

TC-AUGMENTATION OF GRAPHS

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ABSTRACT. A graph $G = \{V(G), E(G)\}$ is called a TC-graph if its vertex set admits a partition $V(G) = D_T \cup D_C$, where D_T is a total dominating set of G and D_C is a connected dominating set of G . In this study, we introduce the notion of TC-augmentation as follows: let $\mathcal{E} \subseteq E(\overline{G})$ be a set of edges, and define the augmentation $G_{\mathcal{E}}$ of G as the supergraph of G obtained by adding the edges in \mathcal{E} to G , that is, $G_{\mathcal{E}} = (V(G), E(G) \cup \mathcal{E})$. If $G_{\mathcal{E}}$ is a TC-graph, then it is called a TC-augmentation of G . Furthermore, the TC-augmentation number $tc(G)$ of a graph G is defined as the minimum cardinality of \mathcal{E} such that $G_{\mathcal{E}}$ is a TC-augmentation of G . This research investigates the TC-augmentation number of selected graph families, including star graphs $K_{1,n}$ and bi-star graphs $BS_{m,n}$, their complements, and graphs obtained through unary operations—particularly the Mycielskian construction applied to these families.

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1. INTRODUCTION

Graph theory provides a fundamental framework for modeling and analyzing complex systems in areas such as computer science, telecommunications, and social network analysis. By representing systems as vertices and edges, graph-theoretic models capture structural relationships in a precise mathematical form. Within this framework, domination theory has emerged as a central topic due to its wide-ranging applications in network monitoring, resource allocation, routing, and fault-tolerant system design [10].

The modern development of domination theory began with the seminal work of Cockayne and Hedetniemi in 1977, which unified earlier results on dominating sets and established a strong theoretical foundation for the field [2]. Since then, numerous variants of domination have been introduced to address specific structural and practical considerations, including total domination, independent domination, and connected domination [7]. These variants model different aspects of coverage, redundancy, and connectivity in networks.

As networks grow increasingly complex, recent research has focused on how domination parameters change when graphs are modified, particularly through edge additions or augmentation. Augmentation problems seek to determine the minimum number of edges required to endow a graph with a desired domination property, offering insight into both structural graph theory and network optimization [5]. Closely related are partitioning problems, which ask whether the vertex set of a graph can be divided into subsets satisfying specified domination conditions. Such partitions are often difficult to achieve, especially when multiple domination requirements must be satisfied simultaneously.

In this direction, Delgado, Desormeaux, and Haynes [3] introduced TI-partition and TI-graph, whose vertices can be partitioned into a total dominating set and an independent dominating set, along with the associated notion of TI-augmentation [8]. Motivated by these developments, the present study introduces a related but distinct domination-based structure: partitions of the vertex set of a graph into a total dominating set and a connected dominating set. A graph admitting such a partition is called *TC-graph*.

This research formally introduces the concept of *TC-augmentation number* of a graph, defined as the minimum number of edges that must be added to a graph so that it becomes a TC-graph. This notion extends existing work on domination partitions and augmentation problems by simultaneously incorporating total domination and connectivity constraints. The study investigates this parameter initially for several families of graphs, including star graphs $K_{1,n}$, bi-star graphs $BS_{m,n}$, their complements, and graphs obtained via the Mycielskian construction, contributing new insights to domination theory and laying groundwork for further research on structural graph transformations involving domination and connectivity.

Let $G = (V(G), E(G))$ be a simple undirected graph. Consider the following definitions by Chartrand et al: [1] The *complement* of a graph G , denoted by \overline{G} , is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$. For a nonempty subset S of $V(G)$, the *subgraph induced by S* , denoted by $G[S]$, has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G . Two vertices u and v in a graph G are *connected* if G contains a $u - v$ path. The graph G itself is connected if every two vertices of G are connected. A graph G that is not connected is a *disconnected graph*. The *degree* of a vertex v in a graph G , denoted by $\deg(v)$, is the number of vertices adjacent to v . A vertex of degree 0 is an *isolated vertex*, while a vertex of degree 1 is an *end vertex* or *leaf*. The smallest degree in G is the *minimum degree* $\delta(G)$. A set S of vertices of G is a *dominating set* of G if every vertex of $V(G) \setminus S$ is dominated by at least one vertex of S , that is, every vertex of $V(G) \setminus S$ is adjacent to at least one vertex of S . Moreover, a set $D_C \subseteq V(G)$ is a *connected dominating set* (CD-set) if it is a dominating set and the $G[D_C]$ is connected [11]. A set $D_T \subseteq V(G)$ is a *total dominating set* (TD-set) if every vertex of G is adjacent to some vertex in D_T [8]. A *k-partition* of G , denoted by (V_1, \dots, V_k) , is a family V_1, \dots, V_k of pairwise disjoint subsets with union $V(G)$ [12].

Let G be a graph with vertex set $V(G) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$. The *Mycielski graph* $\mu(G)$ is constructed as follows: (i) Introduce a new set of vertices $V' = \{v'_0, v'_1, v'_2, \dots, v'_{n-1}\}$ called the *shadow vertices* of G ; (ii) For every edge $(v_i, v_j) \in E(G)$, add the edges (v'_i, v'_j) and (v'_j, v_i) . Equivalently, each shadow vertex v'_i is adjacent to the neighbors of v_i in G ; and (iii) Add a new vertex w , called the *apex vertex*, and join w to every shadow vertex in V' [4].

