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ON ROUGH APPROXIMATIONS OF FUZZY SEMIBIPOLAR SOFT BI-IDEALS AND BI-FILTERS IN ORDERED SEMIGROUPS

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Abstract. This paper develops a rough approximation framework for fuzzy semibipolar soft structures on ordered semigroups by means of approximation spaces induced by set-valued functions. Within this framework, we introduce fuzzy semibipolar soft bi-ideals and fuzzy semibipolar soft bi-filters as proper generalizations of classical fuzzy semibipolar soft ideals and filters, respectively. After establishing their fundamental algebraic and order-theoretic properties, upper and lower rough approximation operators are formulated and examined under several types of set-valued-function-based approximation spaces. A series of preservation statements of if–then type is established, showing that fuzzy semibipolar soft bi-ideals and bi-filters are preserved under rough approximations whenever the underlying approximation spaces satisfy suitable order and inclusion conditions. Several illustrative examples are provided to clarify the theoretical findings and to validate the effectiveness of the proposed framework. Overall, this study enriches the theory of fuzzy semibipolar soft sets on ordered semigroups and lays a solid foundation for further research on rough approximation models in uncertainty-oriented algebra and related decision-making applications.

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1. Introduction

Rough set theory, originally proposed by Pawlak [1], has become a cornerstone of approximate reasoning in situations where precise classification of objects is unattainable. Its fundamental idea is based on the construction of lower and upper approximations of a target set through information

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granules induced by an underlying relation. These granules determine the resolution at which objects are perceived and play a decisive role in shaping the behavior of rough approximation operators [2,3]. Over the past decades, this granule-oriented viewpoint has evolved significantly and has been actively developed in connection with granular computing and decision-theoretic rough set models [4]. Recent studies further confirm that granulation remains central in modern rough frameworks, including multilevel and three-way rough models, where approximation quality is closely tied to the choice of granules [5,6].

Within this broad context, algebraic structures equipped with order relations provide a natural setting for studying rough approximations under algebraic constraints. Ordered groupoids and ordered semigroups allow binary operations to interact coherently with partial orders, thereby enabling approximation models that respect both algebraic composition and order-theoretic hierarchy. Classical, fuzzy, soft, and rough variants of ideals, filters, and congruences have been extensively investigated in such structures, revealing fundamental connections between algebraic regularity and uncertainty modeling. Nevertheless, many existing rough fuzzy ideal and rough fuzzy filter models are based on fixed membership descriptions and do not adequately incorporate parameterization, despite its recognized importance since the introduction of soft set theory by Molodtsov [7]. As a consequence, their ability to support flexible and context-dependent reasoning remains limited.

Considerable progress has been achieved in integrating roughness with fuzzy algebraic structures across a wide range of systems. Rough approximations of fuzzy ideals and fuzzy filters have been studied in various algebraic frameworks, including BCK/BCI-algebras, ordered semigroups, and related implicational structures. These studies clarify how lower and upper rough operators preserve ideal-type and filter-type properties under algebraic and order-theoretic constraints [8–11]. Despite these advances, most existing results in fuzzy and rough fuzzy mathematics rely on approximation mechanisms induced solely by fuzzy membership functions. In contrast, rough approximation models based on soft sets or soft parameterization—where approximation functions explicitly depend on parameter sets—remain relatively unexplored. This observation indicates that rough approximations induced by soft-set-based granules constitute an open and promising research direction. In parallel, the development of fuzzy semibipolar soft sets by Prasertpong [12] in 2022 introduced a powerful framework that integrates fuzzy membership evaluation, semibipolar representation of positive and negative evidence, and soft parameterization. Within this framework, fuzzy semibipolar soft ideals were subsequently formulated and investigated in the setting of ordered groupoids. Continuing along this line of research, fuzzy semibipolar soft filters on ordered groupoids were proposed in [13] in 2024, providing refined models for directional filtering behavior under bipolar uncertainty. Compared with classical fuzzy and intuitionistic fuzzy approaches, semibipolar soft models explicitly distinguish supportive and opposing tendencies across parameters, which makes them particularly suitable for ordered algebraic environments and decision-oriented applications.

Motivated by these developments and the absence of a systematic rough approximation framework for fuzzy semibipolar soft structures, this paper investigates rough approximations of the newly introduced fuzzy semibipolar soft bi-ideals and fuzzy semibipolar soft bi-filters in ordered semigroups. The approximation process is formulated through approximation spaces induced by set-valued functions, which act as information granules. Within this framework, both upper and lower rough approximation operators are constructed and analyzed under various order- and inclusion-based conditions. The paper is organized as follows. Section 2 reviews the necessary preliminaries, including fundamental concepts of fuzzy sets, soft sets, fuzzy semibipolar soft sets, and ordered semigroups. Section 3 presents the main theoretical results of the paper. In particular, fuzzy semibipolar soft bi-ideals and fuzzy semibipolar soft bi-filters are introduced as proper generalizations of classical fuzzy semibipolar soft ideals and filters, respectively. A series of preservation theorems is established, showing that these structures are stable under upper and lower rough approximations in suitable set-valued-function-based approximation spaces. Concrete examples and counterexamples are provided to illustrate the obtained results and to demonstrate that the converse implications do not hold in general. Finally, the last section summarizes the main contributions of the paper and discusses possible directions for future research.

2. Preliminaries

To make this paper self-contained, we present some preliminary notions in this section. Throughout the paper, let S and U denote two non-empty sets.

