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GENERALIZATION OF SOME VOLTERRA-FREDHOLM TYPE DELAYED INTEGRAL INEQUALITIES WITH POWER AND THEIR APPLICATIONS

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Abstract. The goal of this study is to investigate some new upper bounds on solutions to specific nonlinear retarded integral inequalities of the Volterra-Fredholm type with power. Our findings offer a fresh and fascinating Volterra-Fredholm type with power, which is a potent generalization of the well-known Gronwall-Bellman type integral inequalities from the literature. The boundedness of the solutions to the retarded Volterra-Fredholm type integral equation with power is studied using an application.

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1. Introduction

T.H. Gronwall [7] demonstrated an amazing inequality in 1919 that has garnered and is still receiving a lot of attention in the literature. New Gronwall-type integral inequalities have been established in recent years by a number of authors (see [1], [5], [9], [17]). Many authors have recently expressed interest in extending the Gronwall-Bellman inequality to other forms, such as nonlinear retarded integral inequalities with power (see [10], [13], [14], [15], [18]) and integral inequalities of Volterra-Fredholm type (see [2], [3], [6], [11]).

The following useful linear Volterra-Fredholm type integral inequality with retardation was established by Pachpatte [16] in 2004.

$$r(t) \leq k + \int_{h(\alpha)}^{h(t)} a(t,s)\left[f(s)r(s) + \int_{h(\alpha)}^{s} c(s,\sigma)r(\sigma)d\sigma\right]ds + \int_{h(\alpha)}^{h(\beta)} b(t,s)r(s)ds. \tag{1}$$

Ma and Pečarić [12] examined the nonlinear retarded Volterra-Fredholm integral inequality presented below in 2008.

$$r(t) \quad \leq \quad k + \int_{\lambda(t_0)}^{\lambda(t)} \sigma_1(s)\left[f(s)\omega\left(r(s)\right) + \int_{\lambda(t_0)}^{s} \sigma_2(\tau)\omega\left(r(\tau)\right)d\tau\right]ds$$

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$$+\int_{\lambda(t_0)}^{\lambda(T)} \sigma_1(s)\left[f(s)\omega\left(r(s)\right)+\int_{\lambda(t_0)}^{s}\sigma_2(\tau)\omega\left(r(\tau)\right)d\tau\right]ds \tag{2}$$

The following nonlinear Volterra-Fredholm integral inequality was established by Kender et al. [8] in 2014.

$$r^p(t) \leq c+\int_\alpha^t a(t,s)\left[r(s)+\int_\alpha^s f(s,\sigma)r(\sigma)d\sigma\right]ds+\int_\alpha^\beta g(t,s)r^p(s)ds. \tag{3}$$

The following delayed integral inequality with power was developed by Boudeliou [4] in 2024.

$$r(t) \leq a(t)+\int_0^{\alpha(t)} f(s)r(s)ds+\int_0^{\alpha(t)} g(s)\left[r^m(s)+\int_0^s h(\tau)r^n(\tau)d\tau\right]^p ds. \tag{4}$$

As previously stated, some academics extend Gronwall's inequality to the Volterra-Fredholm type, while others extend it to integral inequalities with power. To create new delayed integral inequalities of Volterra-Fredholm type with powers, we propose to unify the two forms previously discussed. To estimate the unknown function, we used several analytical methods. Our results generalize some findings obtained in [8], [10], [12], and [16]. The results obtained can be used to study the boundedness of certain power-delay Volterra-Fredholm integral equations. An example application is provided.

2. Main results

Throughout this paper, we use the following notations: $I=[x_0,T]$, and
$\Delta=\left\{(x,s)\in I^2 : x_0\leq s\leq x\leq T\right\}$.

The following lemma is very useful in the proof procedures of our main results.

Lemma 2.1 ([6]). *Assume that $a\geq 0$, $p\geq q\geq 0$, and $p\neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}a+\frac{p-q}{p}.$$

Theorem 2.2. *Let $r(x), a(x), b(x) \in C\left(I,\mathbb{R}_+\right)$, $f(x,s), c(x,s)\in C\left(\Delta,\mathbb{R}_+\right)$, and $f(x,s), c(x,s)$ be nondecreasing in x for each $s\in I$. $\lambda(x)\in C^1\left(I,I\right)$ be nondecreasing function on I, with $\lambda(x)\leq x$. Let $0\leq q\leq p$, $0\leq m\leq p$, $0\leq n\leq p$, $p\neq 0$, and $0<\theta\leq 1$. If $r(x)$ satisfies*

$$\begin{aligned} r^p(x) \quad &\leq \quad a(x)+\int_{\lambda(x_0)}^{\lambda(x)} f\left(x,s\right)\left[b(s)r^q(s)+\int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds \\ &\quad +\int_{\lambda(x_0)}^{\lambda(T)} c(x,s)r^n(\tau)ds, \end{aligned} \tag{5}$$

for $x\in I$, where p,q,m and n are constants and

$$K(T)=\frac{n}{p}\exp\left(\theta\int_{\lambda(x_0)}^{\lambda(T)} f\left(X,s\right)\left[\frac{q}{p}b(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right]ds\right)\int_{\lambda(x_0)}^{\lambda(T)} c(X,s)ds<1, \tag{6}$$

then

$$r(x) \leq \left(a(x) + \frac{A(x)}{1-K(T)} \exp\left(\theta \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left[\frac{q}{p} b(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right] ds \right) \right)^{\frac{1}{p}}, \tag{7}$$

where

$$\begin{aligned} A(x) &= \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left(\theta \left[b(s) \left(\frac{q}{p} a(s) + \frac{p-q}{p} \right) + \int_{\lambda(x_0)}^{s} c(s,\tau) \left(\frac{m}{p} a(\tau) + \frac{p-m}{p} \right) d\tau \right] \right. \\ &\quad \left. + 1 - \theta \right) ds + \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) \left[\frac{n}{p} a(s) + \frac{p-n}{p} \right] ds. \end{aligned} \tag{8}$$

