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INTUITIONISTIC FUZZY HYPER p-IDEALS IN HYPER BCK-ALGEBRAS

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Abstract. This research studies intuitionistic fuzzy hyper p-ideals in hyper BCK-algebras and compares them with intuitionistic fuzzy hyper BCK-ideals. We define intuitionistic fuzzy (weak, strong) hyper p-ideals and characterize them in terms of level subsets and hyper homomorphic pre-images. We analyze their interrelationships and identify the strongest intuitionistic fuzzy relation on a hyper BCK-algebra, advancing the theory of fuzzy mathematical structures.

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1. Introduction

Imai and Iséki's pioneering work [5] introduced BCK-algebras, abstracting the properties of set difference. Iséki subsequently extended this concept to BCI-algebras. Notably, BCK-algebras are grounded in BCK logic, while BCI-algebras are rooted in BCI logic. This interplay between algebraic structures and logical systems enables translation between well-formed formulas and theorems in logic and algebra, offering valuable insights into the underlying logics.

The concept of hyperstructures was introduced by Marty [14] at the 8th Congress of Scandinavian Mathematicians in 1934. Today, hyperstructures are widely applied in both pure and applied mathematics. In 1985, Bhattacharya and Mukherjee [2] established relations for fuzzy groups. In 1966, Imai and Iséki [5] proposed a set of axioms, known as BCK-algebras, while investigating the properties of

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set difference. Later, Jun et al. [6,9] applied hyperstructures to BCK-algebras in 1999-2000, introducing the concept of hyper BCK-algebras, a generalization of BCK-algebras, and exploring related properties. Around the same time, applying the concept to K-algebras, Borzooei et al. introduced hyper K-algebras in 2000, as described in [3]. In 2000, Jun et al. [8] explored the properties of fuzzy strong hyper BCK-ideals. In 1965, Zadeh [20] introduced the concept of fuzzy sets, a tool for handling uncertainty. Building on this, Jun and Xin [7] applied fuzzy set theory to hyper BCK-algebras in 2001. In 2020, Muhiuddin et al. [15] explored bipolar-valued fuzzy soft hyper BCK-ideals in hyper BCK-algebras. In 2021, Khademan et al. [10] introduced various types of fuzzy soft positive implicative hyper BCK-ideals. Atanassov [1] presented intuitionistic fuzzy sets in 1986, broadening the scope of fuzzy sets. These sets assign both membership and non-membership values to elements, providing a more comprehensive representation than traditional fuzzy sets.

Several intuitionistic structures have been proposed to better model uncertainty. In 2020, Seo et al. [19] introduced multipolar intuitionistic fuzzy hyper BCK-ideals in hyper BCK-algebras. Satyanarayana et al. further developed the theory by studying various types of ideals in BCK and hyper BCK-algebras [4,12,16–18].

This paper investigates the intuitionistic fuzzification of (weak, strong) hyper p-ideals in hyper BCK-algebras and examines their associated structural properties.

Acronyms

- $\mathcal{HBCKA}$: hyper BCK-algebra
- $\mathcal{HBCKI}$: hyper BCK-ideal
- $\mathcal{HPI}$: hyper p-ideal
- $\mathcal{IFS}$: intuitionistic fuzzy set
- $\mathcal{IFHBCKI}$: intuitionistic fuzzy hyper BCK-ideal
- $\mathcal{IFHPI}$: intuitionistic fuzzy hyper p-ideal
- $\mathcal{IFSHPI}$: intuitionistic fuzzy strong hyper p-ideal
- $\mathcal{IFWHPI}$: intuitionistic fuzzy weak hyper p-ideal

2. Preliminaries

To establish a foundation for this article, we present essential concepts in the following section.

Let U be a nonempty set endowed with hyper-operation, i.e., $\star$ is a function from $\mathsf{U} \times \mathsf{U}$ to $Q^*(\mathsf{U}) = Q(\mathsf{U})/\{\emptyset\}$. For any two subsets $\mathfrak{C}$ and $\mathfrak{G}$ of K, $(\mathfrak{C} \star \mathfrak{G})$ is denoted by $\bigcup_{(\mathfrak{m}\in\mathfrak{C},\mathfrak{n}\in\mathfrak{G})} \mathfrak{m} \star \mathfrak{n}$. We shall use $\gamma_1 \star \gamma_2$ instead of $\gamma_1 \star \{\gamma_2\}$, $\{\gamma_1\} \star \gamma_2$ or $\{\gamma_1\} \star \{\gamma_2\}$.

Definition 2.1. [9] *By an $\mathcal{HBCKA}$ $(\mathsf{U}, \star, 0)$, we mean a nonempty set U endowed with a hyper operation $\star$ and a constant 0 fulfilling the below axioms:*

$(\mathcal{HBCKA}1)$ $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \ll \gamma_1 \star \gamma_2$,

$(\mathcal{HBCKA}2)$ $(\gamma_1 \star \gamma_2) \star \gamma_3 = (\gamma_1 \star \gamma_3) \star \gamma_2$,

$(\mathcal{HBCKA}3)$ $\gamma_1 \star \mathsf{U} \ll \{\gamma_1\}$,

$(\mathcal{HBCKA}4)$ $\gamma_1 \ll \gamma_2$ *and* $\gamma_2 \ll \gamma_1 \Rightarrow \gamma_1 = \gamma_2$, *for all* $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$.

We can define a relation $\ll$ on U by letting $\gamma_1 \ll \gamma_2$ if and only if $0 \in \gamma_1 \star \gamma_2$, and for every $\mathfrak{C}, \mathfrak{G} \subseteq \mathsf{U}$, $\mathfrak{C} \ll \mathfrak{G}$ is defined for all $\mathfrak{m} \in \mathfrak{C}$ there exists $\mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{m} \ll \mathfrak{n}$. In such a case, we call the relation $\ll$ the hyper-order in U.

Clearly, every BCK-algebra $(\mathsf{U}, \star, 0)$ can be viewed as an $\mathcal{HBCKA}$ with the operation $\gamma_1 \star \gamma_2 = \{\gamma_1 \circ \gamma_2\}$.

A notable example of an $\mathcal{HBCKA}$ is the set of nonnegative real numbers, denoted by $\mathsf{U} = [0, \infty)$, equipped with a specific operation

$$\gamma_1 \star \gamma_2 := \begin{cases} [0, \gamma_1] & \text{if } \gamma_1 < \gamma_2 \\ [0, \gamma_2] & \text{if } \gamma_1 > \gamma_2 \neq 0 \\ \{\gamma_1\} & \text{if } \gamma_2 = 0. \end{cases}$$

Proposition 2.1. [9] *In every $\mathcal{HBCKA}$ U, the below properties are satisfied:*(P_1) $\gamma_1 \star \gamma_2 \ll \{\gamma_1\}$, (P_2) $\gamma_1 \star 0 \ll \{\gamma_1\}$, (P_3) $0 \star 0 = \{0\}$, (P_4) $0 \ll \gamma_1$, (P_5) $\mathfrak{C} \ll \mathfrak{C}$, (P_6) $\mathfrak{C} \subseteq \mathfrak{G} \Rightarrow \mathfrak{C} \ll \mathfrak{G}$, (P_7) $0 \star \gamma_1 = \{0\}$, (P_8) $0 \star \mathfrak{C} = 0$, (P_9) $\mathfrak{C} \star \{0\} = \{0\}$ *implies that* $\mathfrak{C} = \{0\}$, (P_10) $\gamma_2 \ll \gamma_3$ *implies that* $\gamma_1 \star \gamma_3 \ll \gamma_1 \star \gamma_2$, *for all* $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$, *and for any nonempty subsets* $\mathfrak{C}, \mathfrak{G}$ *of* U.

For the fundamental study related to hyper BCK-subalgebras and (weak, strong, reflexive) hyper BCK-ideals, refer to [9]. Hereafter, U denotes an $\mathcal{HBCKA}$.

Lemma 2.1. [9] *For every $\mathcal{HBCKA}$, the following hold:*

i. Any strong $\mathcal{HBCKI}$ is an $\mathcal{HBCKI}$.

ii. Any $\mathcal{HBCKI}$ is a weak $\mathcal{HBCKI}$.

Lemma 2.2. [8] *For any reflexive $\mathcal{HBCKI}$ $\mathfrak{G}$ of U, if* $\gamma_1 \star \gamma_2 \cap \mathfrak{G} \neq \emptyset$ *implies that* $\gamma_1 \star \gamma_2 \ll \mathfrak{G}$ *for all* $\gamma_1, \gamma_2 \in \mathsf{U}$.

Proposition 2.2. [6] *If $\mathfrak{C}$ is a subset of U and $\mathfrak{G}$ is any hyper BCK-ideal of U, such that $\mathfrak{C} \ll \mathfrak{G}$ then $\mathfrak{C} \subseteq \mathfrak{G}$.*

Definition 2.2. [13] *For an $\mathcal{HBCKA}$ U, a non-empty subset $\mathfrak{G} \subseteq \mathsf{U}$, containing 0 is known as*

i. a weak hyper p-ideal of U *if* $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \subseteq \mathfrak{G}$ *and* $\gamma_2 \in \mathfrak{G} \Rightarrow \gamma_1 \in \mathfrak{G}$.

ii. a hyper p-ideal of U *if* $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \ll \mathfrak{G}$ *and* $\gamma_2 \in \mathfrak{G} \Rightarrow \gamma_1 \in \mathfrak{G}$.

iii. a strong hyper p-ideal of U *if* $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathfrak{G} \neq \emptyset$ *and* $\gamma_2 \in \mathfrak{G} \Rightarrow \gamma_1 \in \mathfrak{G}$.

Theorem 2.1. [13] *Every (strong, weak) hyper p-ideal is a (strong, weak) hyper BCK-ideal.*

Remark. Generally, every (strong, weak) hyper BCK-ideal is not a (strong, weak) hyper p-ideal (see [13]).

Theorem 2.2. [13] *For every $\mathcal{HBCKA}$, the following hold:*

$i.$ *Any hyper p-ideal is a weak hyper p-ideal.*

$ii.$ *Any strong hyper p-ideal is a hyper p-ideal.*

Remark. Generally, the converse of Theorem 2.2 isn't hold (see, [13]).

For a detailed study of fuzzy (weak, strong) hyper BCK-ideals, one must consult [7].

Theorem 2.3. [7] *For every $\mathcal{HBCKA}$, the following hold:*

$i.$ *Every fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal.*

$ii.$ *Every fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal.*

Definition 2.3. [13] *For an $\mathcal{HBCKA}$ U, a fuzzy set $\xi^{\mathcal{N}}$ in U is said to be*

$i.$ *a fuzzy weak hyper p-ideal of U if for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,*

$$\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1) \geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\},$$

$ii.$ *a fuzzy hyper p-ideal of U if $\gamma_1 \ll \gamma_2$ implies that $\xi^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(\gamma_2)$, and for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,*

$$\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\},$$

$iii.$ *a fuzzy strong hyper p-ideal of U if for all $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,*

$$\inf_{\mathfrak{m} \in \gamma_1 \star \gamma_1} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\gamma_1) \geq \min\{\sup_{\mathfrak{n} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{n}), \xi^{\mathcal{N}}(\gamma_2)\}.$$

3. $\mathcal{IFHP}$-Ideals of Hyper BCK-Algebras

We present the concept of $\mathcal{IF}(\mathcal{W}, \mathcal{S})\mathcal{HPI}$s and explore their properties.