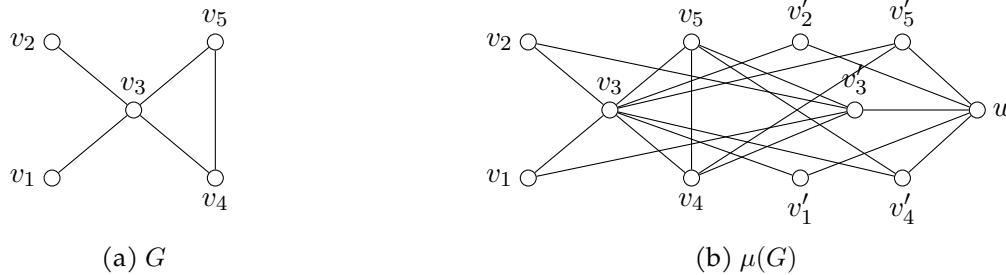


FIGURE 1. The graph G and its Mycielski graph $\mu(G)$.

2. THE TC-AUGMENTATION OF G

Definition 1. A graph $G = \{V(G), E(G)\}$ is a **TC-graph** if $V(G)$ admits a partition (D_T, D_C) in which D_T is a total dominating set (TD-set) and D_C is a connected dominating set (CD-set). Such a partition is called a **TC-partition** of $V(G)$.

Example 1. The graph G in Figure 2 is a TC-graph for the partition (D_T, D_C) where $D_T = \{v_1, v_9, v_5\}$ is a TD-set and $D_C = \{v_2, v_3, v_4, v_6, v_7, v_8\}$ is a CD-set.

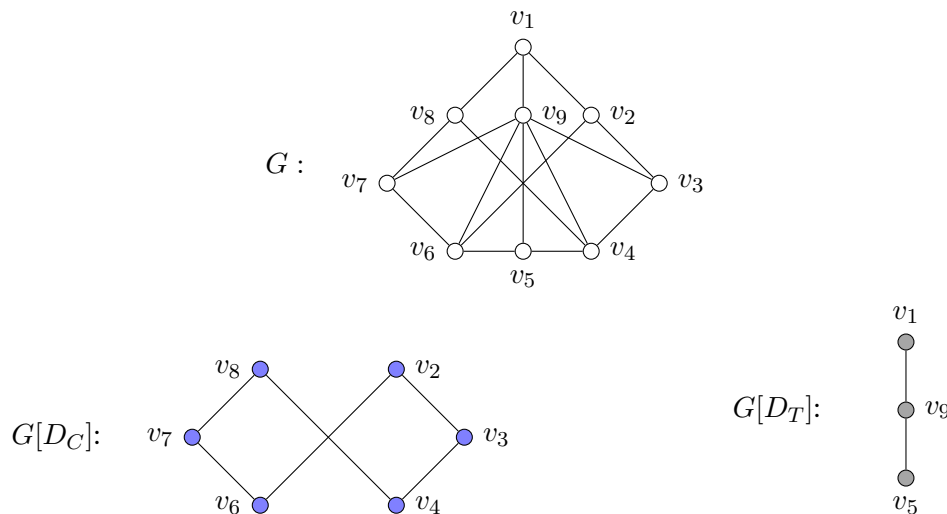


FIGURE 2. A TC-graph G .

Definition 2. Let $\mathcal{E} \subseteq E(\overline{G})$. The graph $G_{\mathcal{E}} = (V(G), E(G) \cup \mathcal{E})$ is a **TC-augmentation** of G if $G_{\mathcal{E}}$ is a TC-graph.

Definition 3. The **TC-augmentation number** of a graph G , denoted $\text{tc}(G)$, is the minimum number of edges that must be added to G so that the resulting graph is a TC-augmentation of G , that is,

$$\text{tc}(G) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq E(\overline{G}), G_{\mathcal{E}} \text{ is a TC-graph}\}.$$

Remark 1. For any graph G , one has $\text{tc}(G) \geq 0$, and equality holds if and only if G is a TC-graph.

Example 2. Consider the graph H in Figure 3. For any partition (P, Q) of $V(H)$, if without loss of generality, $v_3 \in P$ and $v_1, v_2 \in Q$, then Q fails to be a connected nor a total dominating set. If, on the other hand, v_3 and at least one of v_1 or v_2 belong to the same subset in the partition say P , then Q fails to be a dominating set since Q cannot dominate v_1 or v_2 . Hence, H cannot admit a TC-partition so that H is not a TC-graph.

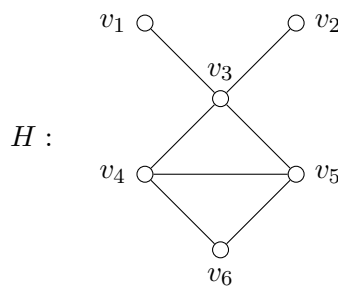


FIGURE 3. A graph H .

Now, let $\mathcal{E} = \{v_1v_2\} \subseteq E(\overline{H})$. Then the resulting graph $H_{\mathcal{E}} = (V(H), E(H) \cup \mathcal{E})$, illustrated in Figure 4, is a TC-graph and thus a TC-augmentation of H . It admits the TC-partition (D_T, D_C) of $V(H_{\mathcal{E}})$, where $D_T = \{v_1, v_2, v_5, v_6\}$ forms a TD-set of $H_{\mathcal{E}}$ and $D_C = \{v_3, v_4\}$ forms a CD-set of $H_{\mathcal{E}}$.