Proof. Define a positive and nondecreasing function $z(x)$ on I by

$$\begin{aligned} z(x) &= \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left[b(s) r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau) r^m(\tau) d\tau \right]^\theta ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) r^n(s) ds, \end{aligned}$$

for $x \in I$, then we have

$$r(x) \leq (a(x) + z(x))^{\frac{1}{p}}. \tag{9}$$

Using the inequality (9) and Lemma 2.1, we get

$$\begin{aligned} z(x) &\leq \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left[b(s) (a(s) + z(s))^{\frac{q}{p}} + \int_{\lambda(x_0)}^{s} c(s,\tau) (a(\tau) + z(\tau))^{\frac{m}{p}} d\tau \right]^\theta ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) (a(s) + z(s))^{\frac{n}{p}} ds \\ &\leq \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left(\theta \left[b(s) (a(s) + z(s))^{\frac{q}{p}} + \int_{\lambda(x_0)}^{s} c(s,\tau) (a(\tau) + z(\tau))^{\frac{m}{p}} d\tau \right] + 1 - \theta \right) ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) (a(s) + z(s))^{\frac{n}{p}} ds \\ &\leq \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left(\theta \left[b(s) \left[\frac{q}{p} (a(s) + z(s)) + \frac{p-q}{p} \right] \right. \right. \\ &\quad \left. \left. + \int_{\lambda(x_0)}^{s} c(s,\tau) \left[\frac{m}{p} (a(\tau) + z(\tau)) + \frac{p-m}{p} \right] d\tau \right] + 1 - \theta \right) ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) \left[\frac{n}{p} (a(s) + z(s)) + \frac{p-n}{p} \right] ds, \\ &\leq A(x) + \theta \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left[\frac{q}{p} b(s) z(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau) z(\tau) d\tau \right] ds + \frac{n}{p} \int_{\lambda(x_0)}^{\lambda(T)} c(x,s) z(s) ds. \end{aligned}$$

Since $A(x)$ is a positive and nondecreasing function, define a positive and nondecreasing function $v(x)$ for $x_0 \leq x \leq X \leq T$ by

$$v(x) = A(X) + \theta \int_{\lambda(x_0)}^{\lambda(x)} f(X,s) \left[\frac{q}{p} b(s)z(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau)z(\tau)d\tau \right] ds + \frac{n}{p} \int_{\lambda(x_0)}^{\lambda(T)} c(X,s)z(s)ds,$$

$$v(x_0) = A(X) + \frac{n}{p} \int_{\lambda(x_0)}^{\lambda(T)} c(X,s)z(s)ds, \tag{10}$$

then

$$z(x) \leq v(x), \tag{11}$$

and

$$\begin{aligned} v'(x) &= \theta f(X,\lambda(x)) \left[\frac{q}{p} b(\lambda(x))z(\lambda(x)) + \frac{m}{p} \int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)z(\tau)d\tau \right] \lambda'(x) \\ &\leq \theta f(X,\lambda(x))\, v(\lambda(x)) \left[\frac{q}{p} b(\lambda(x)) + \frac{m}{p} \int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)d\tau \right] \lambda'(x), \end{aligned}$$

or

$$\frac{v'(x)}{v(x)} \leq \theta f(X,\lambda(x)) \left[\frac{q}{p} b(\lambda(x)) + \frac{m}{p} \int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)d\tau \right] \lambda'(x).$$

By setting $x = s$ in in the last inequality and integrating it with respect to s from x_0 to x, making the change of variable $s = \lambda(x)$, we get

$$v(x) \leq v(x_0) \exp\left(\theta \int_{\lambda(x_0)}^{\lambda(x)} f(X,s) \left[\frac{q}{p} b(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau)d\tau \right] ds \right),$$

From (11) we have

$$z(x) \leq v(x_0) \exp\left(\theta \int_{\lambda(x_0)}^{\lambda(x)} f(X,s) \left[\frac{q}{p} b(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau)d\tau \right] ds \right). \tag{12}$$

Substituting (12) into (10) and combing with condition (6), we get

$$\begin{aligned} v(x_0) &\leq A(X) + \frac{n}{p} z(\lambda(T)) \int_{\lambda(x_0)}^{\lambda(T)} c(X,s)ds \\ &\leq A(X) + \frac{n}{p} z(T) \int_{\lambda(x_0)}^{\lambda(T)} c(X,s)ds \\ &\leq A(X) + v(x_0)K(T). \end{aligned}$$

Then

$$v(x_0) \leq \frac{A(X)}{1-K(T)}$$

Substituting the last inequality into (12), we get

$$z(x) \leq \frac{A(X)}{1-K(T)} \exp\left(\theta \int_{\lambda(x_0)}^{\lambda(x)} f(X,s) \left[\frac{q}{p} b(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau)d\tau \right] ds \right).$$