Definition 3.1. *For an $\mathcal{HBCKA}$ U, an intuitionistic fuzzy set($\mathcal{IFS}$) $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U is called*

$i.$ *an $\mathcal{IFWHPI}$ of U if for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,*

$$\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1) \geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\},$$

$$\upsilon^{\mathcal{N}}(0) \leq \upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\},$$

$ii.$ *an $\mathcal{IFHPI}$ of U if $\gamma_1 \ll \gamma_2$ implies that $\xi^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(\gamma_2)$, and $\upsilon^{\mathcal{N}}(\gamma_1) \leq \upsilon^{\mathcal{N}}(\gamma_2)$ and for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,*

$$\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\},$$

$$\upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\},$$

iii. an $\mathcal{IFSHPI}$ of U *if for all* $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,

$$\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\gamma_1) \geq \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{n}), \xi^{\mathcal{N}}(\gamma_2)\},$$

$$\sup_{\mathfrak{x}\in\gamma_1\star\gamma_1} \upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{y}), \upsilon^{\mathcal{N}}(\gamma_2)\}.$$

Theorem 3.1. *Any intuitionistic fuzzy (weak, strong) hyper p-ideal of* U *is an intuitionistic fuzzy (weak, strong) hyper BCK-ideal of* U.

Proof. Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFHPI}$ of U. Then $\forall\, \gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$, we get

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\inf_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

Now, putting $h = 0$, then

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\inf_{\mathfrak{m}\in(\gamma_1\star 0)\star(\gamma_2\star 0)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\sup_{\mathfrak{x}\in(\gamma_1\star 0)\star(\gamma_2\star 0)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}, \end{aligned}$$

thus

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\inf_{\mathfrak{m}\in(\gamma_1\star\gamma_2)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_2)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

□

Remark. Generally, the converse of Theorem 3.1 is not valid. See the following examples.

Example 3.1. *Let* $\mathsf{U} = \{0, \gamma_1, \gamma_2\}$ *be an* $\mathcal{HBCKA}$ *with the binary operation* $\star$ *as follows*

Table 1. Hyper BCK-algebra

$\star$	0	γ_1	γ_2
0	$\{0\}$	$\{0\}$	$\{0\}$
γ_1	$\{\gamma_1\}$	$\{0, \gamma_1\}$	$\{0, \gamma_1\}$
γ_2	$\{\gamma_2\}$	$\{\gamma_2\}$	$\{0, \gamma_1\}$

Define an $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U by: $\xi^{\mathcal{N}}(0) = 0.5$, $\xi^{\mathcal{N}}(\gamma_1) = 0.3$, $\xi^{\mathcal{N}}(\gamma_2) = 0.1$, and $\upsilon^{\mathcal{N}}(0) = 0.2$, $\upsilon^{\mathcal{N}}(\gamma_1) = 0.3$, $\upsilon^{\mathcal{N}}(\gamma_2) = 0.4$. One can readily confirm that $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFWHBCKI}$ but not an $\mathcal{IFWHPI}$ of U as

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) = 0.3 &< 0.5 = \min\{\inf_{\gamma_1\in(\gamma_1\star\gamma_1)\star(0\star\gamma_1)} \xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(0)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) = 0.3 &> 0.2 = \max\{\sup_{\gamma_1\in(\gamma_1\star\gamma_1)\star(0\star\gamma_1)} \upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(0)\}. \end{aligned}$$

Example 3.2. *Consider the hyper BCK-algebra* $\mathsf{U} = \{0, \gamma_1, \gamma_2\}$ *with the binary operation* $\star$ *defined as in Table 1 of Example 3.1.*

Define an $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U by: $\xi^{\mathcal{N}}(0) = 0.3$, $\xi^{\mathcal{N}}(\gamma_1) = 0.05$, $\xi^{\mathcal{N}}(\gamma_2) = 0.02$, and $\upsilon^{\mathcal{N}}(0) = 0$, $\upsilon^{\mathcal{N}}(\gamma_1) = 0.25$, $\upsilon^{\mathcal{N}}(\gamma_2) = 0.6$. Clearly $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFHBCKI}$ but not an $\mathcal{IFHPI}$ of U since

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) = 0.05 &< 0.3 = \min\{\inf_{\gamma_1 \in (\gamma_1 \star \gamma_1) \star (0 \star \gamma_1)} \xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(0)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) = 0.25 &> 0 = \max\{\sup_{\gamma_1 \in (\gamma_1 \star \gamma_1) \star (0 \star \gamma_1)} \upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(0)\}. \end{aligned}$$

Example 3.3. *Let* $\mathsf{U} = \{0, \gamma_1, \gamma_2, \gamma_3\}$ *be an* $\mathcal{HBCKA}$ *with the binary operation* $\star$ *as follows:*

TABLE 2. Hyper BCK-algebra

$\star$	0	γ_1	γ_2	γ_3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
γ_1	$\{\gamma_1\}$	$\{0, \gamma_1\}$	$\{0, \gamma_1\}$	$\{0, \gamma_1\}$
γ_2	$\{\gamma_2\}$	$\{\gamma_2\}$	$\{0, \gamma_1\}$	$\{0, \gamma_1\}$
γ_3	$\{\gamma_3\}$	$\{\gamma_3\}$	$\{\gamma_3\}$	$\{0, \gamma_1\}$

Define an $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U by: $\xi^{\mathcal{N}}(0) = \xi^{\mathcal{N}}(\gamma_1) = 0.4$, $\xi^{\mathcal{N}}(\gamma_1) = 0.03$, $\xi^{\mathcal{N}}(\gamma_2) = 0.02$, and $\upsilon^{\mathcal{N}}(0) = 0$, $\upsilon^{\mathcal{N}}(\gamma_1) = 0.3$, $\upsilon^{\mathcal{N}}(\gamma_2) = 0.4$. Evidently $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFSHBCKI}$ but not an $\mathcal{IFSHPI}$ of U since

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_2) = 0.02 &< 0.4 = \min\{\sup_{\gamma_1 \in (\gamma_2 \star \gamma_2) \star (0 \star \gamma_2)} \xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\gamma_1)\}, \\ \upsilon^{\mathcal{N}}(\gamma_2) = 0.4 &> 0 = \max\{\inf_{\gamma_1 \in (\gamma_2 \star \gamma_2) \star (0 \star \gamma_2)} \upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\gamma_1)\}. \end{aligned}$$

Theorem 3.2. *For every* $\mathcal{HBCKA}$*, the following hold:*

i. Every $\mathcal{IFHPI}$ *is an* $\mathcal{IFWHPI}$*.*
ii. Every $\mathcal{IFSHPI}$ *is an* $\mathcal{IFHPI}$*.*

Proof. (i) Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFHPI}$ of U. Since, for any $\mathcal{IFHPI}$ is an $\mathcal{IFHBCKI}$ (by Theorem 3.1) and any $\mathcal{IFHBCKI}$ is an $\mathcal{IFWHBCKI}$ (by Theorem 2.3(i)), hence $\mathcal{N}$ is also an $\mathcal{IFWHBCKI}$ of U. Thus, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ satisfies $\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1)$, and $\upsilon^{\mathcal{N}}(0) \leq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Also being an $\mathcal{IFHPI}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ satisfies

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\sup_{\mathfrak{r} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{r}), \upsilon^{\mathcal{N}}(\gamma_2)\} \end{aligned}$$

$\forall\, \gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$. Therefore, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFWHPI}$ of U.

(ii) Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFSHPI}$ of U. Since any $\mathcal{IFSHPI}$ is an $\mathcal{IFSHBCKI}$ (by Theorem 3.1) and every $\mathcal{IFSHBCKI}$ is an $\mathcal{IFHBCKI}$ (by Theorem 2.3(ii)), we obtain $\mathcal{N}$ is also an $\mathcal{IFHBCKI}$ of U. Thus, for all $\gamma_1, \gamma_2 \in \mathsf{U}$, if $\gamma_1 \ll \gamma_2$, then $\xi^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(\gamma_2)$ and $\upsilon^{\mathcal{N}}(\gamma_1) \leq \upsilon^{\mathcal{N}}(\gamma_2)$. Also being an $\mathcal{IFSHPI}$, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ satisfies for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$,

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\sup_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\inf_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

Since $\sup_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\mathfrak{n})$ and $\inf_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \upsilon^{\mathcal{N}}(\mathfrak{y})$ for all $\mathfrak{n}, \mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$. We get

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\sup_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\} \geq \min\{\xi^{\mathcal{N}}(\mathfrak{n}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\inf_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\} \leq \max\{\upsilon^{\mathcal{N}}(\mathfrak{y}), \upsilon^{\mathcal{N}}(\gamma_2)\}, \end{aligned}$$

for all $\mathfrak{n}, \mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$. $\xi^{\mathcal{N}}(\mathfrak{n}) \geq \inf_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m})$ and
$\upsilon^{\mathcal{N}}(\mathfrak{y}) \leq \sup_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x})$ for all $\mathfrak{n}, \mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, thus we have

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\xi^{\mathcal{N}}(\mathfrak{n}), \xi^{\mathcal{N}}(\gamma_2)\} \geq \min\{\inf_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\upsilon^{\mathcal{N}}(\mathfrak{y}), \upsilon^{\mathcal{N}}(\gamma_2)\} \leq \max\{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

That is,

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\inf_{\mathfrak{m}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

□

Remark. Evidently, the converse of Theorem 3.2 is not valid (see the following example).

Example 3.4. *Let* $\mathsf{U} = \{0, \gamma_1, \gamma_2\}$ *be an* $\mathcal{HBCKA}$ *with the binary operation* $\star$ *as follows:*

TABLE 3. Hyper BCK-algebra

$\star$	0	γ_1	γ_2
0	$\{0\}$	$\{0\}$	$\{0\}$
γ_1	$\{\gamma_1\}$	$\{0, \gamma_1\}$	$\{0, \gamma_1\}$
γ_2	$\{\gamma_2\}$	$\{\gamma_2\}$	$\{0, \gamma_1, \gamma_2\}$

Define an $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U by: $\xi^{\mathcal{N}}(0) = 0.6$, $\xi^{\mathcal{N}}(\gamma_1) = 0.03$, $\xi^{\mathcal{N}}(\gamma_2) = 0.2$, and $\upsilon^{\mathcal{N}}(0) = 0.15$, $\upsilon^{\mathcal{N}}(\gamma_1) = 0.4$, $\upsilon^{\mathcal{N}}(\gamma_2) = 0.3$. Then $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFWHPI}$ but not an $\mathcal{IFHPI}$ of U as: $\gamma_1 \ll \gamma_2$ but $\xi^{\mathcal{N}}(\gamma_1) = 0.03 < 0.2 = \xi^{\mathcal{N}}(\gamma_2)$, and $\upsilon^{\mathcal{N}}(\gamma_1) = 0.4 > 0.3 = \upsilon^{\mathcal{N}}(\gamma_2)$.