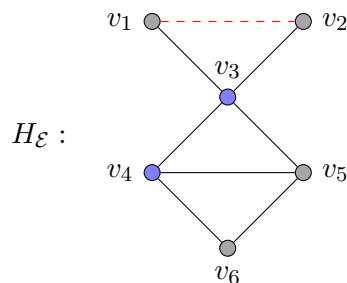


FIGURE 4. A TC-augmentation $H_{\mathcal{E}}$ of the graph H .

With $H_{\mathcal{E}}$ being a TC-augmentation of H such that $|\mathcal{E}| = 1$, one can see that $\text{tc}(H) = 1$.

Remark 2. If G has an isolated vertex, then G is not a TC-graph since for any partition of $V(G)$, the set in the partition not containing the isolated vertex will fail to be a dominating, connected, nor total dominating set.

Theorem 1. *If G is a disconnected graph, then G is not a TC-graph.*

Proof. Let G be a disconnected graph and B_1 and B_2 be components of G , and (P, Q) be a partition of $V(G)$. If $V(B_1) \subseteq P$ and $V(B_2) \subseteq Q$, then neither P nor Q can be a dominating set of G . If $V(B_1) \cap P \neq \emptyset$ and $V(B_2) \cap P \neq \emptyset$ or $V(B_1) \cap Q \neq \emptyset$ and $V(B_2) \cap Q \neq \emptyset$, then P or Q , respectively, cannot induce a connected subgraph of G . Thus, G cannot have a TC-partition. \square

Theorem 2. *Let G be a disconnected graph with TC-graph components B_1, \dots, B_n where $D_C^{(i)}$ is the CD-set of B_i , for each $i = 1, \dots, n$. If $\mathcal{E} = \{x_i x_{i+1} : i = 1, \dots, n - 1, x_i \in D_C^{(i)}\}$, then $G_{\mathcal{E}}$ is a TC-augmentation of G and $tc(G) = n - 1$.*

Proof. Observe that $\bigcup_{i=1}^n D_C^{(i)}$ is a dominating set, while $\bigcup_{i=1}^n D_T^{(i)}$ is a TD-set of G . Since $G[\bigcup_{i=1}^n D_C^{(i)}]$ has n connected components, adding exactly $n - 1$ edges to $G[\bigcup_{i=1}^n D_C^{(i)}]$ connecting a vertex x_i of $D_C^{(i)}$ to a vertex x_{i+1} of $D_C^{(i+1)}$ for $i = 1, \dots, n - 1$ produces a connected subgraph of G . Thus, for $\mathcal{E} = \{x_i x_{i+1} : i = 1, \dots, n, x_i \in D_C^{(i)}\}$, $G_{\mathcal{E}}$ is a TC-augmentation of G , with (D_C, D_T) as the TC-partition of $V(G)$ where $D_C = \bigcup_{i=1}^n D_C^{(i)}$ and $D_T = \bigcup_{i=1}^n D_T^{(i)}$. This means that

$$tc(G) \leq n - 1. \quad (\star)$$

Let (P, Q) be a partition of $V(G)$. Observe that for each TC-graph component B_i of G , exactly one of the following is true:

- (i) $V(B_i) \subseteq P$;
- (ii) $V(B_i) \subseteq Q$;
- (iii) $V(B_i) \cap P \neq \emptyset$ and $V(B_i) \cap Q \neq \emptyset$.

Now, let r, s, t be the number of components of G of types (i), (ii), and (iii), respectively. Then $r + s + t = n$. Without loss of generality, in order for P to be a CD-set, we need to add at least $r + t - 1$ edges so that $G[P]$ becomes connected and at least another s edges for P to dominate the vertices in type (ii) components. Thus, for P to be a CD-set, we need to add at least $(r + t - 1) + s = n - 1$ edges. Hence, with (\star) , one has $tc(G) = n - 1$. \square

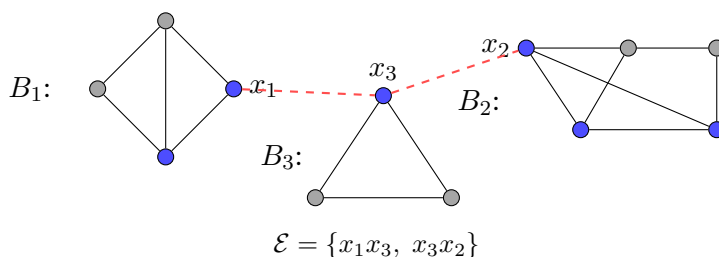


FIGURE 5. A disconnected graph G with TC-graph components and its TC-augmentation.

Theorem 3. Every connected graph G with a leaf vertex is not a TC-graph.

Proof. Let G be a connected graph with a leaf vertex u , and let v be the unique neighbor of u . Consider any partition of $V(G)$.

Case 1: If $G \cong P_2$, then $V(G) = \{u, v\}$. Observe that u and v belong to different subsets of the partition, so that the subset containing u fails to be a total dominating set since u has no neighbor other than v .

Case 2: If $G \not\cong P_2$, then $N(v) \setminus u \neq \emptyset$. If u and v belong to the same subset in the partition, then the other subset fails to be a dominating set. In particular, u can only be dominated by v , so any subset not containing v cannot dominate u . Hence, no TC-partition of $V(G)$ exists, and therefore G is not a TC-graph. \square

Remark 3. In view of Theorem 3, the graph families path graphs, helm graphs, star graphs, bi-star graphs, sunlet graphs, and all trees contain leaf vertices; hence, none of these are TC-graphs.