Since X is chosen arbitrarily, then

$$z(x) \leq \frac{A(x)}{1-K(T)} \exp\left(\theta \int_{\lambda(x_0)}^{\lambda(x)} f(x,s) \left[\frac{q}{p} b(s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau)d\tau \right] ds \right). \tag{13}$$

Combining (13) with (9) yields (7). □

Remark 2.3. *For $p = q = m = n = \theta = 1$, $a(x) = k$, then inequality (5) reduces to inequality (1).*

Remark 2.4. *For $q = \theta = 1, n = p, \lambda(x) = x, b(x) = 1$, and $c(s, \tau) = 0$, then Theorem 2.2 reduces to Theorem 2.4 in [8]. Morever, if $q = \theta = m = 1, n = p, \lambda(x) = x, b(x) = 1, a(x)$ is a constant, then Theorem 2.2 reduces to Theorem 2.5 in [8].*

Theorem 2.5. *Let $r, \lambda, f, b, c, p, q, m, \theta$ be as in Theorem 2.2, where $0 \le q \le m \le p$ and $r_0 > 0$ is a constant. If $r(x)$ satisfies*

$$\begin{aligned} r^p(x) &\le r_0 + \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[b(s)r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[b(s)r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds, \end{aligned} \tag{14}$$

for $x \in I$, and

$$\begin{aligned} H(r) &= (2r - r_0)^{\frac{p-m}{p}} - \frac{p-m}{p-q} r^{\frac{p-\theta q}{p}} - \left(\frac{p-m}{p-q}\right)\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left[\theta b(x,s) + 1 - \theta\right]ds \\ &\quad -\theta\left(\frac{p-m}{p-q}\right)\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left(\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right)ds - \frac{m-q}{p-q}, \end{aligned} \tag{15}$$

is increasing, and $H(r) = 0$ has a solution δ for $r \ge r_0$, then

$$\begin{aligned} r(x) &\le \Bigg[\frac{p-m}{p-q}\delta^{\frac{p-\theta q}{p}} + \frac{m-q}{p-q} + \frac{p-m}{p}\int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[\theta b(s) + 1 - \theta\right]ds \\ &\quad +\theta\left(\frac{p-m}{p-q}\right)\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left(\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right)ds\Bigg]^{\frac{1}{p-m}}. \end{aligned} \tag{16}$$

Proof. Fix any arbitrary $X \in I$, then for $x_0 \le x \le X \le T$, define a positive and nondecreasing function $z(x)$ on I by

$$\begin{aligned} z(x) &= r_0 + \int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left[b(s)r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[b(s)r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds, \end{aligned}$$

for $x \in I$, then we have

$$r(x) \le z^{\frac{1}{p}}(x), \tag{17}$$

and

$$z(x_0) = r_0 + \int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[b(s)r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^\theta ds.$$

Using Lemma 2.1 and the inequality (17), we get

$$\begin{aligned} z'(x) &\leq f\left(X,\lambda(x)\right)\left[b(\lambda(x))z^{\frac{q}{p}}(\lambda(x))+\int_{\lambda(x_0)}^{\lambda(x)}c(\lambda(x),\tau)z^{\frac{m}{p}}(\tau)d\tau\right]^{\theta}\lambda'(x)\\ &\leq z^{\frac{\theta q}{p}}(\lambda(x))f\left(X,\lambda(x)\right)\left[b(\lambda(x))+z^{\frac{m-q}{p}}(\lambda(x))\int_{\lambda(x_0)}^{\lambda(x)}c(\lambda(x),\tau)d\tau\right]^{\theta}\lambda'(x), \end{aligned}$$

or

$$\frac{z'(x)}{z^{\frac{\theta q}{p}}(x)}\leq f\left(X,\lambda(x)\right)\left[b(\lambda(x))+z^{\frac{m-q}{p}}(\lambda(x))\int_{\lambda(x_0)}^{\lambda(x)}c(\lambda(x),\tau)d\tau\right]^{\theta}\lambda'(x), \tag{18}$$

Setting $x=s$ in (18), and then integrating with respect to s from x_0 to x, making the change of variable $s=\lambda(x)$, we get

$$z^{\frac{p-\theta q}{p}}(x)\leq z^{\frac{p-\theta q}{p}}(x_0)+\frac{p-\theta q}{p}\int_{\lambda(x_0)}^{\lambda(x)}f\left(X,s\right)\left[b\left(s\right)+z^{\frac{m-q}{p}}(s)\int_{\lambda(x_0)}^{s}c(s,\tau)d\tau\right]^{\theta}ds.$$

Using Lemma 2.1, the last inequality can be reformulated as follows

$$\begin{aligned} z^{\frac{p-\theta q}{p}}(x) &\leq z^{\frac{p-\theta q}{p}}(x_0)+\frac{p-\theta q}{p}\int_{\lambda(x_0)}^{\lambda(x)}f\left(X,s\right)\left[\theta b\left(s\right)+\theta z^{\frac{m-q}{p}}(s)\int_{\lambda(x_0)}^{s}c(s,\tau)d\tau+1-\theta\right]ds\\ &\leq z^{\frac{p-\theta q}{p}}(x_0)+\frac{p-\theta q}{p}\int_{\lambda(x_0)}^{\lambda(x)}f\left(X,s\right)\left[\theta b\left(s\right)+1-\theta\right]ds\\ &\quad+\theta\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)}z^{\frac{m-q}{p}}(s)f\left(X,s\right)\left(\int_{\lambda(x_0)}^{s}c(s,\tau)d\tau\right)ds\\ &\leq A(x)+\theta\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)}z^{\frac{m-q}{p}}(s)f\left(X,s\right)\left(\int_{\lambda(x_0)}^{s}c(s,\tau)d\tau\right)ds, \end{aligned}$$

where

$$A(x)=z^{\frac{p-\theta q}{p}}(x_0)+\frac{p-\theta q}{p}\int_{\lambda(x_0)}^{\lambda(x)}f\left(X,s\right)\left[\theta b\left(s\right)+1-\theta\right]ds. \tag{19}$$

