. Convergence results

In order to pass to the limit in (4.6), the following convergence results are necessary (see [6] and [24]).

Lemma 4.8. *For $i = 1, ..., N$, as $\epsilon \to +\infty$, we have*

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{4.16}$$

Proposition 4.9. *There exists a measurable function $u : \Omega \longrightarrow \mathbb{R}$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_\epsilon \longrightarrow u \text{ in measure and a.e. in } \ \Omega \text{ as } \epsilon \longrightarrow 0. \tag{4.17}$$

Proof. For the proof of (4.17), we refer to [6] (see also [24]).

As for $k > 0$, T_k is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in Ω. Finally, using Lemma 1.1 we deduce that for all $k > 0$, $T_k(u) \in dom(\beta)$ a.e. in Ω. Since $T_k(u) \in dom(\beta)$, we get $u \in dom(\beta)$ *a.e.* in Ω and as $dom(\beta)$ is bounded, then $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$. □

Proposition 4.10. *Assume (1.2)-(1.8). If $u_\epsilon \in E$ is a weak solution of (4.5) then.*

(i) *for all $i = 1, ..., N$, $\frac{\partial u_\epsilon}{\partial x_i}$ converges in measure to the weak partial gradient of u;*

(ii) *for all $i = 1, ..., N$ and $k > 0$, $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon)\right)$ converges to $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u)\right)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(.)}(\Omega)$ weakly.*

(iii) *For $i = 1, ..., N$, $a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$ in $L^1(\Omega)$ and a.e. $x \in \Omega$.*

Proof. (see [22], Proposition 3.3) □

We have the following lemmas (see [23,27,28]).

Lemma 4.11. *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, for any $i = 1, ..., N$,*

$$\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \longrightarrow \frac{\partial}{\partial x_i}(h(u)\varphi) \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0, \text{ for any } i = 1, ..., N.$$

Lemma 4.12. *We have*

$$\lim_{\epsilon \to 0} \int_\Omega h(u_\epsilon)\xi d\mu_\epsilon = \int_\Omega h(u)\xi d\mu, \tag{4.18}$$

$$\lim_{\epsilon \to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\xi dx = 0 \tag{4.19}$$

and

$$\lim_{\epsilon \to 0} \int_{\partial\Omega} \gamma_\epsilon(u_\epsilon)h(u_\epsilon)\varphi d\sigma = \int_{\partial\Omega} \gamma(u)h(u)\varphi d\sigma, \tag{4.20}$$

for any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Now, we pass to the limit in $\beta_\epsilon(u_\epsilon)$.

Since, for any $k > 0$, $(h_k(u_\epsilon)\beta_\epsilon(u_\epsilon))_{\epsilon>0}$, $(h_k(u_\epsilon)\gamma_\epsilon(u_\epsilon))_{\epsilon>0}$ are bounded respectively in $L^1(\Omega)$ and $L^1(\partial\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$ and $t_k \in \mathcal{M}_b(\partial\Omega)$ such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \rightharpoonup z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0$$

and

$$h_k(u_\epsilon)\gamma_\epsilon(u_\epsilon) \rightharpoonup t_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \varphi dz_k + \int_{\partial\Omega} \varphi dt_k = -\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h_k(u)\varphi]dx - \epsilon \int_\Omega \varphi |u|^{P_M(x)-2} u h_k(u)dx$$
$$+ \int_\Omega \varphi h_k(u)d\mu + \int_{\partial\Omega} \varphi h_k(u)dg$$

which implies that $z_k \in \mathcal{M}_b^{p_m(.)}(\Omega)$, $t_k \in \mathcal{M}_b^{p_m(.)}(\partial\Omega)$ and, for any $k \leq l$,

$$z_k = z_l \ ; \ \ t_k = t_l \ \text{ on } [|T_k(u)| < k].$$

Let us consider the Radon measure defined by

$$\begin{cases} z = z_k, \ \text{ on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*, \\ \\ z = 0 \ \text{ on } \displaystyle\bigcap_{k \in \mathbb{N}^*} [|T_k(u)| = k] \end{cases} \tag{4.21}$$