Corollary 1. If G is a TC-graph, then $\delta(G) \geq 2$.

Proof. Suppose, on the contrary, that $\delta(G) < 2$. Then G contains either an isolated vertex or a leaf vertex. By Remark 2 and Theorem 3, G is not a TC-graph. \square

Remark 4. The converse of Corollary 1 is not true in general. In particular, the cycle graph C_n for $n \geq 5$ admits no TC-partition, since any partition forces one part to be disconnected or fails to be a dominating set.

3. THE TC-AUGMENTATION OF A STAR GRAPH $K_{1,n}$

Definition 4. [6] A *star graph*, denoted by $K_{1,n}$, is a graph of order $n + 1$ in which one vertex, called the apex v_0 , has degree n , and the remaining n vertices each have degree 1, that is,

$$V(K_{1,n}) = \{v_0, u_1, \dots, u_n\},$$

$$E(K_{1,n}) = \{\{v_0, u_i\} : 1 \leq i \leq n\}.$$

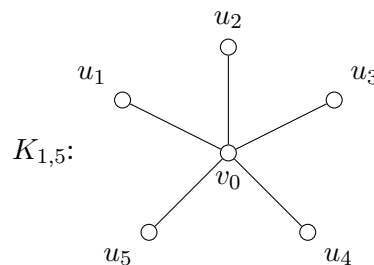


FIGURE 6. The Star Graph $K_{1,5}$.

Theorem 4. The TC-augmentation number of a star graph $K_{1,n}$ with $n \geq 2$ is

$$\text{tc}(K_{1,n}) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. By Theorem 3, for all $n \geq 2$, the star graph $K_{1,n}$ is not a TC-graph.

Consider $G = K_{1,n}$ with vertex set $V(G) = \{v_0, u_1, \dots, u_n\}$ where the vertex v_0 is the apex and u_1, \dots, u_n are the leaf vertices. Let $D_C, D_T \subseteq V(G)$ such that

$$D_C = \{v_0\}, \quad D_T = \{u_i : i = 1, 2, \dots, n\}.$$

Define $\mathcal{E}^* \subseteq E(\overline{G})$ by

$$\mathcal{E}^* = \begin{cases} \{u_1u_2, u_3u_4, \dots, u_{n-1}u_n\}, & \text{if } n \text{ is even,} \\ \{u_1u_2, u_3u_4, \dots, u_{n-2}u_{n-1}, u_nu_1\}, & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, D_C is a CD-set and D_T is a TD-set via the augmentation \mathcal{E}^* . Consequently, (D_C, D_T) forms a TC-partition of $V(G_{\mathcal{E}^*})$ so that $G_{\mathcal{E}^*}$ is a TC-graph and

$$\text{tc}(G) \leq |\mathcal{E}^*| = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$\text{tc}(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

Next, let (P, Q) be any partition of $V(G)$, with both subsets containing at least one leaf vertex. Without loss of generality, let $v_0 \in P$ and k be the number of leaf vertices in P so that $n - k$ is the number of leaf vertices in Q . Observe that P is both a CD-set and TD-set.

If we put P to be the CD-set, then we need k edges to be added to G connecting the k leaf vertices in P to some vertices in Q so that Q becomes a dominating set. Subsequently, we need at least $\lceil \frac{n-k}{2} \rceil$ edges to connect the vertices of Q for it to be a TD-set. In total, we need a minimum of $\lceil \frac{n-k}{2} \rceil + k$ edges for Q to be a TD-set and for (P, Q) to be a TC-partition. Observe that as we increase the value of k , the sum

$$\left\lceil \frac{n-k}{2} \right\rceil + k = \begin{cases} \frac{n+k+1}{2}, & \text{if } n-k \text{ is odd,} \\ \frac{n+k}{2}, & \text{if } n-k \text{ is even} \end{cases}$$

also increases. Thus, the minimum value of such sum is attained when $k = 1$, that is

$$\left\lceil \frac{n+1}{2} \right\rceil = \begin{cases} \frac{n+2}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

These values are greater than $|\mathcal{E}^*| = \left\lceil \frac{n}{2} \right\rceil$.

Moreover, if we assign P as the TD-set, we need to add k edges in G to connect the k leaf vertices in P to some vertices in Q and then add at least $(n - k) - 1$ edges to connect all the $n - k$ vertices of Q . In total, we need to add $k + (n - k) - 1 = n - 1$ edges for Q to be CD-set. But $n - 1 > \lceil \frac{n}{2} \rceil = |\mathcal{E}^*|$. Since (P, Q) is arbitrary, we see that

$$|\mathcal{E}^*| = \lceil \frac{n}{2} \rceil = \min\{|\mathcal{E}| : G_{\mathcal{E}} \text{ is a TC-graph}\}.$$