Definition 2.1. [14] *A function $f : U \to [0,1]$ is called a* fuzzy subset *of U. Throughout this paper, $\mathscr{F}\mathscr{P}(U)$ denotes the family of all fuzzy subsets of U. In particular, $1_U, 0_U \in \mathscr{F}\mathscr{P}(U)$ are defined by $1_U(u) = 1$ and $0_U(u) = 0$ for all $u \in U$.*

For $f, g \in \mathscr{F}\mathscr{P}(U)$, define $f\widetilde{\wedge}g$, $f\widetilde{\vee}g$, $f\widetilde{+}g$, and $f \simeq g$ in $\mathscr{F}\mathscr{P}(U)$ by

$$(f\widetilde{\wedge}g)(u) := \min\{f(u), g(u)\},$$

$$(f\widetilde{\vee}g)(u) := \max\{f(u), g(u)\},$$

$$(f\widetilde{+}g)(u) := \min\{1,\ f(u) + g(u)\},$$

$$(f \simeq g)(u) := \max\{0,\ f(u) - g(u)\}, \qquad u \in U.$$

Moreover, for $f, g \in \mathscr{F}\mathscr{P}(U)$, we write $f\widetilde{\leq}g$ if $f(u) \leq g(u)$ for all $u \in U$, and $f\widetilde{\geq}g$ means $g\widetilde{\leq}f$.

Remark 2.1. [14] *According to Definition 2.1, the fuzzy subset 1_U is the greatest element of $\mathscr{F}\mathscr{P}(U)$, while 0_U is the least element of $\mathscr{F}\mathscr{P}(U)$.*

Proposition 2.1. [15] *Let $\{f_i : i \in I\}$ be a non-empty collection of all fuzzy subsets of U. Define*

$$\widetilde{\bigwedge}_{i\in I} f_i : U \to [0,1] | u \mapsto (\widetilde{\bigwedge}_{i\in I} f_i)(u) := \inf\{f_i(u) : i \in I\}$$

and

$$\widetilde{\bigvee}_{i\in I} f_i : U \to [0,1] | u \mapsto (\widetilde{\bigvee}_{i\in I} f_i)(u) := \sup\{f_i(u) : i \in I\}.$$

Then $\widetilde{\bigwedge}_{i\in I} f_i, \widetilde{\bigvee}_{i\in I} f_i \in \mathscr{F}\mathscr{P}(U)$. In addition,

$$\widetilde{\bigwedge}_{i\in I} f_i = \inf\{f_i : i \in I\} \quad \textit{and} \quad \widetilde{\bigvee}_{i\in I} f_i = \sup\{f_i : i \in I\}.$$

Throughout this paper, $\mathscr{P}(U)$ denotes the power set of U.

Definition 2.2. [7] *Let U be an initial universe of objects and let S be a nonempty set of parameters. A* soft set *over U with respect to S is a pair (F, S), where F is a mapping given by*

$$F : S \longrightarrow \mathscr{P}(U).$$

For each parameter $a \in S$, the set $F(a) \subseteq U$ represents the set of a-approximate elements of the soft set (F, S).

Definition 2.3. [12] *A triple $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S)$ is called a fuzzy semibipolar soft set over U with respect to S if $\acute{\mathcal{F}} : S \to \mathscr{F}\mathscr{P}(U)$ and $\grave{\mathcal{F}} : S \to \mathscr{F}\mathscr{P}(U)$ are functions such that*

(P1) $\acute{\mathcal{F}}(a)(u) \neq \grave{\mathcal{F}}(a)(u)$ *and*

(P2) $\acute{\mathcal{F}}(a)(u) + \grave{\mathcal{F}}(a)(u) = 1$

for all $a \in S, u \in U$. Here, $\mathscr{S}(U \sim S)$ denotes a collection of all fuzzy semibipolar soft sets over U with respect to S.

Definition 2.4. [12] *Let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S), \mathcal{G} := (\acute{\mathcal{G}}, \grave{\mathcal{G}}, S) \in \mathscr{S}(U \sim S)$. $\mathcal{F}$ is a fuzzy semibipolar soft subset of $\mathcal{G}$, denoted by $\mathcal{F} \widetilde{\subseteq} \mathcal{G}$, if $\acute{\mathcal{F}}(a) \widetilde{\leq} \acute{\mathcal{G}}(a)$ and $\grave{\mathcal{F}}(a) \widetilde{\geq} \grave{\mathcal{G}}(a)$ for all $a \in S$. We say that $\mathcal{F}$ is equal to $\mathcal{G}$ if $\mathcal{F} \widetilde{\subseteq} \mathcal{G}$ and $\mathcal{G} \widetilde{\subseteq} \mathcal{F}$.*

Definition 2.5. [16] *Let S be a nonempty set equipped with a binary operation $\cdot$ and a partial order $\leq_S$.*

(1) *The triple $(S, \cdot, \leq_S)$ is called an* ordered groupoid *if $(S, \cdot)$ is a groupoid and the order $\leq_S$ is compatible with the operation, that is, for all $a, b, c \in S$,*

$$a \leq_S b \;\Rightarrow\; ac \leq_S bc \text{ and } ca \leq_S cb.$$

(2) *The triple $(S, \cdot, \leq_S)$ is called an* ordered semigroup *if $(S, \cdot)$ is a semigroup (i.e., the operation $\cdot$ is associative) and $(S, \cdot, \leq_S)$ forms an ordered groupoid.*

Definition 2.6. [12] *Let $(S, \cdot, \leq_S)$ be an ordered groupoid and $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. Then $\mathcal{F}$ is called a fuzzy semibipolar soft ideal if*

(P1) *For all* $a, b \in S$,

$$\acute{\mathcal{F}}(ab) \widetilde{\geq} \acute{\mathcal{F}}(a) \widetilde{\vee} \acute{\mathcal{F}}(b) \quad \textit{and} \quad \grave{\mathcal{F}}(ab) \widetilde{\leq} \grave{\mathcal{F}}(a) \widetilde{\wedge} \grave{\mathcal{F}}(b).$$

Equivalently, for all $a, b \in S$,

(1) $\acute{\mathcal{F}}(ab) \widetilde{\geq} \acute{\mathcal{F}}(a)$ *and* $\grave{\mathcal{F}}(ab) \widetilde{\leq} \grave{\mathcal{F}}(a)$,

(2) $\acute{\mathcal{F}}(ab) \widetilde{\geq} \acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(ab) \widetilde{\leq} \grave{\mathcal{F}}(b)$.