and

$$\begin{cases} t = t_k, \ \text{ on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*, \\ \\ t = 0 \ \text{ on } \displaystyle\bigcap_{k \in \mathbb{N}^*} [|T_k(u)| = k]. \end{cases} \tag{4.22}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\varphi dz + \int_{\partial\Omega} h(u)\varphi dt = -\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u}{\partial x_i}) \frac{\partial}{\partial x_i}[h(u)\varphi]dx - \epsilon \int_\Omega \varphi |u|^{P_M(x)-2} u h(u) dx$$

$$+ \int_\Omega \varphi h(u) d\mu + \int_{\partial\Omega} \varphi h(u) dg,$$

for any $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Indeed, let $k_0 > 0$ be such that $supp(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}
\int_\Omega h(u)\xi dz + \int_{\partial\Omega} h(u)\xi dt &= \int_\Omega h(u)\xi dz_{k_0} + \int_{\partial\Omega} h(u)\xi dt_{k_0} \\
&= -\lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u_\epsilon}{\partial x_i}) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx - \lim_{\epsilon\to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\varphi dx \\
&+ \lim_{\epsilon\to 0} \int_\Omega u_\epsilon h(u_\epsilon)\varphi d\sigma + \lim_{\epsilon\to 0} \int_{\partial\Omega} g_\epsilon h(u_\epsilon)\varphi d\sigma \\
&= -\lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx - \lim_{\epsilon\to 0} \epsilon \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon h(u_\epsilon)\varphi dx \\
&+ \lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon + \lim_{\epsilon\to 0} \int_{\partial\Omega} h(u_\epsilon)\varphi dg_\epsilon \\
&= -\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u}{\partial x_i}) \frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} h(u)\varphi dg.
\end{aligned}$$

Moereover, we have the following lemma (see [27], Lemma 4.7).

Lemma 4.13. *The Radon-Nikodym decomposition of the measure z given by (4.21) with respect to $\mathcal{L}^N$,*

$$z = b\mathcal{L}^N + \nu \ \textit{ with } \nu \perp \mathcal{L}^N$$

satisfies the following properties

$$\begin{cases}
b \in \beta(u)\mathcal{L}^N - a.e. \textit{ in } \Omega, \ b \in \ L^1(\Omega), \ \nu \ \in \mathcal{M}_b^{p_m(.)}(\Omega), \\
\nu^+ \textit{ is concentrated on } \ [u = M], \\
\nu^- \textit{ is concentrated on } \ [u = m].
\end{cases}$$

By using Lemma 4.13, we obtain the same decomposition result for the measure t:

$$t = w\mathcal{L}^N + \lambda \text{ with } \ \lambda \perp \mathcal{L}^N$$

satisfies the following properties

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \partial\Omega, \ w \in \ L^1(\partial\Omega), \ \lambda \ \in \mathcal{M}_b^{p_m(.)}(\partial\Omega), \\ \\ \lambda^+ \text{ is concentrated on } \ [u = M], \\ \\ \lambda^- \text{ is concentrated on } \ [u = m]. \end{cases}$$

To end the proof of the Theorem 2.6, we consider $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\varphi$ as a test function in (4.6) to get

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\varphi dx + \epsilon \int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon h(u_\epsilon)\varphi dx + \int_{\partial\Omega} \gamma_\epsilon(u_\epsilon)h(u_\epsilon)\varphi d\sigma$$

$$= \int_\Omega h(u_\epsilon)\varphi d\mu_\epsilon + \int_{\partial\Omega} h(u_\epsilon)\varphi dg_\epsilon. \tag{4.23}$$

The first term of (4.23) can be written as

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx = \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi]dx,$$

for $l_0 > 0$; so, by lemmas (4.11) and (4.12), we have

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx &= \lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\ &= \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial T_{l_0+1}(u)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\xi]dx \\ &= \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\xi]dx. \end{aligned}$$

By using convergence results in (4.23), we get

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx + \lim_{\epsilon\to 0} \int_{\partial\Omega} \gamma_\epsilon(u_\epsilon)h(u_\epsilon)\xi d\sigma &= \int_\Omega h(u)\xi d\mu + \int_{\partial\Omega} h(u)\varphi dg \\ &\quad - \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\xi]dx \\ &= \int_\Omega h(u)\varphi dz + \int_{\partial\Omega} h(u)\varphi dt \\ &= \int_\Omega bh(u)\varphi dx + \int_\Omega h(u)\varphi d\nu \\ &\quad + \int_{\partial\Omega} wh(u)\varphi d\sigma + \int_{\partial\Omega} h(u)\varphi d\lambda. \end{aligned}$$