Therefore,

$$tc(G) = \lceil \frac{n}{2} \rceil.$$

□

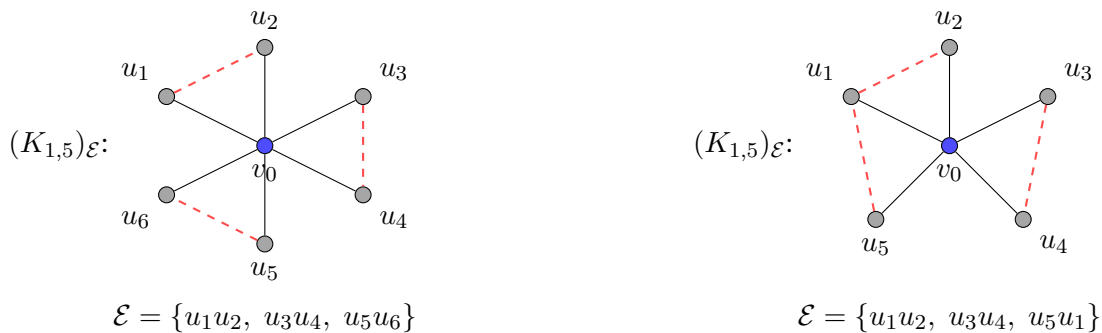


FIGURE 7. An augmented star graphs $(K_{1,5})_{\mathcal{E}}$ and $(K_{1,6})_{\mathcal{E}}$.

Remark 5. The complement of the star graph $K_{1,n}$ consists of a complete graph K_n induced by $\{u_1, \dots, u_n\}$ together with an isolated vertex v_0 . Hence, in view of Remark 2 the complement $\overline{K_{1,n}} \cong K_n \cup K_1$ is not a TC-graph.

Proposition 1. For $n \geq 2$, the TC-augmentation number of the complement $\overline{K_{1,n}}$ is

$$tc(\overline{K_{1,n}}) = 2.$$

Proof. Let $G = K_{1,n}$, where $n \geq 2$. By Remark 5, $\overline{G} \cong K_n \cup K_1$, is not a TC-graph. Moreover, for any partition of $V(G)$, adding a single edge to \overline{G} to dominate v_0 produces a leaf vertex, and by Theorem 3, the resulting graph still cannot be a TC-graph. Hence, $tc(\overline{G}) \geq 2$. (★)

Now, let $D_C, D_T \subseteq V(\overline{G})$ where

$$D_C = \{u_i : 2 \leq i \leq n\} \quad \text{and} \quad D_T = \{v_0, u_1\}.$$

Let $\mathcal{E} \subseteq E(G)$ such that $\mathcal{E} = \{v_0u_1, v_0u_n\}$. Since the vertices $\{u_1, \dots, u_n\}$ induce a complete graph, the set D_C induces a connected subgraph of $\overline{G}_{\mathcal{E}}$. Moreover, every vertex outside D_C has a neighbor in D_C via the augmentation. Thus, D_C is a CD-set of $\overline{G}_{\mathcal{E}}$. Furthermore, the vertices v_0 and u_1 are adjacent via the added edge v_0u_1 , so each vertex in D_T has a neighbor in D_T . Every vertex u_i , for $2 \leq i \leq n$ is adjacent to u_1 , and hence is dominated by D_T . Therefore, D_T is a TD-set of $\overline{G}_{\mathcal{E}}$. Consequently, (D_C, D_T)

forms a TC-partition of $V(\overline{G}_\mathcal{E})$ so that $\overline{G}_\mathcal{E}$ is TC-graph. This shows that $tc(\overline{G}) \leq 2$, and hence, with (\star) , one has $tc(\overline{G}) = 2$. □

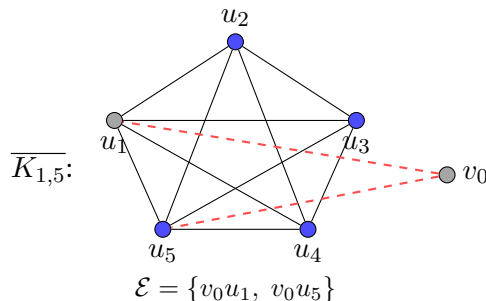


FIGURE 8. An augmented complement $\overline{K}_{1,5}$ with $tc(\overline{K}_{1,5}) = 2$

Proposition 2. For $n \geq 2$, the Mycielskian graph $\mu(K_{1,n})$ is a TC-graph.

Proof. Let $G = K_{1,n}$ with center v_0 and leaves u_1, \dots, u_n . Let $\mu(G)$ be the Mycielskian of G with shadow vertices v'_0, u'_1, \dots, u'_n and apex vertex w .

We claim that $V(\mu(G))$ admits a TC-partition (D_C, D_T) where

$$D_C = \{w, v'_0, u'_1\} \quad \text{and} \quad D_T = V(\mu(G)) \setminus D_C.$$

Since w is adjacent to every shadow vertex, in particular $wv'_0 \in E(\mu(G))$ and $wu'_1 \in E(\mu(G))$, so $\mu(G)[D_C]$ is connected. Moreover, v'_0 is adjacent to every original leaf u_i , u'_1 is adjacent to v_0 , and w is adjacent to every shadow vertex u'_i ($i \geq 2$). Hence every vertex in $V(\mu(G)) \setminus D_C$ has a neighbor in D_C , so D_C is a CD-set of $\mu(G)$.