Example 3.5. *Let* $\mathsf{U} = \{0, \gamma_1, \gamma_2\}$ *be an* $\mathcal{HBCKA}$ *with the binary operation* $\star$ *as follows:*

Table 4. Hyper BCK-algebra

$\star$	0	γ_1	γ_2
0	$\{0\}$	$\{0\}$	$\{0\}$
γ_1	$\{\gamma_1\}$	$\{0, \gamma_1\}$	$\{\gamma_1\}$
γ_2	$\{\gamma_2\}$	$\{\gamma_2\}$	$\{0, \gamma_2\}$

Define an $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ in U by: $\xi^{\mathcal{N}}(0) = \xi^{\mathcal{N}}(\gamma_1) = 0.5$,$\xi^{\mathcal{N}}(\gamma_2) = 0.04$, and $\upsilon^{\mathcal{N}}(0) = \upsilon^{\mathcal{N}}(\gamma_1) = 0.01$, $\upsilon^{\mathcal{N}}(\gamma_2) = 0.7$. Then $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFHPI}$ but it is not an $\mathcal{IFSHPI}$ of U as follows:

$$\xi^{\mathcal{N}}(\gamma_2) = 0.04 \quad < \quad 0.5 = \min\{\sup_{\mathfrak{m} \in (\gamma_2 \star \gamma_2) \star (\gamma_1 \star \gamma_2)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_1)\},$$
$$\upsilon^{\mathcal{N}}(\gamma_2) = 0.7 \quad > \quad 0.01 = \max\{\inf_{\mathfrak{x} \in (\gamma_2 \star \gamma_2) \star (\gamma_1 \star \gamma_2)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_1)\}.$$

Theorem 3.3. *an* $\mathcal{IFS}$ $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ *in* U, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ *is an intuitionistic fuzzy* (*weak, strong*) *hyper p-ideal of* U *if and only if for all* $\mathfrak{s}, \mathfrak{v} \in [0, 1]$, $\xi^{\mathcal{N}}_{\mathfrak{s}}$, *and* $\upsilon^{\mathcal{N}}_{\mathfrak{v}}$ *are non-empty* (*weak, strong*) *hyper p-ideal of* U.

Proof. Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFHPI}$ of U. Since $\mathcal{N}_{(\mathfrak{s},\mathfrak{v})} = (\xi^{\mathcal{N}}_{\mathfrak{s}}, \upsilon^{\mathcal{N}}_{\mathfrak{v}})$ is nonempty, so for any γ_1 in $\mathcal{N}_{(\mathfrak{s},\mathfrak{v})}$, $\xi^{\mathcal{N}}(\gamma_1) \geq \mathfrak{s}$ and $\upsilon^{\mathcal{N}}(\gamma_1) \leq \mathfrak{v}$. Since every $\mathcal{IFHPI}$ is also an $\mathcal{IFWHPI}$ (by Theorem 3.2(i)), so $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is also an $\mathcal{IFWHPI}$ of U. Hence, $\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1) \geq \mathfrak{s}$ and $\upsilon^{\mathcal{N}}(0) \leq \upsilon^{\mathcal{N}}(\gamma_1) \leq \mathfrak{v}$ for all $\gamma_1 \in \mathsf{U}$, imply 0 in $\mathcal{N}_{(\mathfrak{s},\mathfrak{v})}$. Let $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \ll \mathcal{N}_{(\mathfrak{s},\mathfrak{v})}$, and $\gamma_2 \in \mathcal{N}_{(\mathfrak{s},\mathfrak{v})}$, then for all $\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, there exists $\mathfrak{n} \in \xi^{\mathcal{N}}_{\mathfrak{s}}$ such that $\mathfrak{m} \ll \mathfrak{n}$. So $\xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\mathfrak{n}) \geq \mathfrak{s}$ for all $\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$. Hence, $\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \mathfrak{s}$. Also $\xi^{\mathcal{N}}(\gamma_2) \geq \mathfrak{s}$, as $\gamma_2 \in \xi^{\mathcal{N}}_{\mathfrak{s}}$. Thus,

$$\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\} \geq \min\{\mathfrak{s}, \mathfrak{s}\} = \mathfrak{s}$$

imply $\gamma_1 \in \xi^{\mathcal{N}}_{\mathfrak{s}}$, and $\forall\, \mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, there exist $\mathfrak{y} \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$ such that $\mathfrak{x} \ll \mathfrak{y}$. So $\upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \upsilon^{\mathcal{N}}(\mathfrak{y}) \leq \mathfrak{v}$ $\forall\, \mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$. Hence, $\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \mathfrak{v}$. Also $\upsilon^{\mathcal{N}}(\gamma_2) \leq \mathfrak{v}$, as $\gamma_2 \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$. Therefore,

$$\upsilon^{\mathcal{N}}(\gamma_1) \quad \leq \quad \max\{\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\} \leq \max\{\mathfrak{v}, \mathfrak{v}\} = \mathfrak{v}$$

implies $\gamma_1 \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$. Thus, $\xi^{\mathcal{N}}_{\mathfrak{s}}$ and $\upsilon^{\mathcal{N}}_{\mathfrak{v}}$ are $\mathcal{HPI}$s of U. In contrast, suppose that $\xi^{\mathcal{N}}_{\mathfrak{s}}$ and $\upsilon^{\mathcal{N}}_{\mathfrak{v}}$ are non-empty $\mathcal{HPI}$ of U for all $\mathfrak{s}, \mathfrak{v} \in [0, 1]$. Let $\gamma_1 \ll \gamma_2$ for some $\gamma_1, \gamma_2 \in \mathsf{U}$ and let $\xi^{\mathcal{N}}(\gamma_2) = \mathfrak{s}$. Then $\gamma_2 \in \xi^{\mathcal{N}}_{\mathfrak{s}}$. So $\gamma_1 \ll \gamma_2 \in \xi^{\mathcal{N}}_{\mathfrak{s}}$ implies $\gamma_1 \ll \xi^{\mathcal{N}}_{\mathfrak{s}}$. Being an $\mathcal{HPI}$, $\xi^{\mathcal{N}}_{\mathfrak{s}}$ is also an $\mathcal{HBCKI}$ of U (By Theorem 2.1) thus by Proposition 2.2, $\gamma_1 \in \xi^{\mathcal{N}}_{\mathfrak{s}}$. therefore $\xi^{\mathcal{N}}(\gamma_1) \geq \mathfrak{s} = \xi^{\mathcal{N}}(\gamma_2)$. That is, $\gamma_1 \ll \gamma_2$ implies

$\xi^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \mathsf{U}$. Let $\upsilon^{\mathcal{N}}(\gamma_2) = \mathfrak{v}$. Then $\gamma_2 \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$. So $\gamma_1 \ll \gamma_2 \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$ implies $\gamma_1 \ll \upsilon^{\mathcal{N}}_{\mathfrak{v}}$. Being a hyper p-ideal, $\upsilon^{\mathcal{N}}_{\mathfrak{v}}$ is also a hyper BCK-ideal of U (By Theorem 2.1) thus by Proposition 2.2, $\gamma_1 \in \upsilon^{\mathcal{N}}_{\mathfrak{v}}$. Therefore $\upsilon^{\mathcal{N}}(\gamma_1) \leq \mathfrak{v} = \upsilon^{\mathcal{N}}(\gamma_2)$. That is, $\gamma_1 \ll \gamma_2$ implies $\upsilon^{\mathcal{N}}(\gamma_1) \leq \upsilon^{\mathcal{N}}(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \mathsf{U}$. Moreover, for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$, let $\mathfrak{w} = \min\{\inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\gamma_2)\}$. Then $\xi^{\mathcal{N}}(\gamma_2) \geq \mathfrak{w}$ implies that $\gamma_2 \in \xi^{\mathcal{N}}_{\mathfrak{w}}$, and for all $\mathfrak{r} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$ $\xi^{\mathcal{N}}(\mathfrak{r}) \geq \inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{j}) \geq \mathfrak{w}$, imply $\mathfrak{r} \in \xi^{\mathcal{N}}_{\mathfrak{w}}$. Hence, $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \subseteq \xi^{\mathcal{N}}_{\mathfrak{w}}$. By Proposition 2.1$(P_6)$, $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \subseteq \xi^{\mathcal{N}}_{\mathfrak{w}}$ implies $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \ll \xi^{\mathcal{N}}_{\mathfrak{w}}$, which along with $\gamma_2 \in \xi^{\mathcal{N}}_{\mathfrak{w}} \Rightarrow \gamma_1 \in \xi^{\mathcal{N}}_{\mathfrak{w}}$. And let $\mathfrak{d} = \max\{\sup_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\gamma_2)\}$. Then $\upsilon^{\mathcal{N}}(\gamma_2) \leq \mathfrak{d}$ implies $\gamma_2 \in \upsilon^{\mathcal{N}}_{\mathfrak{d}}$ and for all $\mathfrak{e} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, $\upsilon^{\mathcal{N}}(\mathfrak{e}) \leq \sup_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{a}) \leq \mathfrak{d}$, implies $\mathfrak{e} \in \upsilon^{\mathcal{N}}_{\mathfrak{d}}$. Hence, $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \subseteq \upsilon^{\mathcal{N}}_{\mathfrak{d}}$. By Proposition 2.1$(P_6)$, $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \subseteq \upsilon^{\mathcal{N}}_{\mathfrak{d}}$ implies $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \ll \upsilon^{\mathcal{N}}_{\mathfrak{d}}$, which along with $\gamma_2 \in \upsilon^{\mathcal{N}}_{\mathfrak{d}} \Rightarrow \gamma_1 \in \upsilon^{\mathcal{N}}_{\mathfrak{d}}$. Therefore, we obtain

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \mathfrak{w} = \min\{\inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &\leq \mathfrak{d} = \max\{\sup_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

□

Theorem 3.4. *If $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an intuitionistic fuzzy (weak, strong) hyper p-ideal of U, then $\mathcal{P} = \{\gamma_1 \in \mathsf{U} | \xi^{\mathcal{N}}(\gamma_1) = \xi^{\mathcal{N}}(0),$ and $\upsilon^{\mathcal{N}}(\gamma_1) = \upsilon^{\mathcal{N}}(0)\}$ is a (weak, strong) hyper p-ideal of U.*

Proof. Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFSHPI}$ of U. Clear, $0 \in \mathcal{P}$. Let $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P} \neq \emptyset$ and $\gamma_2 \in \mathcal{P}$ for some $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$. Then there exist $\gamma_1 \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P}$ such that $\xi^{\mathcal{N}}(\gamma_1) = \xi^{\mathcal{N}}(0)$ and $\upsilon^{\mathcal{N}}(\gamma_1) = \upsilon^{\mathcal{N}}(0)$. Also $\xi^{\mathcal{N}}(\gamma_1) = \xi^{\mathcal{N}}(0)$ and $\upsilon^{\mathcal{N}}(\gamma_1) = \upsilon^{\mathcal{N}}(0)$. Then