Passing to the limit in (4.23) as $\epsilon \to 0$, we get

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\varphi]dx + \int_\Omega bh(u)\varphi dx + \int_\Omega h(u)\varphi d\nu + \int_{\partial\Omega} wh(u)\varphi d\sigma + \int_{\partial\Omega} h(u)\varphi d\lambda.$$

$$= \int_\Omega h(u)\varphi d\mu + \int_{\partial\Omega} h(u)\varphi dg. \tag{4.24}$$

It yields that (u, b, w) is a solution of the problem (1.1).

To end the proof of Theorem 2.6, we prove (4.2). We take $\xi = T_1(u_\epsilon - T_n(u_\epsilon))$ as a test function in (4.6) to get

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon)))dx + \int_\Omega \beta_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon))dx$$
$$+\epsilon \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} \gamma_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon))d\sigma$$
$$= \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon + \int_{\partial\Omega} T_1(u_\epsilon - T_n(u_\epsilon))dg_\epsilon. \tag{4.25}$$

Since

$$\int_\Omega \beta_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} \gamma_\epsilon(u_\epsilon) T_1(u_\epsilon - T_n(u_\epsilon))d\sigma \geq 0$$

and $\frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon))) = \frac{\partial u_\epsilon}{\partial x_i}\chi_{[n<|u_\epsilon|<n+1]}$, we have from equality (4.25),

$$\sum_{i=1}^N \int_{[n<|u_\epsilon|<n+1]} a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial u_\epsilon}{\partial x_i} dx \leq \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon + \int_{\partial\Omega} T_1(u_\epsilon - T_n(u_\epsilon))dg_\epsilon. \tag{4.26}$$

We have (see [28])

$$\lim_{n\to+\infty} \lim_{\epsilon\to 0} \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon = 0$$

and

$$\lim_{n\to+\infty} \lim_{\epsilon\to 0} \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))dg_\epsilon = 0.$$

Then, using (1.5), and letting $n \to +\infty$, $\epsilon \to 0$ respectively in (4.26), we get

$$\lim_{n\to+\infty} \lim_{\epsilon\to 0} \frac{1}{C} \sum_{i=1}^N \int_{[n<|u_\epsilon|<n+1]} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx = \lim_{n\to+\infty} \frac{1}{C} \sum_{i=1}^N \int_{[n<|u|<n+1]} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \leq 0. \tag{4.27}$$

The connexion between our notion of solution and the entropic formulation is given in the following Theorem. In particular, as the domains of β and γ are bounded, this equivalent formulation is very useful for the proof of uniqueness of solution to problem (1.1). We reason as authors in [27] to get the following results.

Theorem 4.14. *If (u, b, w) is a solution of (1.1) in the sense of Theorem 2.6, then (u, b, w) is a solution in the following sense : for any $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \in dom\beta$,*

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - \varphi)dx + \int_\Omega b T_k(u - \varphi)dx + \int_{\partial\Omega} w T_k(u - \varphi)d\sigma$$
$$\leq \int_\Omega T_k(u - \varphi)d\mu + \int_{\partial\Omega} T_k(u - \varphi)dg, \text{ for any } k > 0. \tag{4.28}$$

The result of the uniqueness of solution to problem (1.1) is the following.

Theorem 4.15. *Let* (u_1, b_1, w_1) *and* (u_2, b_2, w_2) *be two solutions of* (1.1)*. Then*

$$\begin{cases} u_1 = u_2 \ a.e. \ \ in \ \Omega, \\ \\ b_1 = b_2 \ a.e. \ \ in \ \Omega, \\ \\ w_1 = w_2 \ a.e. \ \ in \ \partial\Omega. \end{cases}$$

Proof. For u_1, we choose $\varphi = u_2$ as a test function in (4.28) to get

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_1}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_1 - u_2)dx + \int_\Omega b_1 T_k(u_1 - u_2)dx + \int_{\partial\Omega} w_1 T_k(u_1 - u_2)d\sigma$$

$$\leq \int_\Omega T_k(u_1 - u_2)d\mu + \int_{\partial\Omega} T_k(u_1 - u_2)dg, \text{ for any } k > 0. \tag{4.29}$$