The set D_T contains $V(K_{1,n})$, so each original vertex has a neighbor in D_T . Also, for $i \geq 2$, each shadow vertex $u'_i \in D_T$ is adjacent to $v_0 \in D_T$. Finally, each vertex of D_C has a neighbor in D_T say w is adjacent to u'_2, v'_0 is adjacent to u_2 , and u'_1 is adjacent to v_0 . Therefore every vertex of $\mu(G)$ has a neighbor in D_T , and D_T is a TD-set of $\mu(G)$. Therefore, (D_C, D_T) is a TC-partition of $V(\mu(G))$ so that $\mu(G)$ is a TC-graph for all $n \geq 2$. □

Remark 6. In view of Remark 1, for $n \geq 2$, $tc(\mu(K_{1,n})) = 0$.

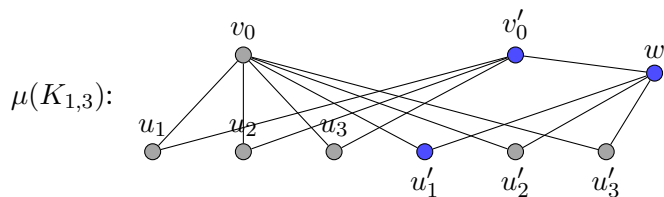


FIGURE 9. A TC-partition of the Mycielskian graph $\mu(K_{1,3})$.

4. THE TC-AUGMENTATION OF A BI-STAR GRAPH $BS_{m,n}$

Definition 5. [9] Let $m, n \geq 2$. The bi-star graph $BS_{m,n}$ is a graph with vertex set

$$V(BS_{m,n}) = \{u_0, v_0\} \cup \{u_1, \dots, u_m\} \cup \{v_1, \dots, v_n\},$$

and edge set

$$E(BS_{m,n}) = \{u_0v_0\} \cup \{u_0u_i : 1 \leq i \leq m\} \cup \{v_0v_j : 1 \leq j \leq n\},$$

where u_0 and v_0 are the central vertices of the two stars.

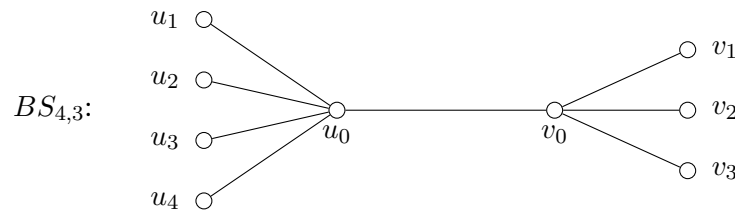


FIGURE 10. The bi-star graph $BS_{4,3}$.

Theorem 5. The TC-augmentation number of the bi-star graph $BS_{m,n}$ with $m, n \geq 2$ is

$$\text{tc}(BS_{m,n}) = \left\lceil \frac{m+n}{2} \right\rceil.$$

Proof. By Theorem 3, the bi-star graph $BS_{m,n}$ is not a TC-graph for all $m, n \geq 2$.

Let $G = BS_{m,n}$ with $V(G) = \{u_0, v_0\} \cup \{u_1, \dots, u_m\} \cup \{v_1, \dots, v_n\}$, and $E(G) = \{u_0v_0\} \cup \{u_0u_i : 1 \leq i \leq m\} \cup \{v_0v_j : 1 \leq j \leq n\}$, where $m, n \geq 2$.

Consider $D_C, D_T \subseteq V(G)$ such that

$$D_C = \{u_0, v_0\} \quad \text{and} \quad D_T = \{u_1, \dots, u_m\} \cup \{v_1, \dots, v_n\}.$$

Without loss of generality, let $m > n$. Define $\mathcal{E}' \subseteq E(\overline{G})$ as follows:

- If $m - n$ is even,

$$\mathcal{E}' = \{\{u_1v_1, u_2v_2, \dots, u_nv_n\} \cup \{u_{n+1}u_{n+2}, u_{n+3}u_{n+4}, \dots, u_{m-1}u_m\}\}$$

- If $m - n$ is odd,

$$\mathcal{E}' = \{\{u_1v_1, u_2v_2, \dots, u_nv_n\} \cup \{u_{n+1}u_{n+2}, u_{n+3}u_{n+4}, \dots, u_{m-2}u_{m-1}, u_mv_n\}\}$$

Observe that $G[D_C]$ is connected since $u_0v_0 \in E(G)$, and D_C dominates G because every leaf vertex u_i and v_j is adjacent to either u_0 or v_0 . Hence, D_C is a CD-set of G . Moreover, D_T is a TD-set, because each center vertex u_0 and v_0 has a neighbor in D_T , and each leaf vertex in D_T has a neighbor in D_T

via the augmentation \mathcal{E}' . Therefore, (D_C, D_T) is a TC-partition of $V(G_{\mathcal{E}'})$, and so $G_{\mathcal{E}'}$ is a TC-graph. Consequently,

$$\text{tc}(G) \leq |\mathcal{E}'| = \begin{cases} n + \frac{m-n}{2}, & \text{if } m-n \text{ is even} \\ n + \frac{m-n+1}{2}, & \text{if } m-n \text{ is odd.} \end{cases}$$

Therefore, $\text{tc}(G) \leq |\mathcal{E}'| = n + \lceil \frac{m-n}{2} \rceil = \lceil \frac{m+n}{2} \rceil$.

Now, let (P, Q) be any partition of $V(G)$ and let:

P_u be the number of leaf vertices $u_i \in P$;

P_v be the number of leaf vertices $v_j \in P$;

Q_u be the number of leaf vertices $u_i \in Q$; and

Q_v be the number of leaf vertices $v_j \in Q$.