(P2) *For all* $a, b \in S$, $a \leq_S b$ *implies* $\acute{\mathcal{F}}(a) \widetilde{\geq} \acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(a) \widetilde{\leq} \grave{\mathcal{F}}(b)$.

Here, $\mathscr{I}(U \sim S)$ *denotes the collection of all fuzzy semibipolar soft ideals over* U *with respect to* S.

Definition 2.7. [13] *Let* $(S, \cdot, \leq_S)$ *be an ordered groupoid and* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. *Then* $\mathcal{F}$ *is called a fuzzy semibipolar soft subgroupoid if*

$$\acute{\mathcal{F}}(ab) \widetilde{\geq} \acute{\mathcal{F}}(a) \widetilde{\wedge} \acute{\mathcal{F}}(b) \quad \textit{and} \quad \grave{\mathcal{F}}(ab) \widetilde{\leq} \grave{\mathcal{F}}(a) \widetilde{\vee} \grave{\mathcal{F}}(b)$$

for all $a, b \in S$. *Here,* $\mathscr{G}(U \sim S)$ *denotes the collection of all fuzzy semibipolar soft subgroupoids over* U *with respect to* S.

Definition 2.8. [13] *Let* $(S, \cdot, \leq_S)$ *be an ordered groupoid, and let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. *Then* $\mathcal{F}$ *is called a fuzzy semibipolar soft filter if*

(P1) *For all* $a, b \in S$,

$$\acute{\mathcal{F}}(ab) = \acute{\mathcal{F}}(a) \widetilde{\wedge} \acute{\mathcal{F}}(b) \quad \textit{and} \quad \grave{\mathcal{F}}(ab) = \grave{\mathcal{F}}(a) \widetilde{\vee} \grave{\mathcal{F}}(b).$$

Equivalently, for all $a, b \in S$,

(1) $\mathcal{F} \in \mathscr{G}(U \sim S)$,

(2) $\acute{\mathcal{F}}(ab) \widetilde{\leq} \acute{\mathcal{F}}(a)$ *and* $\grave{\mathcal{F}}(ab) \widetilde{\geq} \grave{\mathcal{F}}(a)$,

(3) $\acute{\mathcal{F}}(ab) \widetilde{\leq} \acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(ab) \widetilde{\geq} \grave{\mathcal{F}}(b)$.

(P2) *For all* $a, b \in S$, $a \leq_S b$ *implies* $\acute{\mathcal{F}}(a) \widetilde{\leq} \acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(a) \widetilde{\geq} \grave{\mathcal{F}}(b)$.

Here, $\mathscr{F}(U \sim S)$ *denotes the collection of all fuzzy semibipolar soft filters over* U *with respect to* S.

3. Main Results

In this section, we construct a rough approximation model induced by set-valued functions on ordered semigroups. This model provides a unified and systematic framework for analyzing the validity of conditional propositions appearing in the main results. The proposed approximation spaces serve as algebraic environments in which upper and lower rough operators are defined and examined. Within this setting, we study, in a step-by-step manner, how the introduced rough approximations preserve the structural properties of fuzzy semibipolar soft bi-ideals and bi-filters under various order- and inclusion-based conditions. Throughout this section, $(S, \cdot, \leq_S)$ denotes an ordered semigroup.

Definition 3.1. *Let $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ be a set-valued function. Then the pair $(\mathscr{S}(U \sim S), \mathfrak{h})$ is called a $\mathfrak{h}$-approximation space.*

Proposition 3.1. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $\mathfrak{h}$-approximation space, and let $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. Define two mappings $\mathbb{U}$ and $\mathbb{L}$ from S to $\mathscr{F}\mathscr{P}(U)$ by*

$$\mathbb{U}(a) = \widetilde{\bigvee}_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b), \qquad \mathbb{L}(a) = \widetilde{\bigwedge}_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b), \quad a \in S.$$

Then $\mathbb{U}(a) \widetilde{+} \mathbb{L}(a) = 1_U$.

Proof. Let $a \in S$ and $u \in U$ be given. Then, by Definition 2.1 and Proposition 2.1,

$$\mathbb{U}(a)(u) = \sup_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b)(u) \quad \text{and} \quad \mathbb{L}(a)(u) = \inf_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b)(u).$$

Since $\grave{\mathcal{F}}(\alpha) = 1_U \simeq \acute{\mathcal{F}}(\alpha)$ for all $\alpha \in S$, that is, $\grave{\mathcal{F}}(\alpha)(\beta) = 1 - \acute{\mathcal{F}}(\alpha)(\beta)$ for all $\alpha \in S, \beta \in U$, it follows that

$$\begin{aligned} \mathbb{L}(a)(u) &= \inf_{b \in \mathfrak{h}(a)} \big(1 - \acute{\mathcal{F}}(b)(u)\big) \\ &= 1 - \sup_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b)(u) = 1 - \mathbb{U}(a)(u). \end{aligned}$$

Therefore

$$\begin{aligned} (\mathbb{U}(a) \widetilde{+} \mathbb{L}(a))(u) &= \min\{1, \mathbb{U}(a)(u) + \mathbb{L}(a)(u)\} \\ &= \mathbb{U}(a)(u) + \mathbb{L}(a)(u) \\ &= \mathbb{U}(a)(u) + (1 - \mathbb{U}(a)(u)) = 1. \end{aligned}$$