Since $A(x)$ is a positive and nondecreasing function, define a positive and nondecreasing function $v(x)$ for $x_0\leq x\leq X\leq T$ by

$$v(x)=A(X)+\theta\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)}z^{\frac{m-q}{p}}(s)f\left(X,s\right)\left(\int_{\lambda(x_0)}^{s}c(s,\tau)d\tau\right)ds,$$

and $v(x_0)=A(X)$. For $0<\theta\leq 1$, $q>0$, we have

$$z^{\frac{p-q}{p}}(x)\leq z^{\frac{p-\theta q}{p}}(x)\leq v(x),$$

so

$$z(x)\leq v^{\frac{p}{p-q}}(x). \tag{20}$$

Differentiating $v(x)$, we obtain

$$v'(x)\leq\theta\left(\frac{p-\theta q}{p}\right)v^{\frac{m-q}{p-q}}(\lambda(x))f\left(X,\lambda(x)\right)\left(\int_{\lambda(x_0)}^{\lambda(x)}c\left(\lambda(x),\tau\right)d\tau\right)\lambda'(x),$$

or

$$\frac{v'(x)}{v^{\frac{m-q}{p-q}}(x)} \leq \theta \left(\frac{p-\theta q}{p} \right) f\left(X, \lambda(x)\right) \left(\int_{\lambda(x_0)}^{\lambda(x)} c\left(\lambda(x), \tau\right) d\tau \right) \lambda'(x).$$

Integrating the last inequality, we get

$$v^{\frac{p-m}{p-q}}(x) \leq v^{\frac{p-m}{p-q}}(x_0) + \theta \left(\frac{p-m}{p-q} \right) \left(\frac{p-\theta q}{p} \right) \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left(\int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right) ds. \tag{21}$$

From (20) and (21), we have

$$z(x) \leq \left\{ A^{\frac{p-m}{p-q}}(X) + \theta \left(\frac{p-m}{p-q} \right) \left(\frac{p-\theta q}{p} \right) \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left(\int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right) ds \right\}^{\frac{p}{p-m}}. \tag{22}$$

Now we give an estimation of $z(x_0)$, which is given according to the unknown function $r(x)$. We have

$$2z(x_0) - r_0 = z(T) \leq \left\{ A^{\frac{p-m}{p-q}}(X) + \theta \left(\frac{p-m}{p-q} \right) \left(\frac{p-\theta q}{p} \right) \int_{\lambda(x_0)}^{\lambda(T)} f\left(X, s\right) \left(\int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right) ds \right\}^{\frac{p}{p-m}},$$

or

$$\left(2z(x_0) - r_0\right)^{\frac{p-m}{p}} \leq A^{\frac{p-m}{p-q}}(X) + \theta \left(\frac{p-m}{p-q} \right) \left(\frac{p-\theta q}{p} \right) \int_{\lambda(x_0)}^{\lambda(T)} f\left(X, s\right) \left(\int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right) ds.$$

Taking the fact that $\frac{p-m}{p-q} \in (0,1]$, then using (19) and Lemma 2.1, we have

$$A^{\frac{p-m}{p-q}}(x) \leq \frac{p-m}{p-q} \left(z^{\frac{p-\theta q}{p}}(x_0) + \frac{p-\theta q}{p} \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left[\theta b\left(s\right) + 1 - \theta \right] ds \right) + \frac{m-q}{p-q}, \tag{23}$$

and therefore

$$\begin{aligned} \left(2z(x_0) - r_0\right)^{\frac{p-m}{p}} \quad &\leq \quad \frac{p-m}{p-q} \left(z^{\frac{p-\theta q}{p}}(x_0) + \frac{p-\theta q}{p} \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left[\theta b(s) + 1 - \theta \right] ds \right) \\ &\quad + \frac{m-q}{p-q} + \theta \left(\frac{p-\theta q}{p} \right) \left(\frac{p-m}{p-q} \right) \int_{\lambda(x_0)}^{\lambda(T)} f\left(X, s\right) \left(\int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right) ds, \end{aligned}$$

using (15), then the last inequality can be restated as follows

$$H\left(z(x_0)\right) \leq 0 = H\left(\delta\right),$$

Since H is increasing, and therefore from the last inequality, we obtain

$$z(x_0) \leq \delta.$$

Substituting the last inequality into (23), we get

$$A^{\frac{p-m}{p-q}}(x) \leq \frac{p-m}{p-q} \left(\delta^{\frac{p-\theta q}{p}} + \frac{p-\theta q}{p} \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left[\theta b(s) + 1 - \theta \right] ds \right) + \frac{m-q}{p-q},$$

then (22), will be written as follows

$$z(x) \quad \leq \quad \left\{ \frac{p-m}{p-q} \left(\delta^{\frac{p-\theta q}{p}} + \frac{p-q}{p} \int_{\lambda(x_0)}^{\lambda(x)} f\left(X, s\right) \left[\theta b(s) + 1 - \theta \right] ds \right) + \frac{m-q}{p-q} \right.$$

$$+\theta\left(\frac{p-m}{p-q}\right)\left(\frac{p-\theta q}{p}\right)\int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left(\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right)ds\Bigg\}^{\frac{p}{p-m}}. \tag{24}$$