Consider the following cases:

Case 1: Suppose both central vertex $u_0, v_0 \in P$. Then, P is both a CD-set and TD-set of G . Putting P as the CD-set, we need $P_u + P_v$ edges to connect the $P_u + P_v$ leaf vertices in P to some vertices in Q for Q to be a dominating set. Moreover, for Q to be TD-set, we need to add at least $\lceil \frac{Q_u + Q_v}{2} \rceil$ edges to connect the vertices of Q . This implies we need to add

$$(P_u + P_v) + \left\lceil \frac{Q_u + Q_v}{2} \right\rceil = (P_u + P_v) + \left\lceil \frac{(m - P_u) + (n - P_v)}{2} \right\rceil = \left\lceil \frac{P_u + P_v + m + n}{2} \right\rceil$$

edges for (P, Q) to be a TC-partition. Since m and n are fixed, the minimum number of edges is attained when $P_u = P_v = 0$.

Now, if we put P to be the TD-set of G , we need to add $P_u + P_v$ edges in G to connect the $P_u + P_v$ leaf vertices in P to some vertices in Q for Q to be a dominating set and then add at least $(Q_u + Q_v) - 1$ edges to connect all the $Q_u + Q_v$ vertices of Q for Q to be a CD-set. In total, we need at least

$$(P_u + P_v) + (Q_u + Q_v - 1) = m + n - 1$$

edges for (P, Q) to be a TC-partition.

Case 2: Let $u_0 \in P$ and $v_0 \in Q$. For P and Q to be both dominating sets, we can pair v vertices in P to u vertices in Q , and connect the remaining undominated leaf vertices to any vertex of the other subset. To do this, we need at least $\max\{P_u, Q_v\} = \max\{m - Q_u, n - P_v\}$.

Now, if we assign P to be the CD-set, then we need to add at least P_v edges to connect all the vertices in P . Subsequently, we need to add at least $\lceil \frac{Q_u}{2} \rceil$ edges to connect the Q_u leaf vertices of Q to form a TD-set. In total, we need to add

$$\max\{m - Q_u, n - P_v\} + P_v + \left\lceil \frac{Q_u}{2} \right\rceil$$

edges for (P, Q) to be a TC-partition. Note that if $P = \{u_0, u_1, \dots, u_m\}$ and $Q = \{v_0, v_1, \dots, v_n\}$, we need to add at least $\max\{m, n\}$ edges in order for both subsets, P, Q to be a CD-set and TD-set and for (P, Q) to be a TC-partition. Thus, if $P_u = P_v = 0$, then we have $\max\{m, n\}$, and if on the other hand $Q_u = m$ and $P_v = n$, then we have $n + \lceil \frac{m}{2} \rceil$. In both cases,

$$\max\{m, n\} \geq \left\lceil \frac{m+n}{2} \right\rceil, \text{ and } n + \left\lceil \frac{m}{2} \right\rceil \geq \left\lceil \frac{m+n}{2} \right\rceil.$$

Hence, for any partition (P, Q) of $V(G)$, we see that

$$|\mathcal{E}'| = \left\lceil \frac{m+n}{2} \right\rceil = \min\{|\mathcal{E}| : G_{\mathcal{E}} \text{ is a TC-graph}\},$$

and so

$$\text{tc}(G) = \left\lceil \frac{m+n}{2} \right\rceil.$$

□

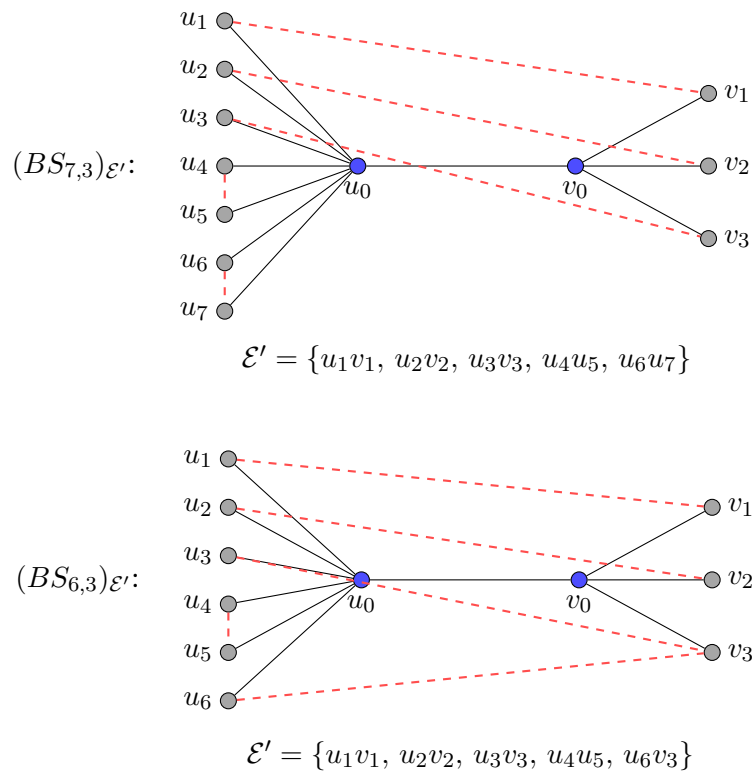


FIGURE 11. An augmented bi-star graphs $(BS_{6,3})_{\mathcal{E}'}$ and $(BS_{7,3})_{\mathcal{E}'}$.