Since $u \in U$ is arbitrary, it follows that $\mathbb{U}(a) \widetilde{+} \mathbb{L}(a) = 1_U$. □

Remark 3.1. *Based on Proposition 3.1, although $\acute{\mathcal{F}}(a)(u) \neq \grave{\mathcal{F}}(a)(u)$ for all $a \in S$ and $u \in U$, it does not necessarily follow that*

$$\sup_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b)(u) \neq \inf_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b)(u).$$

Nevertheless, in many situations these two values are different, depending on the distribution of membership degrees over $\mathfrak{h}(a)$. In this paper, we restrict our attention to the case where

$$\sup_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b)(u) \neq \inf_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b)(u), \quad \textit{for all } a \in S \textit{ and } u \in U.$$

Under this assumption, the pair $(\mathbb{U}, \mathbb{L}, S)$ forms a fuzzy semibipolar soft set, that is, $(\mathbb{U}, \mathbb{L}, S) \in \mathscr{S}(U \sim S)$ throughout the paper. In what follows, we develop a novel rough set model induced by Proposition 3.1.

Definition 3.2. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be an $\mathfrak{h}$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S)$ be an element of $\mathscr{S}(U \sim S)$. The upper rough approximation of $\mathcal{F}$, denoted by $\mathcal{F}^{\mathfrak{h}}$, is defined as*

$$(\acute{\mathcal{F}}^{\mathfrak{h}}, \grave{\mathcal{F}}^{\mathfrak{h}}, S) \in \mathscr{S}(U \sim S),$$

where $\acute{\mathcal{F}}^{\mathfrak{h}}$ and $\grave{\mathcal{F}}^{\mathfrak{h}}$ are given by

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b), \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b), \quad a \in S.$$

The lower rough approximation of $\mathcal{F}$, denoted by $\mathcal{F}_{\mathfrak{h}}$, is defined as

$$(\acute{\mathcal{F}}_{\mathfrak{h}}, \grave{\mathcal{F}}_{\mathfrak{h}}, S) \in \mathscr{S}(U \sim S),$$

where $\acute{\mathcal{F}}_{\mathfrak{h}}$ and $\grave{\mathcal{F}}_{\mathfrak{h}}$ are given by

$$\acute{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{b \in \mathfrak{h}(a)} \acute{\mathcal{F}}(b), \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigvee}_{b \in \mathfrak{h}(a)} \grave{\mathcal{F}}(b), \quad a \in S.$$

Based on this point, we say that $\mathcal{F}$ is a definable fuzzy semibipolar soft set in $(\mathscr{S}(U \sim S), \mathfrak{h})$ if $\mathcal{F}^{\mathfrak{h}} = \mathcal{F}_{\mathfrak{h}}$; otherwise, $\mathcal{F}$ is a rough fuzzy semibipolar soft set in $(\mathscr{S}(U \sim S), \mathfrak{h})$.

For each $\varrho \in [0,1]$, we denote by ϱ_U the fuzzy subset of U defined by $\varrho_U(u) = \varrho$ for all $u \in U$. We now consider the rough case of Definition 3.2 as follows.

Example 3.1. *Let $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ be an ordered semigroup, where the binary operation $\cdot$ on S is given by Table 1 and the order relation $\leq_S$ is defined by*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

TABLE 1. The Cayley table of the semigroup S

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

Define $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ by

$$\mathfrak{h}(\alpha) = \begin{cases} \{a, b, c\}, & \alpha = a, \\ \{d\}, & \alpha = b, \\ \{d, e\}, & \alpha = c, \\ \{d, e\}, & \alpha = d, \\ \{b, c, d\}, & \alpha = e. \end{cases}$$

Hence $(\mathscr{S}(U \sim S), \mathfrak{h})$ *forms a* $\mathfrak{h}$*-approximation space. Now, let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$*, where*

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.8_U, & \alpha = a, \\ 0.6_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.2_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.2_U, & \alpha = a, \\ 0.4_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.8_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

The upper and lower rough approximations of $\mathcal{F}$ *induced by* $\mathfrak{h}$ *are defined, respectively, by*

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta \in \mathfrak{h}(\alpha)} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta \in \mathfrak{h}(\alpha)} \grave{\mathcal{F}}(\beta),$$

and

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta \in \mathfrak{h}(\alpha)} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta \in \mathfrak{h}(\alpha)} \grave{\mathcal{F}}(\beta), \quad \alpha \in S.$$

By direct computation, we obtain the upper rough approximation

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.8_U, & \alpha = a, \\ 0.2_U, & \alpha = b, \\ 0.2_U, & \alpha = c, \\ 0.2_U, & \alpha = d, \\ 0.6_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.2_U, & \alpha = a, \\ 0.8_U, & \alpha = b, \\ 0.8_U, & \alpha = c, \\ 0.8_U, & \alpha = d, \\ 0.4_U, & \alpha = e, \end{cases}$$

and the lower rough approximation

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \begin{cases} 0.4_U, & \alpha = a, \\ 0.2_U, & \alpha = b, \\ 0.1_U, & \alpha = c, \\ 0.1_U, & \alpha = d, \\ 0.2_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \begin{cases} 0.6_U, & \alpha = a, \\ 0.8_U, & \alpha = b, \\ 0.9_U, & \alpha = c, \\ 0.9_U, & \alpha = d, \\ 0.8_U, & \alpha = e. \end{cases}$$

Therefore, $\mathcal{F}$ *is a rough fuzzy semibipolar soft set in the approximation space* $(\mathscr{S}(U \sim S), \mathfrak{h})$*, with upper and lower rough approximations* $\mathcal{F}^{\mathfrak{h}}$ *and* $\mathcal{F}_{\mathfrak{h}}$*, respectively.*

Remark 3.2. *It follows from Definition 3.2 that the relation* $\mathcal{F}_{\mathfrak{h}} \widetilde{\subseteq} \mathcal{F} \widetilde{\subseteq} \mathcal{F}^{\mathfrak{h}}$ *does not hold in general, as illustrated in Example 3.1.*

Definition 3.3. *Let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$*. Then* $\mathcal{F}$ *is called a fuzzy semibipolar soft bi-ideal if*

(P1) $\mathcal{F} \in \mathscr{G}(U \sim S)$.