Similarly for u_2, we choose $\varphi = u_1$ as a test function in (4.28) to get

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_2}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_2 - u_1)dx + \int_\Omega b_2 T_k(u_2 - u_1)dx + \int_{\partial\Omega} w_2 T_k(u_2 - u_1)dx$$

$$\leq \int_\Omega T_k(u_2 - u_1)d\mu + \int_{\partial\Omega} T_k(u_2 - u_1)dg, \text{ for any } k > 0. \tag{4.30}$$

Adding these two inequalities yields

$$\sum_{i=1}^{N} \int_\Omega \left(a_i(x, \frac{\partial u_1}{\partial x_i}) - a_i(x, \frac{\partial u_2}{\partial x_i})\right) \frac{\partial}{\partial x_i} T_k(u_1 - u_2)dx + \int_\Omega (b_1 - b_2) T_k(u_1 - u_2)dx$$

$$+ \int_{\partial\Omega} (w_1 - w_2) T_k(u_1 - u_2)d\sigma \leq 0. \tag{4.31}$$

Therefore, as in [18] the result follows. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] L. Alvarez, P.L. Lions, J.M. Morel, Image Selective Smoothing and Edge Detection by Nonlinear Diffusion. II, SIAM J. Numer. Anal. 29 (1992), 845–866. https://doi.org/10.1137/0729052.

[2] M. Bendahmane, M. Langlais, M. Saad, On Some Anisotropic Reaction-Diffusion Systems with L1-Data Modeling the Propagation of an Epidemic Disease, Nonlinear Anal.: Theory Methods Appl. 54 (2003), 617–636. https://doi.org/10.1016/s0362-546x(03)00090-7.

[3] L. Boccardo, T. Gallouët, Non-Linear Elliptic and Parabolic Equations Involving Measure Data, J. Funct. Anal. 87 (1989), 149–169. https://doi.org/10.1016/0022-1236(89)90005-0.

[4] I. Boccardo, T. Gallouet, Nonlinear Elliptic Equations with Right Hand Side Measures, Commun. Partial. Differ. Equ. 17 (1992), 189–258. https://doi.org/10.1080/03605309208820857.

[5] L. Boccardo, T. Gallouët, L. Orsina, Existence and Uniqueness of Entropy Solutions for Nonlinear Elliptic Equations with Measure Data, Ann. Inst. Henri Poincaré C, Anal. Non Linéaire 13 (1996), 539–551. https://doi.org/10.1016/s0294-1449(16)30113-5.

[6] B.K. Bonzi, S. Ouaro, F.D.Y. Zongo, Entropy Solutions for Nonlinear Elliptic Anisotropic Homogeneous Neumann Problem, Int. J. Differ. Equ. 2013 (2013), 476781. https://doi.org/10.1155/2013/476781.

[7] M.M. Boureanu, V.D. Rădulescu, Anisotropic Neumann Problems in Sobolev Spaces with Variable Exponent, Nonlinear Anal.: Theory Methods Appl. 75 (2012), 4471–4482. https://doi.org/10.1016/j.na.2011.09.033.

[8] H. Brezis. Analyse Fonctionnelle, Théorie et Applications, Masson, Paris, 1983.

[9] H. Brezis, Opérateurs Maximaux Monotones et Semigroupes de Contraction dans les Espaces de Hilbert, North Holland, Amsterdam, 1973.

[10] Y. Chen, S. Levine, M. Rao, Variable Exponent, Linear Growth Functionals in Image Restoration, SIAM J. Appl. Math. 66 (2006), 1383–1406. https://doi.org/10.1137/050624522.

[11] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized Solutions of Elliptic Equations with General Measure Data, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 28 (1999), 741–808. https://www.numdam.org/item/ASNSP_1999_4_28_4_741_0.

[12] L. Diening, Theoretical and Numerical Results for Electrorheological Fluids, PhD. Thesis, University of Frieburg, Germany, 2002.

[13] G. Dolzmann, N. Hungerbühler, S. Müller, Non-Linear Elliptic Systems with Measure-Valued Right Hand Side, Math. Z. 226 (1997), 545–574. https://doi.org/10.1007/pl00004354.