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &\geq \min\{\sup_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\} \\ &\geq \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\gamma_2)\} \\ &= \min\{\xi^{\mathcal{N}}(0), \xi^{\mathcal{N}}(0)\} = \xi^{\mathcal{N}}(0) \end{aligned}$$

implies $\xi^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(0)$ and

$$\begin{aligned} \upsilon^{\mathcal{N}}(\gamma_1) &\leq \max\{\inf_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\} \\ &\leq \max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\gamma_2)\} \\ &= \max\{\upsilon^{\mathcal{N}}(0), \upsilon^{\mathcal{N}}(0)\} = \upsilon^{\mathcal{N}}(0) \end{aligned}$$

implies $\upsilon^{\mathcal{N}}(\gamma_1) \leq \upsilon^{\mathcal{N}}(0)$. Being an $\mathcal{IFSHPI}$, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is also an $\mathcal{IFWHPI}$ of U (by Theorem 3.2), so it satisfies $\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1)$ and $\upsilon^{\mathcal{N}}(0) \geq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Hence, $\xi^{\mathcal{N}}(0) = \xi^{\mathcal{N}}(\gamma_1)$ and $\upsilon^{\mathcal{N}}(0) = \upsilon^{\mathcal{N}}(\gamma_1)$, and so $\gamma_1 \in \mathsf{U}$. □

Theorem 3.5. *Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFS}$ in U defined by:*

$$\xi^{\mathcal{N}}(\gamma_1) = \begin{cases} \mathfrak{s} & \text{if} \quad \gamma_1 \in \mathcal{P} \\ 0 & \text{if} \quad \gamma_1 \notin \mathcal{P}, \end{cases} \quad \text{and } \upsilon^{\mathcal{N}}(\gamma_1) = \begin{cases} \mathfrak{v} & \text{if} \quad \gamma_1 \notin \mathcal{P} \\ 0 & \text{if} \quad \gamma_1 \in \mathcal{P}, \end{cases}$$

$\forall\, \mathfrak{s}, \mathfrak{v} \in \mathsf{U}$, where $\mathcal{P} \subseteq \mathsf{U}$ and $\mathfrak{s}, \mathfrak{v} \in (0,1]$. Then, $\mathcal{P}$ is a (weak, strong) hyper p-ideal if and only if $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is a intuitionistic fuzzy (weak, strong) hyper p-ideal.

Proof. Let $\mathcal{P}$ be a strong $\mathcal{HPI}$ of U. Then, for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$, if $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P} \neq \emptyset$ and $\gamma_2 \in \mathcal{P}$, it follows that $\gamma_1 \in \mathcal{P}$. Hence,

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &= \mathfrak{s} = \min\{\sup_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\}, \\ \upsilon^{\mathcal{N}}(\gamma_1) &= \mathfrak{v} = \max\{\inf_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

If $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P} = \emptyset$ and $\gamma_2 \notin \mathcal{P}$, then $\xi^{\mathcal{N}}(\mathfrak{n}) = 0$ and $\upsilon^{\mathcal{N}}(\mathfrak{y}) = 0$ for all $\mathfrak{n}, \mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, and $\xi^{\mathcal{N}}(\gamma_2) = 0$ and $\upsilon^{\mathcal{N}}(\gamma_2) = 0$, hence

$$\begin{aligned} &\min\{\sup_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\} = 0 \leq \xi^{\mathcal{N}}(\gamma_1), \\ &\max\{\inf_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\} = 0 \geq \upsilon^{\mathcal{N}}(\gamma_1). \end{aligned}$$

If $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P} = \emptyset$ and $\gamma_2 \notin \mathcal{P}$, or, $(\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3) \cap \mathcal{P} \neq \emptyset$ and $\gamma_2 \in \mathcal{P}$. Therefore, for both cases

$$\begin{aligned} &\min\{\sup_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\gamma_2)\} = 0 \leq \xi^{\mathcal{N}}(\gamma_1), \\ &\max\{\inf_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\gamma_2)\} = 0 \geq \upsilon^{\mathcal{N}}(\gamma_1). \end{aligned}$$

According to Proposition 2.1(P_1), we have $\gamma_1 \star \gamma_1 \leq \gamma_1\ \forall\, \gamma_1 \in \mathsf{U}$. Then for all $\mathfrak{j}, \mathfrak{a} \in \gamma_1 \star \gamma_1$, $\mathfrak{j} \ll \gamma_1$ and $\mathfrak{a} \ll \gamma_1$. Being a strong $\mathcal{HPI}$ of U, $\mathcal{P} = \xi^{\mathcal{N}}_{\mathfrak{s}}$ is an $\mathcal{HPI}$ of U (by Theorem 2.2(ii)), and hence $\xi^{\mathcal{N}}$ is an $\mathcal{HPI}$ of U (by Theorem 3.3). Hence, $\mathfrak{j} \ll \gamma_1$ implies $\xi^{\mathcal{N}}(\mathfrak{j}) \geq \xi^{\mathcal{N}}(\gamma_1)$ for all $\mathfrak{j} \in \gamma_1 \star \gamma_1 \Rightarrow \inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}(\mathfrak{j}) \geq \xi^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. And $\mathcal{P} = \upsilon^{\mathcal{N}}_{\mathfrak{v}}$ is an $\mathcal{HPI}$ of U (by Theorem 2.2(ii)), and hence $\upsilon^{\mathcal{N}}$ is an $\mathcal{HPI}$ of U (by Theorem 3.3). Hence, $\mathfrak{a} \ll \gamma_1$ implies $\upsilon^{\mathcal{N}}(\mathfrak{a}) \leq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\mathfrak{a} \in \gamma_1 \star \gamma_1 \Rightarrow \sup_{\mathfrak{a} \in (\gamma_1 \star \gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{a}) \leq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Therefore, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFSHPI}$ of U.

Conversely, let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFSHPI}$ of U. Then by Theorem 3.3, for all $\mathfrak{s}, \mathfrak{v} \in (0,1]$, $\xi^{\mathcal{N}}_{\mathfrak{s}} = \upsilon^{\mathcal{N}}_{\mathfrak{v}} = \mathcal{P}$ is a strong $\mathcal{HPI}$ of U. □

Theorem 3.6. *The family of $\mathcal{IFSHPI}$s forms a completely distributive lattice under join and meet operations.*

Proof. Consider a family of U, denoted as $\{\mathcal{N}^{\mathsf{k}} | \mathsf{k} \in \mathsf{K}\}$. Given that $[0,1]$ forms a completely distributive lattice under the standard ordering, it suffices to confirm that $\bigvee_{\mathsf{k} \in \mathsf{K}} \mathcal{N}\mathsf{k}$ and $\bigwedge_{\mathsf{k} \in \mathsf{K}} \mathcal{N}_{\mathsf{k}}$ are U. For any $\gamma_1 \in \mathsf{U}$, we get

$$\begin{aligned}\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}((\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{m})) &= \inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}(\sup_{k\in K}\xi_k^{\mathcal{N}}(\mathfrak{m})) \\ &= \sup_{k\in K}(\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}\xi_k^{\mathcal{N}}(\mathfrak{m})) \\ &\geq \sup_{k\in K}\xi_k^{\mathcal{N}}(\gamma_1) \\ &= (\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1)\end{aligned}$$

thus, $\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}((\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{m})) \geq (\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1)$ and

$$\begin{aligned}\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}((\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{r})) &= \sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}(\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\mathfrak{r})) \\ &= \inf_{k\in K}(\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}\upsilon_k^{\mathcal{N}}(\mathfrak{r})) \\ &\leq \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_1) \\ &= (\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1)\end{aligned}$$

implies $\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}((\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{r})) \leq (\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1)$. Moreover, for any $\gamma_1, \gamma_2, \gamma_3 \in U$, we have

$$\begin{aligned}(\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1) &= \sup_{k\in K}\xi_k^{\mathcal{N}}(\gamma_1) \\ &\geq \sup_{k\in K}[\min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\xi_k^{\mathcal{N}}(\mathfrak{n}), \xi_k^{\mathcal{N}}(\gamma_2)\}] \\ &= \min\{\sup_{k\in K}(\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\xi_k^{\mathcal{N}}(\mathfrak{n})), \sup_{k\in K}\xi_k^{\mathcal{N}}(\gamma_2)\} \\ &= \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}(\sup_{k\in K}\xi_k^{\mathcal{N}}(\mathfrak{n})), \sup_{k\in K}\xi_k^{\mathcal{N}}(\gamma_2)\} \\ &= \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{n})), (\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_2)\}.\end{aligned}$$

Hence, $(\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1) \geq \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{n})), (\bigvee_{k\in K}\xi_k^{\mathcal{N}})(\gamma_2)\}$, and

$$\begin{aligned}(\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1) &= \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_1) \\ &\leq \inf_{k\in K}[\max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\upsilon_k^{\mathcal{N}}(\mathfrak{y}), \upsilon_k^{\mathcal{N}}(\gamma_2)\}] \\ &= \max\{\inf_{k\in K}(\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\upsilon_k^{\mathcal{N}}(\mathfrak{y})), \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_2)\} \\ &= \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}(\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\mathfrak{y})), \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_2)\} \\ &= \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{y})), (\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_2)\}\end{aligned}$$

imply, $(\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1) \leq \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{y})), (\bigvee_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_2)\}$, Therefore, $\bigvee_{k\in K}\mathcal{N}_k$ is an $\mathcal{IFSHPI}$ of U. Now, we prove that $\bigwedge_{k\in K}\mathcal{N}_k$ is an $\mathcal{IFSHPI}$ of U. For any $\gamma_1 \in U$ we have

$$\begin{aligned}
\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}((\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{m})) &= \inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}(\inf_{k\in K}\xi_k^{\mathcal{N}}(\mathfrak{m})) \\
&= \inf_{k\in K}(\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}\xi_k^{\mathcal{N}}(\mathfrak{m})) \\
&\geq \inf_{k\in K}\xi_k^{\mathcal{N}}(\gamma_1) \\
&= (\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1)
\end{aligned}$$

thus, $\inf_{\mathfrak{m}\in\gamma_1\star\gamma_1}((\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{m})) \geq (\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1)$ and

$$\begin{aligned}
\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}((\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{r})) &= \sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}(\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\mathfrak{r})) \\
&= \inf_{k\in K}(\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}\upsilon_k^{\mathcal{N}}(\mathfrak{r})) \\
&\leq \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_1) \\
&= (\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1)
\end{aligned}$$

implies $\sup_{\mathfrak{r}\in\gamma_1\star\gamma_1}((\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{r})) \leq (\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1)$. Moreover, for any $\gamma_1,\gamma_2,\gamma_3\in U$, we have

$$\begin{aligned}
(\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1) &= \inf_{k\in K}\xi_k^{\mathcal{N}}(\gamma_1) \\
&\geq \inf_{k\in K}[\min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\xi_k^{\mathcal{N}}(\mathfrak{n}),\xi_k^{\mathcal{N}}(\gamma_2)\}] \\
&= \min\{\inf_{k\in K}(\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\xi_k^{\mathcal{N}}(\mathfrak{n})),\inf_{k\in K}\xi_k^{\mathcal{N}}(\gamma_2)\} \\
&= \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}(\inf_{k\in K}\xi_k^{\mathcal{N}}(\mathfrak{n})),\inf_{k\in K}\xi_k^{\mathcal{N}}(\gamma_2)\} \\
&= \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{n})),(\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_2)\}
\end{aligned}$$

Hence, $(\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_1) \geq \min\{\sup_{\mathfrak{n}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\mathfrak{n})),(\bigwedge_{k\in K}\xi_k^{\mathcal{N}})(\gamma_2)\}$ and

$$\begin{aligned}
(\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1) &= \inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_1) \\
&\leq \inf_{k\in K}[\max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\upsilon_k^{\mathcal{N}}(\mathfrak{y}),\upsilon_k^{\mathcal{N}}(\gamma_2)\}] \\
&= \max\{\inf_{k\in K}(\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}\upsilon_k^{\mathcal{N}}(\mathfrak{y})),\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_2)\} \\
&= \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}(\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\mathfrak{y})),\inf_{k\in K}\upsilon_k^{\mathcal{N}}(\gamma_2)\} \\
&= \max\{\inf_{\mathfrak{y}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{y})),(\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_2)\}.
\end{aligned}$$

Thus, $(\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_1) \leq \max\{\inf_{\mathfrak{r}\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)}((\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\mathfrak{y})),(\bigwedge_{k\in K}\upsilon_k^{\mathcal{N}})(\gamma_2)\}$, Therefore, $\bigwedge_{k\in K}\mathcal{N}_k$ is $\mathcal{IFSHPI}$ of U. □

4. Product Operations in Hyper BCK-Algebras

In this section, the product of various types of $\mathcal{IFHHI}$s is examined. Utilizing the transfer principle (see [11]) for $\mathcal{IFS}$s, it is demonstrated that the product of two $\mathcal{IFHHI}$s is likewise an $\mathcal{IFHHI}$.