Since X is chosen arbitrarily, then combining (24) with (17) yields (16). □

Remark 2.6. *For $p=\theta=1, f(x,s)=\sigma(s), q=m$, then Theorem 2.5 reduces to Corollary 2.3 in [13]. Furthermore, if $\lambda(x)=x, q=\theta=m=1$ in the first integral, and for $c(s,\tau)=0, \lambda(x)=x$ in the second integral, then Theorem 2.5 reduces to Theorem 2.5 in [8].*

Theorem 2.7. *Let $a(x), h(x)\in C(I,\mathbb{R}_+), g(x,s)\in C(\Delta,\mathbb{R}_+)$, where g is nondecreasing in x for each $s\in I$, and $\lambda, r, f, b, c, \theta$, are as in Theorem 2.2. Let $0\le q\le p, 0\le m\le p, 0\le i\le p, 0\le j\le p$, and $p\neq 0$ be constants. If $r(x)$ satisfies*

$$\begin{aligned} r^p(x) \quad \le \quad & a(x)+\int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[b(s)r^q(s)+\int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^{\theta} ds \\ & +\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[h(s)r^i(s)+\int_{\lambda(x_0)}^{s} g(s,\tau)r^j(\tau)d\tau\right]^{\theta} ds, \end{aligned} \tag{25}$$

for $x\in I$, and

$$\Phi(x)=\theta\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[\frac{i}{p}h(s)\exp(k(s))+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)\exp(k(\tau))\,d\tau\right]ds<1, \tag{26}$$

then

$$r(x)\le\left[a(x)+\frac{A(x)}{1-\Phi(x)}\exp(k(x))\right]^{\frac{1}{p}}, \tag{27}$$

for $x\in I$, where

$$\begin{aligned} A(x) \quad = \quad & \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\Bigg\{\theta\Bigg[b(s)\left(\frac{q}{p}a(s)+\frac{p-q}{p}\right) \\ & +\int_{\lambda(x_0)}^{s} c(s,\tau)\left(\frac{m}{p}a(\tau)+\frac{p-m}{p}\right)d\tau\Bigg]+1-\theta\Bigg\}ds \\ & +\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\Bigg\{\theta\Bigg[h(s)\left(\frac{i}{p}a(s)+\frac{p-i}{p}\right) \\ & +\int_{\lambda(x_0)}^{s} g(s,\tau)\left(\frac{j}{p}a(\tau)+\frac{p-j}{p}\right)d\tau\Bigg]+1-\theta\Bigg\}ds, \end{aligned} \tag{28}$$

$$k(x)=\theta\int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[\frac{q}{p}b(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right]ds. \tag{29}$$

Proof. Define a positive and nondecreasing function $z(x)$ on I by

$$z(x) \quad = \quad \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[b(s)r^q(s)+\int_{\lambda(x_0)}^{s} c(s,\tau)r^m(\tau)d\tau\right]^{\theta} ds$$

$$+\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[h(s)r^i(s)+\int_{\lambda(x_0)}^{s} g(s,\tau)r^j(\tau)d\tau\right]^\theta ds,$$

for $x \in I$, then we have

$$r(x) \leq [a(x)+z(x)]^{\frac{1}{p}}. \tag{30}$$

Using Lemma 2.1 and inequality (30), we get

$$\begin{aligned}
z(x) \leq\;& \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[b(s)\left[a(s)+z(s)\right]^{\frac{q}{p}}+\int_{\lambda(x_0)}^{s} c(s,\tau)\left[a(\tau)+z(\tau)\right]^{\frac{m}{p}} d\tau\right]^\theta ds \\
&+\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[h(s)\left[a(s)+z(s)\right]^{\frac{i}{p}}+\int_{\lambda(x_0)}^{s} g(s,\tau)\left[a(\tau)+z(\tau)\right]^{\frac{j}{p}} d\tau\right]^\theta ds \\
\leq\;& \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[b(s)\left(\frac{q}{p}\left[a(s)+z(s)\right]+\frac{p-q}{p}\right)\right. \\
&\left.+\int_{\lambda(x_0)}^{s} c(s,\tau)\left(\frac{m}{p}\left[a(\tau)+z(\tau)\right]+\frac{p-m}{p}\right)d\tau\right]^\theta ds \\
&+\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[h(s)\left(\frac{i}{p}\left[a(s)+z(s)\right]+\frac{p-i}{p}\right)\right. \\
&\left.+\int_{\lambda(x_0)}^{s} g(s,\tau)\left(\frac{j}{p}\left[a(\tau)+z(\tau)\right]+\frac{p-j}{p}\right)d\tau\right]^\theta ds, \\
\leq\;& \int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left(\theta\left[b(s)\left(\frac{q}{p}\left[a(s)+z(s)\right]+\frac{p-q}{p}\right)\right.\right. \\
&\left.\left.+\int_{\lambda(x_0)}^{s} c(s,\tau)\left(\frac{m}{p}\left[a(\tau)+z(\tau)\right]+\frac{p-m}{p}\right)d\tau\right]+1-\theta\right) ds \\
&+\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left(\theta\left[h(s)\left(\frac{i}{p}\left[a(s)+z(s)\right]+\frac{p-i}{p}\right)\right.\right. \\
&\left.\left.+\int_{\lambda(x_0)}^{s} g(s,\tau)\left(\frac{j}{p}\left[a(\tau)+z(\tau)\right]+\frac{p-j}{p}\right)d\tau\right]+1-\theta\right) ds,
\end{aligned}$$