[14] F. Ettwein, M. Růžička, Existence of Strong Solutions for Electrorheological Fluids in Two Dimensions: Steady Dirichlet Problem, in: Geometric Analysis and Nonlinear Partial Differential Equations, Springer, Berlin, Heidelberg, 2003: pp. 591–602. https://doi.org/10.1007/978-3-642-55627-2_31.

[15] X. Fan, Anisotropic Variable Exponent Sobolev Spaces and $\overrightarrow{p(x)}$-Laplacian Equations, Complex Var. Elliptic Equ. 56 (2011), 623–642. https://doi.org/10.1080/17476931003728412.

[16] X. Fan, D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617.

[17] I. Ibrango, S. Ouaro, Entropy Solutions for Nonlinear Elliptic Anisotropic Problems With Homogeneous Neumann Boundary Condition, J. Appl. Anal. Comput. 6 (2016), 271–292. https://doi.org/10.11948/2016022.

[18] I. Ibrango, S. Ouaro, Entropy Solution for Doubly Nonlinear Elliptic Anisotropic Problems with Fourier Boundary Conditions, Discuss. Math. Differ. Inclusions, Control. Optim. 35 (2015), 123–124. https://doi.org/10.7151/dmdico.1175.

[19] N. Igbida, S. Ouaro, S. Soma, Elliptic Problem Involving Diffuse Measure Data, J. Differ. Equ. 253 (2012), 3159–3183. https://doi.org/10.1016/j.jde.2012.08.025.

[20] I. Konate, S. Ouaro, Good Radon Measure for Anisotropic Problems With Variable Exponent, Electron. J. Differ. Equ. 2016 (2016), 221.

[21] I. Konate, S. Ouaro, Good Measures for Nonlinear Neumann Anisotropic Problems With Variable Exponent, Int. J. Evol. Equ. 10 (2018), 233–251.

[22] I. Konate, S. Ouaro, Multivalued Anisotropic Problem with Fourier Boundary Condition Involving Diffuse Radon Measure Data and Variable Exponents, Glas. Mat. 54 (2019), 211–232. https://doi.org/10.3336/gm.54.1.09.

[23] I. Konate, S. Ouaro, Nonlinear Multivalued Problems with Variable Exponent and Diffuse Measure Data in Anisotropic Space, Gulf J. Math. 6 (2018), 13–30. https://doi.org/10.56947/gjom.v6i2.130.

[24] B. Kone, S. Ouaro, S. Soma, Weak Solutions for Anisotropic Nonlinear Elliptic Problem With Variable Exponent and Measure Data, Int. J. Evol. Equ 5 (2011), 327–350.

[25] O. Kovacik, J. Rakosnik, On Spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czech. Math. J. 41 (1991), 592–618. http://eudml.org/doc/13956.

[26] M. Mihăilescu, P. Pucci, V. Rădulescu, Eigenvalue Problems for Anisotropic Quasilinear Elliptic Equations with Variable Exponent, J. Math. Anal. Appl. 340 (2008), 687–698. https://doi.org/10.1016/j.jmaa.2007.09.015.

[27] I. Nyanquini, S. Ouaro, S. Soma, Entropy Solution to Nonlinear Multivalued Elliptic Problem With Variable Exponents and Measure Data, Ann. Univ. Craiova Math. Comput. Sci. Ser. 40 (2013), 174–198.

[28] S. Ouaro, A. Ouedraogo, S. Soma, Multivalued Homogeneous Neumann Problem Involving Diffuse Measure Data and Variable Exponent, Nonlinear Dyn. Syst. Theory 16 (2016), 102–114.

[29] C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen, B.P. Dolgin, Electrorheological-Fluid-Based Force Feedback Device, in: Proceedings Volume 3840, Telemanipulator and Telepresence Technologies VI, 1999, pp. 88–99. https://doi.org/10.1117/12.369269.

[30] K. Rajagopal, M. Ružička, Mathematical Modeling of Electrorheological Materials, Contin. Mech. Thermodyn. 13 (2001), 59–78. https://doi.org/10.1007/S001610100034.

[31] M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory, Springer, 2000. https://doi.org/10.1007/bfb0104029.

[32] M. Troisi, Teoremi di Inclusione per Spazi di Sobolev Non Isotropi, Ricerche Mat. 18 (1969), 3–24.