Definition 4.1. *Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFS}$ of U. Then $\mathcal{IFS}$s $\mathcal{N}_1 = (\xi_1^{\mathcal{N}}, \upsilon_1^{\mathcal{N}})$ and $\mathcal{N}_2 = (\xi_2^{\mathcal{N}}, \upsilon_2^{\mathcal{N}})$ on U_1 and U_2 respectively, are defined $\mathcal{N}_1(\gamma_1) = \mathcal{N}((\gamma_1, 0))$, $\mathcal{N}_2(\gamma_2) = \mathcal{N}((0, g))$.*

Theorem 4.1. *Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFS}$ and let $\Upsilon^{\mathcal{N}} = (\xi_\Upsilon^{\mathcal{N}}, \upsilon_\Upsilon^{\mathcal{N}})$ be the strongest intuitionistic fuzzy relation on U. $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFSHPI}$ if and only if $\Upsilon^{\mathcal{N}} = (\xi_\Upsilon^{\mathcal{N}}, \upsilon_\Upsilon^{\mathcal{N}})$ is an $\mathcal{IFSHPI}$ of $\mathsf{U} \times \mathsf{U}$.*

Proof. Let $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ be an $\mathcal{IFSHPI}$ of U. Consider

$$\begin{aligned}
\inf_{(\mathfrak{m},\mathfrak{n}) \in (\gamma_1,\chi)\star(\gamma_1,\chi)} \xi_\Upsilon^{\mathcal{N}}(\mathfrak{m},\mathfrak{n}) &= \inf_{(\mathfrak{m},\mathfrak{n}) \in (\gamma_1\star\gamma_1)\star(\chi\star\chi)} [\min\{\xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\mathfrak{n})\}] \\
&= \min\{\inf_{\mathfrak{m} \in (\gamma_1\star\gamma_1)} \xi^{\mathcal{N}}(\mathfrak{m}), \inf_{\mathfrak{n} \in \chi\star\chi} \xi^{\mathcal{N}}(\mathfrak{n})\} \\
&\geq \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\} \\
&= \xi_\Upsilon^{\mathcal{N}}(\gamma_1, \chi)
\end{aligned}$$

imply $\inf_{(\mathfrak{m},\mathfrak{n}) \in (\gamma_1,\chi)\star(\gamma_1,\chi)} \xi_\Upsilon^{\mathcal{N}}(\mathfrak{m},\mathfrak{n}) \geq \xi_\Upsilon^{\mathcal{N}}(\gamma_1, \chi)$ for all $(\gamma_1, \chi) \in \mathsf{U} \times \mathsf{U}$.

$$\begin{aligned}
\sup_{(\mathfrak{x},\mathfrak{y}) \in (\gamma_1,\chi)\star(\gamma_1,\chi)} \upsilon_\Upsilon^{\mathcal{N}}(\mathfrak{x},\mathfrak{y}) &= \sup_{(\mathfrak{x},\mathfrak{y}) \in (\gamma_1\star\gamma_1)\star(\chi\star\chi)} [\max\{\upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\mathfrak{y})\}] \\
&= \max\{\sup_{\mathfrak{x} \in (\gamma_1\star\gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \sup_{\mathfrak{y} \in \chi\star\chi} \upsilon^{\mathcal{N}}(\mathfrak{y})\} \\
&\leq \max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\} \\
&= \upsilon_\Upsilon^{\mathcal{N}}(\gamma_1, \chi)
\end{aligned}$$

hence $\sup_{(\mathfrak{x},\mathfrak{y}) \in (\gamma_1,\chi)\star(\gamma_1,\chi)} \upsilon_\Upsilon^{\mathcal{N}}(\mathfrak{x},\mathfrak{y}) \leq \upsilon_\Upsilon^{\mathcal{N}}(\gamma_1, \chi)\ \forall\ (\gamma_1, \chi) \in \mathsf{U} \times \mathsf{U}$.
Now, for any (γ_1, χ),$(\gamma_2, \varkappa)$,$(\gamma_3, \eta) \in \mathsf{U} \times \mathsf{U}$, consider $\xi_\Upsilon^{\mathcal{N}}(\gamma_1, \chi) = \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\}$

$$\begin{aligned}
&\geq \min\{\min\{\sup_{\mathfrak{j} \in (\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\gamma_2), \min\{\sup_{\mathfrak{w} \in (\chi\star\eta)\star(\varkappa\star\eta)} \xi^{\mathcal{N}}(\mathfrak{w}), \xi^{\mathcal{N}}(\varkappa)\}\} \\
&= \min\{\min\{\sup_{\mathfrak{j} \in (\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\gamma_2), \sup_{\mathfrak{w} \in (\chi\star\eta)\star(\varkappa\star\eta)} \xi^{\mathcal{N}}(\mathfrak{w}), \xi^{\mathcal{N}}(\varkappa)\}, \\
&\quad \min\{\xi^{\mathcal{N}}(\gamma_2), \xi^{\mathcal{N}}(\varkappa)\}\} \\
&= \min\{\min\{\sup(\xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\mathfrak{w}))\}, \xi_\Upsilon^{\mathcal{N}}(\gamma_2, \varkappa)\},
\end{aligned}$$

where $\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$ and $\mathfrak{w} \in (\chi \star \eta) \star (\varkappa \star \eta)$ implies

$$\xi_\Upsilon^{\mathcal{N}}(\gamma_2, \varkappa) \geq \min\{\sup\{\min(\xi^{\mathcal{N}}(\mathfrak{j}), \xi^{\mathcal{N}}(\mathfrak{w}))\}, \xi_\Upsilon^{\mathcal{N}}(\gamma_2, \varkappa)\},$$

where $\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, $\mathfrak{w} \in (\chi \star \eta) \star (\varkappa \star \eta)$ implies

$$\xi_\Upsilon^{\mathcal{N}}(\gamma_2, \varkappa) \geq \min\{\sup(\xi_\Upsilon^{\mathcal{N}}(\mathfrak{j}, \mathfrak{w})), \xi_\Upsilon^{\mathcal{N}}(\gamma_2, \varkappa)\},$$

where

$$\begin{aligned}(\mathfrak{j}, \mathfrak{w}) &\in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta)) \\ &= (((\gamma_1, \chi) \star (\gamma_3, \eta)) \star ((\gamma_2, \varkappa) \star (\gamma_3, \eta))).\end{aligned}$$

Now, consider

$$\begin{aligned}\upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_1, \chi) &= \max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\} \\ &\leq \max\{\max\{\inf_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\gamma_2), \\ &\quad \max\{\inf_{\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)} \upsilon^{\mathcal{N}}(\mathfrak{d}), \upsilon^{\mathcal{N}}(\varkappa)\}\}\} \\ &= \max\{\max\{\inf_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\gamma_2), \\ &\quad \inf_{\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)} \upsilon^{\mathcal{N}}(\mathfrak{d}), \upsilon^{\mathcal{N}}(\varkappa)\}, \max\{\upsilon^{\mathcal{N}}(\gamma_2), \upsilon^{\mathcal{N}}(\varkappa)\}\} \\ &= \max\{\max\{\inf(\upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\mathfrak{d}))\}, \upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2, \varkappa)\},\end{aligned}$$

where $\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$ and $\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)$ imply

$$\upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2, \varkappa) \leq \max\{\inf\{\max(\upsilon^{\mathcal{N}}(\mathfrak{a}), \upsilon^{\mathcal{N}}(\mathfrak{d}))\}, \upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2, \varkappa)\},$$

where $\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$, $\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)$,
thus $\upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2, \varkappa) \leq \max\{\inf(\upsilon^{\mathcal{N}}_{\Upsilon}(\mathfrak{a}, \mathfrak{d})), \upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2, \varkappa)\}$, where

$$\begin{aligned}(\mathfrak{a}, \mathfrak{d}) &\in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta)) \\ &= (((\gamma_1, \chi) \star (\gamma_3, \eta)) \star ((\gamma_2, \varkappa) \star (\gamma_3, \eta))).\end{aligned}$$

Thus, $\Upsilon^{\mathcal{N}} = (\xi^{\mathcal{N}}_{\Upsilon}, \upsilon^{\mathcal{N}}_{\Upsilon})$ is an $\mathcal{IFSHPI}$ of $\mathsf{U} \times \mathsf{U}$.

Conversely, let $\Upsilon^{\mathcal{N}} = (\xi^{\mathcal{N}}_{\Upsilon}, \upsilon^{\mathcal{N}}_{\Upsilon})$ be an $\mathcal{IFSHPI}$ of $\mathsf{U} \times \mathsf{U}$. Then we have $\inf_{(\mathfrak{m}, \mathfrak{n}) \in (\gamma_1, \chi) \star (\gamma_1, \chi)} \xi^{\mathcal{N}}_{\Upsilon}(\mathfrak{m}, \mathfrak{n}) \geq \xi^{\mathcal{N}}_{\Upsilon}(\gamma_1, \chi)$ for all $(\gamma_1, \chi) \in \mathsf{U} \times \mathsf{U}$,

$$\begin{aligned}&\Rightarrow \inf_{(\mathfrak{m}, \mathfrak{n}) \in (\gamma_1 \star \gamma_1, \chi \star \chi)} [\min\{\xi^{\mathcal{N}}(\mathfrak{m}), \xi^{\mathcal{N}}(\mathfrak{n})\}] \geq \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\} \\ &\Rightarrow \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}(\mathfrak{m}), \inf_{\mathfrak{n} \in (\chi \star \chi)} \xi^{\mathcal{N}}(\mathfrak{n})\} \geq \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\} \\ &\Rightarrow \{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}(\mathfrak{m}), \inf_{\mathfrak{n} \in (\chi \star \chi)} \xi^{\mathcal{N}}(\mathfrak{n})\} \geq \{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\} \\ &\Leftrightarrow \inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\gamma_1), \inf_{\mathfrak{n} \in (\chi \star \chi)} \xi^{\mathcal{N}}(\mathfrak{n}) \geq \xi^{\mathcal{N}}(\chi)\end{aligned}$$

$\forall\, \gamma_1, \chi \in \mathsf{U}$.