using (28) in the right side of the last inequality, we obtain

$$\begin{aligned}
z(x) \leq\;& A(x)+\theta\int_{\lambda(x_0)}^{\lambda(x)} f(x,s)\left[\frac{q}{p}b(s)z(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)z(\tau)d\tau\right] ds \\
&+\theta\int_{\lambda(x_0)}^{\lambda(T)} f(x,s)\left[\frac{i}{p}h(s)z(s)+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)z(\tau)d\tau\right] ds.
\end{aligned}$$

Since $A(x)$ is a nonnegative and nondecreasing function for each $x \in I$, let us fix any arbitrary $X \in I$, then for $x_0 \leq x \leq X \leq T$, we get

$$z(x) \leq A(X)+\theta\int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left[\frac{q}{p}b(s)z(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)z(\tau)d\tau\right] ds$$

$$+\theta\int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[\frac{i}{p}h(s)z(s)+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)z(\tau)d\tau\right]ds. \tag{31}$$

Now define a nonnegative and nondecreasing function $v(x)$ by the right-hand side of (31), then we have

$$v(x)=B(X)+\theta\int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left[\frac{q}{p}b(s)z(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)z(\tau)d\tau\right]ds,$$

where

$$B(X)=A(X)+\theta\int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[\frac{i}{p}h(s)z(s)+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)z(\tau)d\tau\right]ds, \tag{32}$$

then

$$z(x)\le v(x), \tag{33}$$

and $v(x_0)=B(X)$, so we have

$$\begin{aligned} v'(x) &\le \theta f(X,\lambda(x))\left[\frac{q}{p}b(\lambda(x))v(\lambda(x))+\frac{m}{p}\int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)v(\tau)d\tau\right]\lambda'(x)\\ &\le \theta f(X,\lambda(x))v\left(\lambda(x)\right)\left[\frac{q}{p}b(\lambda(x))+\frac{m}{p}\int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)d\tau\right]\lambda'(x), \end{aligned}$$

or

$$\frac{v'(x)}{v(x)}\le\theta f(X,\lambda(x))\left[\frac{q}{p}b(\lambda(x))+\frac{m}{p}\int_{\lambda(x_0)}^{\lambda(x)} c(\lambda(x),\tau)d\tau\right]\lambda'(x),$$

Setting $x=s$ in the last inequality, and then integrating it with respect to s from x_0 to x, making the change of variable $s=\lambda(x)$, we obtain

$$v(x)\le B(X)\exp\left(\theta\int_{\lambda(x_0)}^{\lambda(x)} f(X,s)\left[\frac{q}{p}b(s)+\frac{m}{p}\int_{\lambda(x_0)}^{s} c(s,\tau)d\tau\right]ds\right) \tag{34}$$

since X is arbitrary and using (29), we obtain

$$v(x)\le B(X)\exp\left(k(x)\right) \tag{35}$$

From (33) and (35), we have

$$\begin{aligned} B(X) &\le A(X)+\theta\int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[\frac{i}{p}h(s)v(s)+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)v(\tau)d\tau\right]ds\\ &\le A(X)+\theta B(X)\int_{\lambda(x_0)}^{\lambda(T)} f(X,s)\left[\frac{i}{p}h(s)\exp\left(k(s)\right)\right.\\ &\quad\left.+\frac{j}{p}\int_{\lambda(x_0)}^{s} g(s,\tau)\exp\left(k(\tau)\right)d\tau\right]ds. \end{aligned}$$

Therefore, from (26), we get

$$B(X)\le A(X)+B(X)\Phi(X),$$

or

$$B(X)\le\frac{A(X)}{1-\Phi(X)},$$

since X is chosen arbitrarily, so we have

$$v(x) \leq B(x) \exp\left(k(x)\right) \leq \frac{A(x)}{1-\Phi(x)} \exp\left(k(x)\right). \tag{36}$$

Now from (30), (33), and (36), we obtain the desired inequality (27). □

Remark 2.8. *For $\theta = 1, g(s,\tau) = 0$, then Theorem 2.7 reduces to Theorem 2.5 in [12].*

Remark 2.9. *For $q = \theta = 1$, $c(x,s) = g(x,s) = 0$, $f(x,s) = f(x)$, $f(x,s) = f(s)$, $\lambda(x) = x$, and $i = p$, then Theorem 2.7 reduces to Theorem 2.1 in [8]. Moreover, for $q = \theta = 1$, $c(x,s) = g(x,s) = 0$, $\lambda(x) = x$, and $i = p$, then Theorem 2.7 reduces to Theorem 2.4 in [8].*

We can obtain the following theorem using almost the same procedure in the proof of Theorem 2.7.