(P2) $\acute{\mathcal{F}}(abc) \widetilde{\geq} \acute{\mathcal{F}}(a) \widetilde{\wedge} \acute{\mathcal{F}}(c)$ *and* $\grave{\mathcal{F}}(abc) \widetilde{\leq} \grave{\mathcal{F}}(a) \widetilde{\vee} \grave{\mathcal{F}}(c)$ *for all* $a, b, c \in S$.

(P3) *For all* $a, b \in S, a \leq_S b$ *implies* $\acute{\mathcal{F}}(a)\tilde{\geq}\acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(a)\tilde{\leq}\grave{\mathcal{F}}(b)$.

Here, $\mathscr{BI}(U \sim S)$ *denotes the collection of all fuzzy semibipolar soft bi-ideals over* U *with respect to* S.

Example 3.2. *Let* $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ *be the ordered semigroup given in Example 3.1. In particular, the Cayley table implies that* $\alpha\beta = \alpha$ *for all* $\alpha, \beta \in S$*, and the order relation is*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.8_U, & \alpha = a, \\ 0.6_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.2_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.2_U, & \alpha = a, \\ 0.4_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.8_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

We show that $\mathcal{F} \in \mathscr{BI}(U \sim S)$ *in the sense of Definition 3.3.*

(P1) $\mathcal{F} \in \mathscr{G}(U \sim S)$. *For any* $\alpha, \beta \in S$*, we have* $\alpha\beta = \alpha$*. Hence*

$$\acute{\mathcal{F}}(\alpha\beta) = \acute{\mathcal{F}}(\alpha) \tilde{\geq} \acute{\mathcal{F}}(\alpha)\tilde{\wedge}\acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}(\alpha\beta) = \grave{\mathcal{F}}(\alpha) \tilde{\leq} \grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\beta).$$

(P2) *Let* $\alpha, \beta, \gamma \in S$*. Since* $\alpha\beta\gamma = \alpha$*, we have*

$$\acute{\mathcal{F}}(\alpha\beta\gamma) = \acute{\mathcal{F}}(\alpha) \tilde{\geq} \acute{\mathcal{F}}(\alpha)\tilde{\wedge}\acute{\mathcal{F}}(\gamma), \qquad \grave{\mathcal{F}}(\alpha\beta\gamma) = \grave{\mathcal{F}}(\alpha) \tilde{\leq} \grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\gamma).$$

(P3) *If* $\alpha \leq_S \beta$*, then by the above assignment* $\acute{\mathcal{F}}(\alpha)\tilde{\geq}\acute{\mathcal{F}}(\beta)$ *and* $\grave{\mathcal{F}}(\alpha)\tilde{\leq}\grave{\mathcal{F}}(\beta)$.

Therefore $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BI}(U \sim S)$.

Proposition 3.2. *Let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$*. If* $\mathcal{F} \in \mathscr{I}(U \sim S)$*, then* $\mathcal{F} \in \mathscr{BI}(U \sim S)$.

Proof. Suppose that $\mathcal{F} \in \mathscr{I}(U \sim S)$. Then, it is sufficient to show that condition $(P2)$ in Definition 3.3 is satisfied. Let $a, b \in S$. Then

$$\acute{\mathcal{F}}(abc)\tilde{\geq}\acute{\mathcal{F}}(ab)\tilde{\geq}\acute{\mathcal{F}}(a) \text{ and } \grave{\mathcal{F}}(abc)\tilde{\leq}\grave{\mathcal{F}}(ab)\tilde{\leq}\grave{\mathcal{F}}(a).$$

Moreover,

$$\acute{\mathcal{F}}(abc)\tilde{\geq}\acute{\mathcal{F}}(bc)\tilde{\geq}\acute{\mathcal{F}}(c) \text{ and } \grave{\mathcal{F}}(abc)\tilde{\leq}\grave{\mathcal{F}}(bc)\tilde{\leq}\grave{\mathcal{F}}(c).$$

Thus $\acute{\mathcal{F}}(abc)\tilde{\geq}\acute{\mathcal{F}}(a)\tilde{\wedge}\acute{\mathcal{F}}(c)$ and $\acute{\mathcal{F}}(abc)\tilde{\leq}\acute{\mathcal{F}}(a)\tilde{\vee}\acute{\mathcal{F}}(c)$. Consequently, $\mathcal{F} \in \mathscr{BI}(U \sim S)$. □