Proposition 3. For $m, n \geq 2$, the complement $\overline{BS_{m,n}}$ is a TC-graph.

Proof. Let $G = BS_{m,n}$ with $m, n \geq 2$. In \overline{G} , the leaves induce a complete graph K_{m+n} , the vertex u_0 is adjacent to all v_j and not to any u_i , the vertex v_0 is adjacent to all u_i and not to any v_j , and u_0 and v_0 are nonadjacent. Consider $D_C, D_T \subseteq V(\overline{G})$ where

$$D_C = \{u_1, v_1\} \quad \text{and} \quad D_T = V(\overline{G}) \setminus D_C.$$

Since all leaf vertices induce a clique in \overline{G} , the subgraph $\overline{G}[D_C]$ is connected. Moreover, u_0 is adjacent to $v_1 \in D_C$, v_0 is adjacent to $u_1 \in D_C$, and every leaf outside D_C is adjacent to both u_1 and v_1 . Hence, D_C is a CD-set of \overline{G} . Next, every vertex in D_T has a neighbor in D_T . Indeed, u_0 is adjacent to $v_2 \in D_T$ and v_0 is adjacent to $u_2 \in D_T$, while all remaining vertices of D_T are leaves and therefore adjacent to other leaves in D_T . Furthermore, each vertex of D_C has a neighbor in D_T . Thus, D_T is a TD-set of \overline{G} . Consequently, (D_T, D_C) is a TC-partition of $V(\overline{G})$ so that \overline{G} is a TC-graph and $tc(\overline{G}) = 0$. \square

Proposition 4. For $m, n \geq 2$, the Mycielskian graph $\mu(BS_{m,n})$ is a TC-graph.

Proof. Let $G = BS_{m,n}$ with central vertices u_0 and v_0 , where u_0 is adjacent to u_1, \dots, u_m and v_0 is adjacent to v_1, \dots, v_n , and $u_0v_0 \in E(G)$. Let $\mu(G)$ be the Mycielskian of G , with shadow vertices $u'_0, v'_0, u'_1, \dots, u'_m, v'_1, \dots, v'_n$ and apex vertex w .

We claim that $V(\mu(G))$ admits a TC-partition $\{D_C, D_T\}$ where

$$D_C = \{w, u'_0, v'_0\} \quad \text{and} \quad D_T = V(\mu(G)) \setminus D_C.$$

Clearly, $\mu(G)[D_C]$ is connected since w is adjacent to every shadow vertex. Moreover, each original vertex u_i (resp. v_j) is adjacent to u'_0 (resp. v'_0), each shadow vertex is adjacent to w , and u_0 and v_0 are adjacent to v'_0 and u'_0 , respectively. Hence, every vertex outside D_C has a neighbor in D_C , so D_C is a CD-set of $\mu(G)$.

The set D_T contains G , so every original vertex has a neighbor in D_T . Also, by the Mycielski construction, each shadow vertex u'_i has a neighbor $u_0 \in D_T$ and each v'_j has a neighbor $v_0 \in D_T$. Finally, every vertex in D_C has a neighbor in D_T . Thus, every vertex of $\mu(G)$ has a neighbor in D_T , so D_T is a TD-set of $\mu(G)$. Therefore, (D_C, D_T) is a TC-partition of $V(\mu(G))$, and so $\mu(G)$ is a TC-graph for all $m, n \geq 2$. \square

Remark 7. In view of Remark 1, for $m, n \geq 2$, $tc(\mu(BS_{m,n})) = 0$.

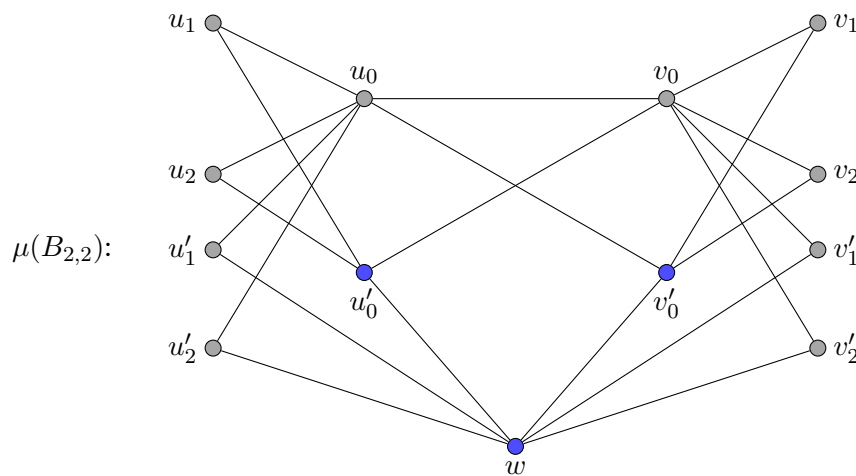


FIGURE 12. A TC-partition of the Mycielskian graph $\mu(B_{2,2})$.

CONCLUDING REMARKS

In this paper, we introduced the concept of TC-graph and the associated TC-augmentation number, that is, the minimum number of edges required for a graph to admit a TC-partition. Exact values of this parameter were obtained for several graph families, including star graphs, bi-star graphs, their complements, and their Mycielskian constructions, illustrating how structural properties influence the existence of TC-partitions. These results extend existing work on domination-based graph partitions and augmentation problems. Future research may explore TC-augmentation for broader graph classes and investigate algorithmic and structural relationships with other domination parameters.

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