Theorem 2.10. *Assume that $r, a, b, c, f, g, \lambda, p, q, m, i$, and θ are defined as in Theorem 2.7, and $l \in C(\Delta, \mathbb{R}_+)$, where f, g, b, c, and l are nondecreasing in x for each $s \in I$. If $r(x)$ satisfies*

$$\begin{aligned} r^p(x) \quad &\leq \quad a(x) + \int_{\lambda(x_0)}^{\lambda(x)} l(x,s) r^i(s) ds + \int_{\lambda(x_0)}^{\lambda(x)} \left[b(x,s) r^q(s) + \int_{\lambda(x_0)}^{s} c(s,\tau) r^m(\tau) d\tau \right]^\theta ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} \left[f(x,s) u^q(s) + \int_{\lambda(x_0)}^{s} g(s,\tau) r^m(\tau) d\tau \right] ds, \end{aligned} \tag{37}$$

for $x \in I$, and for

$$\tilde{\Phi}(x) = \int_{\lambda(x_0)}^{\lambda(T)} \left[\frac{q}{p} f(x,s) \exp\left(\tilde{h}(s)\right) + \frac{m}{p} \int_{\lambda(x_0)}^{s} g(s,\tau) \exp\left(\tilde{h}(\tau)\right) d\tau \right] ds < 1, \tag{38}$$

then

$$r(x) \leq \left[a(x) + \frac{\tilde{A}(x)}{1-\tilde{\Phi}(x)} \exp\left(\tilde{h}(x)\right) \right]^{\frac{1}{p}}, \tag{39}$$

for $x \in I$, where

$$\begin{aligned} \tilde{A}(x) \quad &= \quad \int_{\lambda(x_0)}^{\lambda(x)} l(x,s) \left(\frac{i}{p} a(s) + \frac{p-i}{p} \right) ds + \int_{\lambda(x_0)}^{\lambda(x)} \left\{ \theta \left[b(x,s) \left(\frac{q}{p} a(s) + \frac{p-q}{p} \right) \right.\right. \\ &\quad \left.\left. + \int_{\lambda(x_0)}^{s} c(s,\tau) \left(\frac{m}{p} a(\tau) + \frac{p-m}{p} \right) d\tau \right] + 1 - \theta \right\} ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} \left[f(x,s) \left(\frac{q}{p} a(s) + \frac{p-q}{p} \right) + \int_{\lambda(x_0)}^{s} g(s,\tau) \left(\frac{m}{p} a(\tau) + \frac{p-m}{p} \right) d\tau \right] ds \end{aligned} \tag{40}$$

$$\tilde{h}(x) = \frac{i}{p} \int_{\lambda(x_0)}^{\lambda(x)} l(x,s) ds + \theta \int_{\lambda(x_0)}^{\lambda(x)} \left[\frac{q}{p} b(x,s) + \frac{m}{p} \int_{\lambda(x_0)}^{s} c(s,\tau) d\tau \right] ds. \tag{41}$$

Remark 2.11. *For $p = i = q = m = \theta = 1$, $\lambda(x) = x$, $a(x) = a_0$, and $f = g = 0$, then Theorem 2.10 reduces to Theorem 1.7.2 iii) in [16]. Moreover, for $p = i = 1$, $q, m \in (0,1]$, and $f = g = 0$, then inequality (37) reduces to inequality (4).*

3. Application

In this section, we will illustrate how our main results can be applied to the study of the boundedness of solutions to certain nonlinear delayed Volterra-Fredholm integral equations with power of the form

$$\begin{aligned} y^{p}(x) &= a(x)+\int_{x_0}^{x} H_1\left(s, y\left(\lambda(s)\right), \int_{x_0}^{\lambda(s)} M_1\left(\tau, y(\tau)\right) d\tau\right) ds \\ &\quad + \int_{x_0}^{T} H_2\left(s, y\left(\lambda(s)\right), \int_{x_0}^{\lambda(s)} M_2\left(\tau, y(\tau)\right) d\tau\right) ds, \end{aligned} \tag{42}$$

where $p > 0$ is a constant, $y, a \in C\left(I, \mathbb{R}\right), H_i \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), M_i \in C(I \times \mathbb{R}, \mathbb{R}), i = 1, 2$, and $\lambda(x) \in C^1\left(I, I\right)$, where $\lambda(x)$ be nondecreasing function on I with $\lambda(x) \leq x$, $I = [x_0, T] \subset \mathbb{R}$.

Theorem 3.1. *Assume that the functions $H_i, M_i, i = 1, 2$ in (42) satisfy the conditions*

$$\left|H_1\left(x, y, z\right)\right| \leq f(x)\Big(b(x)\left|y\right|^{q} + |z|\Big)^{\frac{1}{2}}, \tag{43}$$

$$\left|H_2\left(x, y, z\right)\right| \leq f(x)\left(h(x)\left|y\right|^{i} + |z|\right)^{\frac{1}{2}}, \tag{44}$$

$$\left|M_1\left(x, y\right)\right| \leq c(x)\left|y\right|^{m}, \tag{45}$$

$$\left|M_2\left(x, y\right)\right| \leq g(x)\left|y\right|^{j}, \tag{46}$$

where b, c, f, g, q, i, j, and m are defined as in Theorem 2.7. If $y(x)$ is a solution of (42) on I, then for

$$\begin{aligned} \Phi_1(x) &= \frac{1}{2}\int_{\lambda(x_0)}^{\lambda(T)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)} \left[\frac{i}{p} h(\lambda^{-1}(s)) \exp\left(k_1(s)\right)\right. \\ &\quad \left. + \frac{j}{p}\int_{\lambda(x_0)}^{s} g(\tau)\exp\left(k_1(\tau)\right) d\tau\right] ds < 1, \end{aligned}$$

we have

$$|y(x)| \leq \left[|a(x)| + \frac{A_1(x)}{1-\Phi_1(x)} \exp\left(k_1(x)\right)\right]^{\frac{1}{p}}, \tag{47}$$