Remark 3.3. *According to Proposition 3.2, the converse does not hold in general. Indeed, consider the ordered semigroup* S *in Example 3.1 (so that* $\alpha\beta = \alpha$ *for all* $\alpha, \beta \in S$*) and define* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}(a) = 0.9_U,\ \acute{\mathcal{F}}(b) = 0.8_U,\ \acute{\mathcal{F}}(c) = 0.7_U,\ \acute{\mathcal{F}}(d) = 0.6_U,\ \acute{\mathcal{F}}(e) = 0.4_U,$$

and $\grave{\mathcal{F}}(\alpha) = 1_U - \acute{\mathcal{F}}(\alpha)$ *for all* $\alpha \in S$. *Then* $\mathcal{F}$ *satisfies* (P1)–(P3) *of Definition 3.3; hence* $\mathcal{F} \in \mathscr{BI}(U \sim S)$. *However,* $\mathcal{F}$ *is not a fuzzy semibipolar soft ideal in the sense of Definition 2.6. For instance, taking* $\alpha = e$ *and* $\beta = a$, *we have* $\alpha\beta = e$, *and thus*

$$\acute{\mathcal{F}}(\alpha\beta) = \acute{\mathcal{F}}(e) = 0.4_U \,\widetilde{\ngeq}\, \acute{\mathcal{F}}(a) = 0.9_U, \qquad \grave{\mathcal{F}}(\alpha\beta) = \grave{\mathcal{F}}(e) = 0.6_U \,\widetilde{\nleq}\, \grave{\mathcal{F}}(a) = 0.1_U,$$

which violates the absorption requirement of ideals. Hence, $\mathcal{F} \notin \mathscr{I}(U \sim S)$. *Consequently, the notion of fuzzy semibipolar soft bi-ideals constitutes a proper generalization of fuzzy semibipolar soft ideals.*

Definition 3.4. *Let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. *Then* $\mathcal{F}$ *is called a fuzzy semibipolar soft bi-filter if*

- (P1) $\mathcal{F} \in \mathscr{G}(U \sim S)$.
- (P2) $\acute{\mathcal{F}}(aba) \widetilde{\leq} \acute{\mathcal{F}}(a)$ *and* $\grave{\mathcal{F}}(aba) \widetilde{\geq} \grave{\mathcal{F}}(a)$ *for all* $a, b \in S$.
- (P3) *For all* $a, b \in S$, $a \leq_S b$ *implies* $\acute{\mathcal{F}}(a) \widetilde{\leq} \acute{\mathcal{F}}(b)$ *and* $\grave{\mathcal{F}}(a) \widetilde{\geq} \grave{\mathcal{F}}(b)$.

Here, $\mathscr{BF}(U \sim S)$ *denotes the collection of all fuzzy semibipolar soft bi-filters over* U *with respect to* S.

Example 3.3. *Let* $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ *be the ordered semigroup given in Example 3.1, where* $\alpha\beta = \alpha$ *for all* $\alpha, \beta \in S$ *and*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.3_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.7_U, & \alpha = d, \\ 0.9_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.7_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.3_U, & \alpha = d, \\ 0.1_U, & \alpha = e. \end{cases}$$

We verify that $\mathcal{F}$ *is a fuzzy semibipolar soft bi-filter in the sense of Definition 3.4.*

(P1) *For any* $\alpha, \beta \in S$, *since* $\alpha\beta = \alpha$, *we have*

$$\acute{\mathcal{F}}(\alpha\beta) = \acute{\mathcal{F}}(\alpha) \,\widetilde{\geq}\, \acute{\mathcal{F}}(\alpha) \widetilde{\wedge} \acute{\mathcal{F}}(\beta),$$

and

$$\grave{\mathcal{F}}(\alpha\beta) = \grave{\mathcal{F}}(\alpha) \,\widetilde{\leq}\, \grave{\mathcal{F}}(\alpha) \widetilde{\vee} \grave{\mathcal{F}}(\beta).$$

Hence $\mathcal{F} \in \mathscr{G}(U \sim S)$.

(P2) *Let* $\alpha, \beta \in S$. *Since* $\alpha\beta\alpha = \alpha$, *we obtain*

$$\acute{\mathcal{F}}(\alpha\beta\alpha) = \acute{\mathcal{F}}(\alpha) \,\widetilde{\leq}\, \acute{\mathcal{F}}(\alpha), \qquad \grave{\mathcal{F}}(\alpha\beta\alpha) = \grave{\mathcal{F}}(\alpha) \,\widetilde{\geq}\, \grave{\mathcal{F}}(\alpha).$$

Thus condition (P2) *holds.*

(P3) *If* $\alpha \leq_S \beta$, *then by the above assignment*

$$\acute{\mathcal{F}}(\alpha)\widetilde{\leq}\acute{\mathcal{F}}(\beta) \quad \text{and} \quad \grave{\mathcal{F}}(\alpha)\widetilde{\geq}\grave{\mathcal{F}}(\beta).$$

Therefore $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BF}(U \sim S)$.

Proposition 3.3. *Let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. *If* $\mathcal{F} \in \mathscr{F}(U \sim S)$, *then* $\mathcal{F} \in \mathscr{BF}(U \sim S)$.

Proof. Suppose that $\mathcal{F} \in \mathscr{F}(U \sim S)$. Then, it is sufficient to show that condition $(P2)$ in Definition 3.4 is satisfied. Let $a, b \in S$. Then

$$\acute{\mathcal{F}}(aba)\widetilde{\leq}\acute{\mathcal{F}}(ab)\widetilde{\leq}\acute{\mathcal{F}}(a) \text{ and } \grave{\mathcal{F}}(aba)\widetilde{\geq}\grave{\mathcal{F}}(ab)\widetilde{\geq}\grave{\mathcal{F}}(a).$$