$\sup_{(\mathfrak{x}, \mathfrak{y}) \in (\gamma_1, \chi) \star (\gamma_1, \chi)} \upsilon^{\mathcal{N}}_{\Upsilon}(\mathfrak{x}, \mathfrak{y}) \leq \upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_1, \chi)$ for all $(\gamma_1, \chi) \in \mathsf{U} \times \mathsf{U}$

$$\Rightarrow \sup_{(\mathfrak{x}, \mathfrak{y}) \in (\gamma_1 \star \gamma_1, \chi \star \chi)} [\max\{\upsilon^{\mathcal{N}}(\mathfrak{x}), \upsilon^{\mathcal{N}}(\mathfrak{y})\}] \leq \max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\}$$

$$\Rightarrow \max\{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \sup_{\mathfrak{y}\in(\chi\star\chi)} \upsilon^{\mathcal{N}}(\mathfrak{y})\} \leq \max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\}$$

$$\Rightarrow \{\sup_{\mathfrak{x}\in(\gamma_1\star\gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{x}), \sup_{\mathfrak{y}\in(\chi\star\chi)} \upsilon^{\mathcal{N}}(\mathfrak{y})\} \leq \{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\}$$

$$\Leftrightarrow \sup_{\mathfrak{x}\in(\gamma_1\star\gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \upsilon^{\mathcal{N}}(\gamma_1), \sup_{\mathfrak{y}\in(\chi\star\chi)} \upsilon^{\mathcal{N}}(\mathfrak{y}) \leq \upsilon^{\mathcal{N}}(\chi)$$

$\forall\ \gamma_1, \chi \in \mathsf{U}$. Therefore, The initial condition is fulfilled for $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ to be considered an $\mathcal{IFSHPI}$. Note that being an $\mathcal{IFSHPI}$ $\mathsf{U}\times\mathsf{U}$, $\Upsilon^{\mathcal{N}} = (\xi^{\mathcal{N}}_{\Upsilon}, \upsilon^{\mathcal{N}}_{\Upsilon})$ is also an $\mathcal{IFWHPI}$ of $\mathsf{U}\times\mathsf{U}$ (by Theorem 3.2), hence $\Upsilon^{\mathcal{N}} = (\xi^{\mathcal{N}}_{\Upsilon}, \upsilon^{\mathcal{N}}_{\Upsilon})$ satisfies $\Upsilon^{\mathcal{N}}(0,0) \geq \Upsilon^{\mathcal{N}}(\gamma_1, f)$ for all $(0,0),(\gamma_1,f) \in \mathsf{U}\times\mathsf{U}$ implies $\min\{\xi^{\mathcal{N}}(0), \xi^{\mathcal{N}}(0)\} \geq \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\gamma_1)\} \Rightarrow \xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Now, for any (γ_1,χ),$(\gamma_2,\varkappa)$,$(\gamma_3,\eta)$$\in$ $\mathsf{U}\times\mathsf{U}$, $\Upsilon^{\mathcal{N}} = (\xi^{\mathcal{N}}_{\Upsilon}, \upsilon^{\mathcal{N}}_{\Upsilon})$ satisfies $\xi^{\mathcal{N}}_{\Upsilon}(\gamma_1,\chi) \geq \min\{\sup(\xi^{\mathcal{N}}_{\Upsilon}(\mathfrak{e}_0,\mathfrak{z}_0)), \xi^{\mathcal{N}}_{\Upsilon}(\gamma_2,\varkappa)\}$ where

$$\begin{aligned}(\mathfrak{e}_0,\mathfrak{z}_0) &\in (((\gamma_1,\chi)\star(\gamma_3,\eta))\star((\gamma_2,\varkappa)\star(\gamma_3,\eta)))\\ &= ((\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3),(\chi\star\eta)\star(\varkappa\star\eta))\end{aligned}$$

$$\Rightarrow \min\{\xi^{\mathcal{N}}(\gamma_1), \xi^{\mathcal{N}}(\chi)\} \geq \min\{\sup\{\min(\xi^{\mathcal{N}}(e_0)), \xi^{\mathcal{N}}(\mathfrak{z}_0)\}, \min\{\xi^{\mathcal{N}}(\gamma_2), \xi^{\mathcal{N}}(\varkappa)\}\}$$

where $(\mathfrak{e}_0,\mathfrak{z}_0) \in ((\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3),(\chi\star\eta)\star(\varkappa\star\eta))$. Putting $\gamma_1 = \gamma_2 = h = 0$, we get

$$\min\{\xi^{\mathcal{N}}(0), \xi^{\mathcal{N}}(\chi)\} \geq \min\{\sup\{\xi^{\mathcal{N}}(0), \xi^{\mathcal{N}}(\mathfrak{z}_0)\}, \min\{\xi^{\mathcal{N}}(0), \xi^{\mathcal{N}}(\varkappa)\}\}$$

where $(\mathfrak{e}_0,\mathfrak{z}_0) \in (0,((\chi\star\eta)\star(\varkappa\star\eta)))$ implies

$$\xi^{\mathcal{N}}(\chi) \geq \min\{\sup_{\mathfrak{z}_0\in(\chi\star\eta)\star(\varkappa\star\eta)} \xi^{\mathcal{N}}(\mathfrak{z}_o), \xi^{\mathcal{N}}(\varkappa)\},$$

since $\xi^{\mathcal{N}}(0) \geq \xi^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Similarly, putting $\chi = \varkappa = \eta = 0$, we get

$$\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\sup_{\mathfrak{e}_0\in(\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3)} \xi^{\mathcal{N}}(\mathfrak{e}_o), \xi^{\mathcal{N}}(\gamma_2)\}.$$

Again,

$$\upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_1,\chi) \leq \max\{\inf(\upsilon^{\mathcal{N}}_{\Upsilon}(\mathfrak{e}_3,\mathfrak{z}_3)), \upsilon^{\mathcal{N}}_{\Upsilon}(\gamma_2,\varkappa)\}$$

where

$$\begin{aligned}(\mathfrak{e}_3,\mathfrak{z}_3) &\in (((\gamma_1,\chi)\star(\gamma_3,\eta))\star((\gamma_2,\varkappa)\star(\gamma_3,\eta)))\\ &= ((\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3),(\chi\star\eta)\star(\varkappa\star\eta))\end{aligned}$$

imply

$$\max\{\upsilon^{\mathcal{N}}(\gamma_1), \upsilon^{\mathcal{N}}(\chi)\} \leq \max\{\inf\{\max(\upsilon^{\mathcal{N}}(e_3)), \upsilon^{\mathcal{N}}(\mathfrak{z}_3))\}, \max\{\upsilon^{\mathcal{N}}(\gamma_2), \upsilon^{\mathcal{N}}(\varkappa)\}\}$$

where $(\mathfrak{e}_3,\mathfrak{z}_3) \in ((\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3),(\chi\star\eta)\star(\varkappa\star\eta))$. Putting $\gamma_1 = \gamma_2 = h = 0$, we obtain

$$\max\{\upsilon^{\mathcal{N}}(0), \upsilon^{\mathcal{N}}(\chi)\} \leq \max\{\inf\{\max(\upsilon^{\mathcal{N}}(0)), \upsilon^{\mathcal{N}}(\mathfrak{z}_3))\}, \max\{\upsilon^{\mathcal{N}}(0), \upsilon^{\mathcal{N}}(\varkappa)\}\}$$

where $(\mathfrak{e}_3, \mathfrak{z}_3) \in (0, ((\chi \star \eta) \star (\varkappa \star \eta)))$ implies

$$\upsilon^{\mathcal{N}}(\chi) \leq \max\{\inf_{\mathfrak{z}_3 \in (\chi \star \eta) \star (\varkappa \star \eta)} \upsilon^{\mathcal{N}}(\mathfrak{z}_3), \upsilon^{\mathcal{N}}(\varkappa)\},$$

since $\upsilon^{\mathcal{N}}(0) \leq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Similarly, putting $\chi = \varkappa = \eta = 0$, we get

$$\upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\inf_{\mathfrak{e}_3 \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{e}_3), \upsilon^{\mathcal{N}}(\gamma_2)\}.$$

Therefore, $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IFSHPI}$ of U. □

Theorem 4.2. *Let $\psi : \mathsf{U} \to \mathsf{U}'$ be an onto $\mathcal{HBCKA}$s from an $\mathcal{HBCKA}$ U to an $\mathcal{HBCKA}$ U'. $\mu = (\xi^{\mathcal{N}}_{\mu}, \upsilon^{\mathcal{N}}_{\mu})$ is an $\mathcal{IFSHPI}$ of U' then the hyper homomorphic pre-image $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ of $\mu = (\xi^{\mathcal{N}}_{\mu}, \upsilon^{\mathcal{N}}_{\mu})$ under $\psi = (\xi^{\mathcal{N}}_{\psi}, \upsilon^{\mathcal{N}}_{\psi})$ is an $\mathcal{IFSHPI}$ of U.*

Proof. Let $\psi = (\xi^{\mathcal{N}}_{\psi}, \upsilon^{\mathcal{N}}_{\psi})$ be an $\mathcal{IFSHPI}$ of U'. Since $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is a hyper homomorphic pre-image of $\mu = (\xi^{\mathcal{N}}_{\mu}, \upsilon^{\mathcal{N}}_{\mu})$ under $\psi = (\xi^{\mathcal{N}}_{\psi}, \upsilon^{\mathcal{N}}_{\psi})$, so $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is defined by $\mathcal{N} = \mu \star \psi$, that is, $\xi^{\mathcal{N}}(\gamma_1) = \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\gamma_1))$ and $\upsilon^{\mathcal{N}}(\gamma_1) = \upsilon^{\mathcal{N}}_{\mu}(\upsilon^{\mathcal{N}}_{\psi}(\gamma_1))$ for all $\gamma_1 \in K$. Since $\mu = (\xi^{\mathcal{N}}_{\mu}, \upsilon^{\mathcal{N}}_{\mu})$ satisfies

$$\inf_{\xi^{\mathcal{N}}_{\psi}(\mathfrak{m}) \in \xi^{\mathcal{N}}_{\psi}(\gamma_1) \star \xi^{\mathcal{N}}_{\psi}(\gamma_1) = \xi^{\mathcal{N}}_{\psi}(\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\mathfrak{m})) \geq \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\gamma_1))$$

for all $\gamma_1 \in \mathsf{U}$ and $\xi^{\mathcal{N}}_{\psi}(\gamma_1) \in \mathsf{U}'$ implies $\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_1)} \xi^{\mathcal{N}}(\mathfrak{m}) \geq \xi^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. Now,

$$\sup_{\upsilon^{\mathcal{N}}_{\psi}(\mathfrak{x}) \in \upsilon^{\mathcal{N}}_{\psi}(\gamma_1) \star \upsilon^{\mathcal{N}}_{\psi}(\gamma_1) = \upsilon^{\mathcal{N}}_{\psi}(\gamma_1 \star \gamma_1)} \upsilon^{\mathcal{N}}_{\mu}(\upsilon^{\mathcal{N}}_{\psi}(\mathfrak{x})) \leq \upsilon^{\mathcal{N}}_{\mu}(\upsilon^{\mathcal{N}}_{\psi}(\gamma_1))$$