for $x \in I$, where

$$\begin{aligned} A_1(x) &= \int_{\lambda(x_0)}^{\lambda(x)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)} \left\{\frac{1}{2}\left[b\left(\lambda^{-1}(s)\right)\left(\frac{q}{p} a(\lambda^{-1}(s)) + \frac{p-q}{p}\right)\right.\right. \\ &\quad \left.\left. + \int_{\lambda(x_0)}^{s} c(\tau)\left(\frac{m}{p} a(\tau) + \frac{p-m}{p}\right) d\tau\right] + \frac{1}{2}\right\} ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)} \left\{\frac{1}{2}\left[h(\lambda^{-1}(s))\left(\frac{i}{p} a(\lambda^{-1}(s)) + \frac{p-i}{p}\right)\right.\right. \\ &\quad \left.\left. + \int_{\lambda(x_0)}^{s} g(\tau)\left(\frac{j}{p} a(\tau) + \frac{p-j}{p}\right) d\tau\right] + \frac{1}{2}\right\} ds, \end{aligned}$$

$$k_1(x) = \frac{1}{2}\int_{\lambda(x_0)}^{\lambda(x)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)} \left[\frac{q}{p} b\left(\lambda^{-1}(s)\right) + \frac{m}{p}\int_{\lambda(x_0)}^{s} c(\tau)d\tau\right] ds.$$

Proof. The equation (42) can be written as follows

$$\begin{aligned} |y^p(x)| \quad &\leq \quad |a(x)| + \int_{x_0}^{x} \left| H_1\left(s, y\left(\lambda(s)\right), \int_{x_0}^{\lambda(s)} M_1\left(\tau, y(\tau)\right) d\tau\right)\right| ds \\ &\quad + \int_{x_0}^{T} \left| H_2\left(s, y\left(\lambda(s)\right), \int_{x_0}^{\lambda(s)} M_2\left(\tau, y(\tau)\right) d\tau\right)\right| ds, \end{aligned} \tag{48}$$

Using the conditions (43)-(46) in (48), we get

$$\begin{aligned} |y\left(x\right)|^p \quad &\leq \quad |a(x)| + \int_{x_0}^{x} f(s)\left[b(s)\left|y\left(\lambda(s)\right)\right|^q + \int_{x_0}^{\lambda(s)} \left|M_1\left(\tau, y(\tau)\right)\right| d\tau\right]^{\frac{1}{2}} ds \\ &\quad + \int_{x_0}^{T} f(s)\left[h(s)\left|y\left(\lambda(s)\right)\right|^i + \int_{x_0}^{\lambda(s)} \left|M_2\left(\tau, y(\tau)\right)\right| d\tau\right]^{\frac{1}{2}} ds \\ &\leq \quad |a(x)| + \int_{x_0}^{x} f(s)\left[b(s)\left|y\left(\lambda(s)\right)\right|^q + \int_{x_0}^{\lambda(s)} c(\tau)\left|y\left(\tau\right)\right|^m d\tau\right]^{\frac{1}{2}} ds \\ &\quad + \int_{x_0}^{T} f(s)\left[h(s)\left|y\left(\lambda(s)\right)\right|^i + \int_{x_0}^{\lambda(s)} g(\tau)\left|y(\tau)\right|^j d\tau\right]^{\frac{1}{2}} ds, \end{aligned} \tag{49}$$

making a suitable change of variables in (49), we get

$$\begin{aligned} |y\left(x\right)|^p \quad &\leq \quad |a(x)| + \int_{\lambda(x_0)}^{\lambda(x)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)}\left[b\left(\lambda^{-1}(s)\right)\left|y\left(s\right)\right|^q + \int_{\lambda(x_0)}^{s} c(\tau)\left|y\left(\tau\right)\right|^m d\tau\right]^{\frac{1}{2}} ds \\ &\quad + \int_{\lambda(x_0)}^{\lambda(T)} \frac{f(\lambda^{-1}(s))}{\lambda'\left(\lambda^{-1}(s)\right)}\left[h(\lambda^{-1}(s))\left|y\left(s\right)\right|^i + \int_{\lambda(x_0)}^{s} g(\tau)\left|y(\tau)\right|^j d\tau\right]^{\frac{1}{2}} ds, \end{aligned} \tag{50}$$

for any $x \in I$. Now, an application of Theorem 2.7 to (50) yields the required result in (47). The proof is complete. □

Conclusion

T.H. Growall (1919, [7]) introduced the concept of integral inequalities in the early 20th century and presented a notable inequality that has attracted and continues to attract considerable attention in academic literature. A number of Gronwall-type integral inequalities in one or more independent variables have been established by various researchers during the last few decades. The generalization of the Gronwall-Bellman inequality to various forms, such as nonlinear integral inequalities with delay, of Volterra-Fredholm type, and nonlinear retarded integral inequalities with power, has attracted a lot of attention lately.

In this research, we have constructed some novel nonlinear Volterra-Fredholm-type integral inequalities with power that generalize several results found in [8], [10], [12], [13], and [16], in keeping with this trend and advancing the study of integral inequalities. The unknown function in Theorems 2.2, 2.5, 2.7, and 2.10 has been estimated using several analytical techniques. The results obtained can be used to investigate the uniqueness and boundedness of solutions to certain delayed Volterra-Fredholm integral equations with power. In order to examine the boundedness of the solution, an illustrated example is also provided as an application.

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