Therefore $\mathcal{F} \in \mathscr{BF}(U \sim S)$. □

Remark 3.4. *According to Proposition 3.3, the converse does not hold in general. Indeed, consider the ordered semigroup* S *in Example 3.1 and define* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}(a) = 0.1_U,\ \acute{\mathcal{F}}(b) = 0.3_U,\ \acute{\mathcal{F}}(c) = 0.6_U,\ \acute{\mathcal{F}}(d) = 0.7_U,\ \acute{\mathcal{F}}(e) = 0.9_U,$$

and $\grave{\mathcal{F}}(\alpha) = 1_U - \acute{\mathcal{F}}(\alpha)$ *for all* $\alpha \in S$. *Then* $\mathcal{F}$ *satisfies* (P1)–(P3) *of Definition 3.4; hence* $\mathcal{F} \in \mathscr{BF}(U \sim S)$. *On the other hand,* $\mathcal{F}$ *is not a fuzzy semibipolar soft filter in the sense of Definition 2.8. In fact, letting* $x = e$ *and* $y = a$, *we have* $xy = e$, *and thus*

$$\acute{\mathcal{F}}(\alpha\beta) = \acute{\mathcal{F}}(e) = 0.9_U \neq \acute{\mathcal{F}}(e)\widetilde{\wedge}\acute{\mathcal{F}}(a) = 0.9_U\widetilde{\wedge}0.1_U = 0.1_U,$$

which violates condition (P1) *of Definition 2.8. Therefore,* $\mathcal{F} \notin \mathscr{F}(U \sim S)$. *Consequently, the notion of fuzzy semibipolar soft bi-filters constitutes a proper generalization of fuzzy semibipolar soft filters.*

For subsets A and B of S, define

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

This framework serves as a basis for developing the main results under several approximation spaces presented below.

Definition 3.5. *Let* $(\mathscr{S}(U \sim S), \mathfrak{h})$ *be a* $\mathfrak{h}$*-approximation space, and let* $\approxeq, \cong$ *be elements of the family of relations* $\{\subseteq, \supseteq, =\}$. *We say that* $(\mathscr{S}(U \sim S), \mathfrak{h})$ *is a* $(\mathfrak{h}, \approxeq, \cong)$*-approximation space if*

(P1) $\mathfrak{h}(a)\mathfrak{h}(b) \approxeq \mathfrak{h}(ab)$ *for all* $a, b \in S$.

(P2) *For all* $a, b \in S, a \leq_S b$ *implies* $\mathfrak{h}(a) \cong \mathfrak{h}(b)$.

In order to deepen our understanding of how semibipolar soft structures behave within ordered semigroups, it is essential to analyze the underlying relational patterns that govern their interaction with approximation operators. These relationships reveal structural dependencies that are not immediately apparent from the definitions alone.

Theorem 3.1. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, \subseteq, \supseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{I}(U \sim S)$, then $\mathcal{F}^{\mathfrak{h}} \in \mathscr{I}(U \sim S)$.*

Proof. Assume that $\mathcal{F} \in \mathscr{I}(U \sim S)$. Then, we consider the following three arguments.

(1) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) \widetilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \widetilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(a)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) \widetilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \widetilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(a).\end{aligned}$$

(2) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) \widetilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \widetilde{\geq} \widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\beta)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) \widetilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \widetilde{\leq} \widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\beta)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).\end{aligned}$$

(3) Let $a, b \in G$ be given. Suppose that $a \leq_G b$. Then $\mathfrak{h}(a) \supseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) \widetilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) \widetilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}^{\mathfrak{h}} \in \mathscr{I}(U \sim G)$. □

Theorem 3.2. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, \subseteq, \cong)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{G}(U \sim S)$, then $\mathcal{F}^{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.*

Proof. Suppose that $\mathcal{F} \in \mathscr{G}(U \sim S)$. Let $a, b \in S$ be given. Then

$$\begin{aligned}&\acute{\mathcal{F}}^{\mathfrak{h}}(ab) = \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) \widetilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right)\\ &\widetilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a),\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\widetilde{\wedge}\acute{\mathcal{F}}(\beta)\right) = \left(\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right)\right)\widetilde{\wedge}\left(\widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\beta)\right)\right)\\ &= \acute{\mathcal{F}}^{\mathfrak{h}}(a)\widetilde{\wedge}\acute{\mathcal{F}}^{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) \tilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right)\\ &\tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a),\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\beta)\right) = \left(\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right)\right)\tilde{\vee}\left(\widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\beta)\right)\right)\\ &= \grave{\mathcal{F}}^{\mathfrak{h}}(a)\tilde{\vee}\grave{\mathcal{F}}^{\mathfrak{h}}(b).\end{aligned}$$

Hence $\mathcal{F}^{\mathfrak{h}} \in \mathscr{G}(U \sim S)$. □

Theorem 3.3. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, \subseteq, \supseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{BI}(U \sim S)$, then $\mathcal{F}^{\mathfrak{h}} \in \mathscr{BI}(U \sim S)$.*

Proof. Assume that $\mathcal{F} \in \mathscr{BI}(U \sim S)$. Then, we consider the following three arguments.

(1) By Theorem 3.1, we have $\mathcal{F}^{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b, c \in S$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(abc) &= \widetilde{\bigvee}_{d\in\mathfrak{h}(abc)}\left(\acute{\mathcal{F}}(d)\right) \tilde{\geq} \widetilde{\bigvee}_{d\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(d)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\gamma\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\alpha\beta\gamma)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a),\gamma\in\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\alpha)\tilde{\wedge}\acute{\mathcal{F}}(\gamma)\right)\\ &= \left(\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right)\right)\tilde{\wedge}\left(\widetilde{\bigvee}_{\gamma\in\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\gamma)\right)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(a)\tilde{\wedge}\acute{\mathcal{F}}^{\mathfrak{h}}(c)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(abc) &= \widetilde{\bigwedge}_{d\in\mathfrak{h}(abc)}\left(\grave{\mathcal{F}}(d)\right) \tilde{\leq} \widetilde{\bigwedge}_{d\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(d)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\gamma\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\alpha\beta\gamma)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a),\gamma\in\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\gamma)\right)\\ &= \left(\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right)\right)\tilde{\vee}\left(\widetilde{\bigwedge}_{\gamma\in\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\gamma)\right)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(a)\tilde{\vee}\grave{\mathcal{F}}^{\mathfrak{h}}(c)\end{aligned}$$

(3) Let $a, b \in S$ be given. Suppose that $a \leq_S b$. Then $\mathfrak{h}(a) \supseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}^{\mathfrak{h}} \in \mathscr{BI}(U \sim S)$. □

The following example confirms Theorem 3.3 by demonstrating that the upper rough approximation of a fuzzy semibipolar soft bi-ideal is again a fuzzy semibipolar soft bi-ideal.