for all $\gamma_1 \in \mathsf{U}$ and $\upsilon^{\mathcal{N}}_{\psi}(\gamma_1) \in \mathsf{U}'$ implies $\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_1)} \upsilon^{\mathcal{N}}(\mathfrak{x}) \leq \upsilon^{\mathcal{N}}(\gamma_1)$ for all $\gamma_1 \in \mathsf{U}$. For any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$, consider

$$\begin{aligned} \xi^{\mathcal{N}}(\gamma_1) &= \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\gamma_1)) \\ &\geq \min\{\sup_{\xi^{\mathcal{N}}_{\psi}(\mathfrak{n}) \in (\xi^{\mathcal{N}}_{\psi}(\gamma_1) \star \gamma_3') \star (\gamma_2' \star \gamma_3')} \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\mathfrak{n})), \xi^{\mathcal{N}}_{\psi}(\gamma_2')\} \end{aligned}$$

where $\gamma_2', \gamma_3' \in \mathsf{U}'$. Since $\psi : \mathsf{U} \to \mathsf{U}'$ be an onto $\mathcal{HBCKA}$s, for $\gamma_2', \gamma_3' \in \mathsf{U}'$, there exist $\gamma_2, \gamma_3 \in \mathsf{U}$ such that $\xi^{\mathcal{N}}_{\psi}(\gamma_2) = \gamma_2'$ and $\xi^{\mathcal{N}}_{\psi}(\gamma_3) = \gamma_3'$. Therefore, we get

$$\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\sup_{\xi^{\mathcal{N}}_{\psi}(\mathfrak{n}) \in (\xi^{\mathcal{N}}_{\psi}(\gamma_1) \star \xi^{\mathcal{N}}_{\psi}(\gamma_3)) \star (\xi^{\mathcal{N}}_{\psi}(\gamma_2) \star \xi^{\mathcal{N}}_{\psi}(\gamma_3))} \xi^{\mathcal{N}}_{\mu}(\xi^{\mathcal{N}}_{\psi}(\mathfrak{n})), \xi^{\mathcal{N}}_{\psi}(\xi^{\mathcal{N}}_{\psi}(\gamma_2))\},$$

thus $\xi^{\mathcal{N}}(\gamma_1) \geq \min\{\sup_{\mathfrak{n} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi^{\mathcal{N}}(\mathfrak{n}), \xi^{\mathcal{N}}(\gamma_2)\}$ for all $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$. Now, consider

$$\begin{aligned} \upsilon^{\mathcal{N}}(\gamma_1) &= \upsilon^{\mathcal{N}}_{\mu}(\upsilon^{\mathcal{N}}_{\psi}(\gamma_1)) \\ &\leq \max\{\inf_{\upsilon^{\mathcal{N}}_{\psi}(\mathfrak{y}) \in (\upsilon^{\mathcal{N}}_{\psi}(\gamma_1) \star \gamma_3') \star (\gamma_2' \star \gamma_3')} \upsilon^{\mathcal{N}}_{\mu}(\upsilon^{\mathcal{N}}_{\psi}(\mathfrak{y})), \upsilon^{\mathcal{N}}_{\psi}(\gamma_2')\} \end{aligned}$$

where $\gamma_2', \gamma_3' \in \mathsf{U}'$. Since $\psi : \mathsf{U} \to \mathsf{U}'$ be an onto $\mathcal{HBCKA}$s, for $\gamma_2', \gamma_3' \in \mathsf{U}'$, there exist $\gamma_2, \gamma_3 \in \mathsf{U}$ such that $\upsilon_\psi^{\mathcal{N}}(\gamma_2) = \gamma_2'$, and $\upsilon_\psi^{\mathcal{N}}(\gamma_3) = \gamma_3'$. Therefore,

$$\upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\inf_{\upsilon_\psi^{\mathcal{N}}(\mathfrak{y}) \in (\upsilon_\psi^{\mathcal{N}}(\gamma_1) \star \upsilon_\psi^{\mathcal{N}}(\gamma_3)) \star (\upsilon_\psi^{\mathcal{N}}(\gamma_2) \star \upsilon_\psi^{\mathcal{N}}(\gamma_3))} \upsilon_\mu^{\mathcal{N}}(\upsilon_\psi^{\mathcal{N}}(\mathfrak{y})), \upsilon_\psi^{\mathcal{N}}(\upsilon_\psi^{\mathcal{N}}(\gamma_2))\},$$

thus $\upsilon^{\mathcal{N}}(\gamma_1) \leq \max\{\inf_{\mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon^{\mathcal{N}}(\mathfrak{y}), \upsilon^{\mathcal{N}}(\gamma_2)\}$ for all $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}$. □

Additionally, we discuss the product of two $\mathcal{IFHPI}$s, referring to [9] for background on the product of fuzzy hyper BCK-ideals.

Theorem 4.3. *An $\mathcal{IFS}$ $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ that is, $(\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}}) = (\xi_1^{\mathcal{N}}, \upsilon_1^{\mathcal{N}}) \times (\xi_2^{\mathcal{N}}, \upsilon_2^{\mathcal{N}})$ is an intuitionistic fuzzy (weak, strong) hyper p-ideal of $\mathsf{U} = \mathsf{U}_1 \times \mathsf{U}_2$ if and only if $\mathcal{N}_1 = (\xi_1^{\mathcal{N}}, \upsilon_1^{\mathcal{N}})$, and $\mathcal{N}_2 = (\xi_2^{\mathcal{N}}, \upsilon_2^{\mathcal{N}})$ are intuitionistic fuzzy (weak, strong) hyper p-ideals of U_1, and U_2.*

Proof. Let $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ be an $\mathcal{IFHPI}$ of $\mathsf{U} = \mathsf{U}_1 \times \mathsf{U}_2$ and let $\gamma_1 \ll \chi$ for some $\gamma_1, \chi \in \mathsf{U}_1$. Then $(\gamma_1, 0) \ll (\chi, 0) \Rightarrow \xi^{\mathcal{N}}(\gamma_1, 0) = \xi_1^{\mathcal{N}}(\gamma_1) \geq \xi^{\mathcal{N}}(\chi, 0) = \xi_1^{\mathcal{N}}(\chi)$ and $\upsilon^{\mathcal{N}}(\gamma_1, 0) = \upsilon_1^{\mathcal{N}}(\gamma_1) \leq \upsilon^{\mathcal{N}}(\chi, 0) = \upsilon_1^{\mathcal{N}}(\chi)$, that is, $\xi_1^{\mathcal{N}}(\gamma_1) \geq \xi_1^{\mathcal{N}}(\chi)$ and $\upsilon_1^{\mathcal{N}}(\gamma_1) \leq \upsilon_1^{\mathcal{N}}(\chi)$. Moreover, for any $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}_1$, let

$$\mathfrak{s} = \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi_1^{\mathcal{N}}(\mathfrak{m}), \xi_1^{\mathcal{N}}(\gamma_2)\},$$

then for all $\mathfrak{n} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$,

$$\xi_1^{\mathcal{N}}(\mathfrak{n}) \geq \inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi_1^{\mathcal{N}}(\mathfrak{m}) \geq \mathfrak{s},$$

and $\xi_1^{\mathcal{N}}(\gamma_2) \geq \mathfrak{s}$ imply $\xi^{\mathcal{N}}(\mathfrak{n}, 0) \geq \mathfrak{s}$ and $\xi^{\mathcal{N}}(\gamma_2, 0) \geq \mathfrak{s}$ for all $(\mathfrak{n}, 0) \in ((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0))$ $\Rightarrow (\mathfrak{n}, 0) \in \xi_{\mathfrak{s}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \xi_{\mathfrak{s}}^{\mathcal{N}}$, $\Rightarrow ((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0)) \subseteq \xi_{\mathfrak{s}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \xi_{\mathfrak{s}}^{\mathcal{N}} \Rightarrow$ $((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0)) \ll \xi_{\mathfrak{s}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \xi_{\mathfrak{s}}^{\mathcal{N}}$ thus $(\gamma_1, 0) \in \xi_{\mathfrak{s}}^{\mathcal{N}}$, since $\xi_{\mathfrak{s}}^{\mathcal{N}}$ is an $\mathcal{HPI}$ by Theorem 3.3. Hence, $\xi^{\mathcal{N}}(\gamma_1, 0) \geq \mathfrak{s}$. Now, let

$$\mathfrak{v} = \max\{\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon_1^{\mathcal{N}}(\mathfrak{x}), \upsilon_1^{\mathcal{N}}(\gamma_2)\},$$

then for all $\mathfrak{y} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)$,

$$\upsilon_1^{\mathcal{N}}(\mathfrak{y}) \leq \sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon_1^{\mathcal{N}}(\mathfrak{x}) \leq \mathfrak{v},$$

and $\upsilon_1^{\mathcal{N}}(\gamma_2) \geq \mathfrak{v}$ imply $\upsilon^{\mathcal{N}}(\mathfrak{y}, 0) \leq \mathfrak{v}$ and $\upsilon^{\mathcal{N}}(\gamma_2, 0) \leq \mathfrak{v}$ for all $(\mathfrak{y}, 0) \in ((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0))$ implies $(\mathfrak{y}, 0) \in \upsilon_{\mathfrak{v}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \upsilon_{\mathfrak{v}}^{\mathcal{N}}$, implies $((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0)) \subseteq \upsilon_{\mathfrak{v}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \upsilon_{\mathfrak{v}}^{\mathcal{N}}$ which implies $((\gamma_1, 0) \star (\gamma_3, 0)) \star ((\gamma_2, 0) \star (\gamma_3, 0)) \ll \upsilon_{\mathfrak{v}}^{\mathcal{N}}$ and $(\gamma_2, 0) \in \upsilon_{\mathfrak{v}}^{\mathcal{N}}$ hence $(\gamma_1, 0) \in \upsilon_{\mathfrak{v}}^{\mathcal{N}}$, since $\upsilon_{\mathfrak{v}}^{\mathcal{N}}$ is an $\mathcal{HPI}$ by Theorem 3.3. Hence, $\upsilon^{\mathcal{N}}(\gamma_1, 0) \geq \mathfrak{v}$. Therefore,

$$\begin{aligned} \xi_1^{\mathcal{N}}(\gamma_1) &\geq \mathfrak{s} = \min\{\inf_{\mathfrak{m} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi_1^{\mathcal{N}}(\mathfrak{m}), \xi_1^{\mathcal{N}}(\gamma_2)\}, \text{ and} \\ \upsilon_1^{\mathcal{N}}(\gamma_1) &\leq \mathfrak{v} = \max\{\sup_{\mathfrak{x} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon_1^{\mathcal{N}}(\mathfrak{x}), \upsilon_1^{\mathcal{N}}(\gamma_2)\}. \end{aligned}$$

This condition satisfies our requirement. Similarly, it can be shown that $\mathcal{N}_2 = (\xi_2^{\mathcal{N}}, \upsilon_2^{\mathcal{N}})$ is an $\mathcal{IFHPI}$ of U_2.

Conversely, suppose that $\mathcal{N}_1 = (\xi_1^{\mathcal{N}}, \upsilon_1^{\mathcal{N}})$ and $\mathcal{N}_2 = (\xi_2^{\mathcal{N}}, \upsilon_2^{\mathcal{N}})$ are $\mathcal{IFHPI}$s of U_1 and U_2 respectively. For any (γ_1, χ),$(\gamma_2, \varkappa) \in \mathsf{U} = \mathsf{U}_1 \times \mathsf{U}_2$, where $\gamma_1, \gamma_2 \in \mathsf{U}_1$ and $\chi, \varkappa \in \mathsf{U}_2$, let $(\gamma_1, \chi) \ll (\gamma_2, \varkappa)$. Since $(\gamma_1, \chi) \ll (\gamma_2, \varkappa)$ implies $\gamma_1 \ll \gamma_2$ and $\chi \ll \varkappa$ imply that $\xi_1^{\mathcal{N}}(\gamma_1) \geq \xi_1^{\mathcal{N}}(\gamma_2)$ and $\xi_2^{\mathcal{N}}(\chi) \geq \xi_2^{\mathcal{N}}(\varkappa)$

$$\begin{aligned}
&\Rightarrow \quad \min\{\xi_1^{\mathcal{N}}(\gamma_1), \xi_2^{\mathcal{N}}(\chi)\} \geq \min\{\xi_1^{\mathcal{N}}(\gamma_2), \xi_2^{\mathcal{N}}(\varkappa)\} \\
&\Rightarrow \quad (\xi_1^{\mathcal{N}} \times \xi_2^{\mathcal{N}})(\gamma_1, \chi) \geq (\xi_1^{\mathcal{N}} \times \xi_2^{\mathcal{N}})(\gamma_2, \varkappa) \\
&\Rightarrow \quad \xi^{\mathcal{N}}(\gamma_1, \chi) \geq \xi^{\mathcal{N}}(\gamma_2, \varkappa).
\end{aligned}$$

Thus $(\gamma_1, \chi) \ll (\gamma_2, \varkappa)$ implies $\xi^{\mathcal{N}}(\gamma_1, \chi) \geq \xi^{\mathcal{N}}(\gamma_2, \varkappa)$.
Now, $\upsilon_1^{\mathcal{N}}(\gamma_1) \leq \upsilon_1^{\mathcal{N}}(\gamma_2)$ and $\upsilon_2^{\mathcal{N}}(\chi) \leq \upsilon_2^{\mathcal{N}}(\varkappa)$

$$\begin{aligned}
&\Rightarrow \quad \max\{\upsilon_1^{\mathcal{N}}(\gamma_1), \upsilon_2^{\mathcal{N}}(\chi)\} \leq \max\{\upsilon_1^{\mathcal{N}}(\gamma_2), \upsilon_2^{\mathcal{N}}(\varkappa)\} \\
&\Rightarrow \quad (\upsilon_1^{\mathcal{N}} \times \upsilon_2^{\mathcal{N}})(\gamma_1, \chi) \leq (\upsilon_1^{\mathcal{N}} \times \upsilon_2^{\mathcal{N}})(\gamma_2, \varkappa) \\
&\Rightarrow \quad \upsilon^{\mathcal{N}}(\gamma_1, \chi) \leq \upsilon^{\mathcal{N}}(\gamma_2, \varkappa).
\end{aligned}$$

Hence, $(\gamma_1, \chi) \ll (\gamma_2, \varkappa)$ implies $\upsilon^{\mathcal{N}}(\gamma_1, \chi) \leq \upsilon^{\mathcal{N}}(\gamma_2, \varkappa)$. Moreover,
for any $(\gamma_1, \chi), (\gamma_2, \varkappa), (\gamma_3, \eta) \in \mathsf{U}$, where $\gamma_1, \gamma_2, \gamma_3 \in \mathsf{U}_1$ and $\chi, \varkappa, \eta \in \mathsf{U}_2$, $\xi^{\mathcal{N}}(\gamma_1, \chi) = (\xi_1^{\mathcal{N}} \times \xi_2^{\mathcal{N}})(\gamma_1, \chi)$ $= \min\{\xi_1^{\mathcal{N}}(\gamma_1), \xi_2^{\mathcal{N}}(\chi)\}$

$$\begin{aligned}
&\geq \quad \min\{\min\{\inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi_1^{\mathcal{N}}(\mathfrak{j}), \xi_1^{\mathcal{N}}(\gamma_2)\}, \min\{\inf_{\mathfrak{w} \in (\chi \star \eta) \star (\varkappa \star \eta)} \xi_2^{\mathcal{N}}(\mathfrak{w}), \xi_2^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \min\{\min\{\inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \xi_1^{\mathcal{N}}(\mathfrak{j}), \inf_{\mathfrak{w} \in (\chi \star \eta) \star (\varkappa \star \eta)} \xi_2^{\mathcal{N}}(\mathfrak{w})\}, \min\{\xi_1^{\mathcal{N}}(\gamma_2), \xi_1^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \min\{\inf_{\mathfrak{j} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), \mathfrak{w} \in (\chi \star \eta) \star (\varkappa \star \eta)} \{\min\{\xi_1^{\mathcal{N}}(\mathfrak{j}), \xi_2^{\mathcal{N}}(\mathfrak{w})\}\}, \min\{\xi_1^{\mathcal{N}}(\gamma_2), \xi_2^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \min\{\inf_{(\mathfrak{j}, \mathfrak{w}) \in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta))} (\xi_1^{\mathcal{N}} \times \xi_2^{\mathcal{N}})(\mathfrak{j} \times \mathfrak{w}), (\xi_1^{\mathcal{N}} \times \xi_2^{\mathcal{N}})(\gamma_2 \times \varkappa)\} \\
&= \quad \min\{\inf_{(\mathfrak{j}, \mathfrak{w}) \in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta))} \xi^{\mathcal{N}}(\mathfrak{j}, \mathfrak{w}), \xi^{\mathcal{N}}(\gamma_2, \varkappa)\} \\
&\Rightarrow \quad \xi^{\mathcal{N}}(\gamma_1, \chi) \geq \min\{\inf_{(\mathfrak{j}, \mathfrak{w}) \in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta))} \xi^{\mathcal{N}}(\mathfrak{j}, \mathfrak{w}), \xi^{\mathcal{N}}(\gamma_2, \varkappa)\}.
\end{aligned}$$

Now, $\upsilon^{\mathcal{N}}(\gamma_1, \chi) = (\upsilon_1^{\mathcal{N}} \times \upsilon_2^{\mathcal{N}})(\gamma_1, \chi) = \max\{\upsilon_1^{\mathcal{N}}(\gamma_1), \upsilon_2^{\mathcal{N}}(\chi)\}$

$$\begin{aligned}
&\leq \quad \max\{\max\{\inf_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon_1^{\mathcal{N}}(\mathfrak{a}), \upsilon_1^{\mathcal{N}}(\gamma_2)\}, \max\{\inf_{\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)} \upsilon_2^{\mathcal{N}}(\mathfrak{d}), \upsilon_2^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \max\{\max\{\inf_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3)} \upsilon_1^{\mathcal{N}}(\mathfrak{a}), \inf_{\mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)} \upsilon_2^{\mathcal{N}}(\mathfrak{d})\}, \max\{\upsilon_1^{\mathcal{N}}(\gamma_2), \upsilon_1^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \max\{\inf_{\mathfrak{a} \in (\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), \mathfrak{d} \in (\chi \star \eta) \star (\varkappa \star \eta)} \{\max\{\upsilon_1^{\mathcal{N}}(\mathfrak{a}), \upsilon_2^{\mathcal{N}}(\mathfrak{d})\}\}, \max\{\upsilon_1^{\mathcal{N}}(\gamma_2), \upsilon_2^{\mathcal{N}}(\varkappa)\}\} \\
&= \quad \max\{\inf_{(\mathfrak{a}, \mathfrak{d}) \in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta))} (\upsilon_1^{\mathcal{N}} \times \upsilon_2^{\mathcal{N}})(\mathfrak{a} \times \mathfrak{d}), (\upsilon_1^{\mathcal{N}} \times \upsilon_2^{\mathcal{N}})(\gamma_2 \times \varkappa)\} \\
&= \quad \max\{\inf_{(\mathfrak{a}, \mathfrak{d}) \in ((\gamma_1 \star \gamma_3) \star (\gamma_2 \star \gamma_3), (\chi \star \eta) \star (\varkappa \star \eta))} \upsilon^{\mathcal{N}}(\mathfrak{a}, \mathfrak{d}), \upsilon^{\mathcal{N}}(\gamma_2, \varkappa)\},
\end{aligned}$$

$$\Rightarrow \quad \upsilon^{\mathcal{N}}(\gamma_1,\chi) \le \max\{\inf_{(\mathfrak{a},\mathfrak{d})\in((\gamma_1\star\gamma_3)\star(\gamma_2\star\gamma_3),(\chi\star\eta)\star(\varkappa\star\eta))} \upsilon^{\mathcal{N}}(\mathfrak{a},\mathfrak{d}), \upsilon^{\mathcal{N}}(\gamma_2,\varkappa)\}.$$

□

5. Conclusion

Based on our discussion, we conclude that an $\mathcal{IFSHPI}$ is a specific type of $\mathcal{IFHPI}$, which is itself an $\mathcal{IFWHPI}$. Furthermore, if $\mu = (\xi_\mu^{\mathcal{N}}, \upsilon_\mu^{\mathcal{N}})$ represents the strongest intuitionistic fuzzy relation on a hyper BCK-algebra, it forms an $\mathcal{IF}(\mathcal{W},\mathcal{S})\mathcal{HPI}$ when $\mathcal{N} = (\xi^{\mathcal{N}}, \upsilon^{\mathcal{N}})$ is an $\mathcal{IF}(\mathcal{W},\mathcal{S})\mathcal{HPI}$. Additionally, the hyper homomorphic pre-image of an $\mathcal{IF}(\mathcal{W},\mathcal{S})\mathcal{HPI}$ under an onto hyper homomorphism retains this property. The product of two $\mathcal{IF}(\mathcal{W},\mathcal{S})\mathcal{HPI}$s is also an $\mathcal{IF}(\mathcal{W},\mathcal{S})\mathcal{HPI}$.

This research advances fuzzy mathematical structures by integrating $\mathcal{IFS}$s with $\mathcal{HPI}$s in $\mathcal{HBCKA}$s, enabling sophisticated handling of uncertainty and vagueness. It enhances decision-making by capturing indeterminacy and ambiguity, making it relevant for AI, data analysis, and related applications. Introducing $\mathcal{IFHPI}$s deepens understanding of algebraic structures and yields new insights. The findings have interdisciplinary applications in computer science, engineering, economics, and other fields dealing with uncertainty. The comparative analysis with $\mathcal{IFHBCKI}$s guides future research and applications.

Future directions include applying $\mathcal{IFHPI}$s to decision-making, AI, and data analysis, extending them to other algebraic structures, and developing efficient algorithms for computation. Interdisciplinary collaboration and comparative studies with other models can facilitate real-world applications and enhance their utility. Integrating $\mathcal{IFHPI}$s with soft computing techniques and conducting practical case studies can demonstrate their effectiveness in handling uncertainty and complexity, paving the way for further research and applications.

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