Example 3.4. *Let $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ be an ordered semigroup, where the binary operation $\cdot$ on S is given by Table 2.*

TABLE 2. The Cayley table of the semigroup S

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	c	c
d	a	b	c	d	d
e	a	b	c	d	e

The order relation $\leq_S$ on S is defined by

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ by

$$\mathfrak{h}(\alpha) = \{\beta \in S \mid \alpha \leq_S \beta\} = \begin{cases} \{a, b, c, d, e\}, & \alpha = a, \\ \{b, c, d, e\}, & \alpha = b, \\ \{c, d, e\}, & \alpha = c, \\ \{d, e\}, & \alpha = d, \\ \{e\}, & \alpha = e. \end{cases}$$

We verify that $\mathfrak{h}$ satisfies the conditions of a $(\mathfrak{h}, \subseteq, \supseteq)$-approximation space.

(P1) *For all $\alpha, \beta \in S$, $\mathfrak{h}(\alpha)\mathfrak{h}(\beta) \subseteq \mathfrak{h}(\alpha\beta)$. Indeed, by Table 2, $\alpha\beta = \min\{\alpha, \beta\}$ with respect to $\leq_S$. If $\gamma \in \mathfrak{h}(\alpha)$ and $\delta \in \mathfrak{h}(\beta)$, then $\alpha \leq_S \gamma$ and $\beta \leq_S \delta$, so*

$$\alpha\beta = \min\{\alpha, \beta\} \leq_S \min\{\gamma, \delta\} = \gamma\delta,$$

which yields $\gamma\delta \in \mathfrak{h}(\alpha\beta)$. Hence $\mathfrak{h}(\alpha)\mathfrak{h}(\beta) \subseteq \mathfrak{h}(\alpha\beta)$.

(P2) *If $\alpha \leq_S \beta$, then $\mathfrak{h}(\alpha) \supseteq \mathfrak{h}(\beta)$. Indeed, if $\gamma \in \mathfrak{h}(\beta)$, then $\beta \leq_S \gamma$, and hence $\alpha \leq_S \beta \leq_S \gamma$, which implies $\gamma \in \mathfrak{h}(\alpha)$.*

Therefore $(\mathscr{S}(U \sim S), \mathfrak{h})$ is a $(\mathfrak{h}, \subseteq, \supseteq)$-approximation space. Now define $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ by

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.7_U, & \alpha = a, \\ 0.6_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.2_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.3_U, & \alpha = a, \\ 0.4_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.8_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

Define the upper rough approximation $\mathcal{F}^{\mathfrak{h}} := (\acute{\mathcal{F}}^{\mathfrak{h}}, \grave{\mathcal{F}}^{\mathfrak{h}}, S)$ by

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta\in\mathfrak{h}(\alpha)}\acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta\in\mathfrak{h}(\alpha)}\grave{\mathcal{F}}(\beta).$$

A direct computation yields

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.7_U, & \alpha = a, \\ 0.6_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.2_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.3_U, & \alpha = a, \\ 0.4_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.8_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

Finally, it is straightforward to verify that both $\mathcal{F}$ and $\mathcal{F}^{\mathfrak{h}}$ satisfy conditions (P1)–(P3) *of Definition 3.3. Hence,*

$$(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BI}(U \sim S) \quad \text{and} \quad (\acute{\mathcal{F}}^{\mathfrak{h}}, \grave{\mathcal{F}}^{\mathfrak{h}}, S) \in \mathscr{BI}(U \sim S).$$

Theorem 3.4. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, =, \subseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{F}(U \sim S)$, then $\mathcal{F}^{\mathfrak{h}} \in \mathscr{F}(U \sim S)$.*

Proof. Assume that $\mathcal{F} \in \mathscr{F}(U \sim S)$. Then, we consider the following four arguments.

(1) By Theorem 3.1, we have $\mathcal{F}^{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \,\widetilde{\leq}\, \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(a)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \,\widetilde{\geq}\, \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(a).\end{aligned}$$

(3) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \,\widetilde{\leq}\, \widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\beta)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \,\widetilde{\geq}\, \widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\beta)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).\end{aligned}$$

(4) Let $a, b \in G$ be given. Suppose that $a \leq_G b$. Then $\mathfrak{h}(a) \subseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha \in \mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha)\right) \widetilde{\leq} \widetilde{\bigvee}_{\alpha \in \mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha \in \mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\alpha \in \mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).$$

From (1)-(4) above, it follows that $\mathcal{F}^{\mathfrak{h}} \in \mathscr{F}(U \sim G)$. □

Theorem 3.5. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, =, \subseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{BF}(U \sim S)$, then $\mathcal{F}^{\mathfrak{h}} \in \mathscr{BF}(U \sim S)$.*

Proof. Assume that $\mathcal{F} \in \mathscr{BF}(U \sim S)$. Then, we consider the following three arguments.

(1) By Theorem 3.1, we have $\mathcal{F}^{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b \in S$ be given. Then

$$\acute{\mathcal{F}}^{\mathfrak{h}}(aba) = \widetilde{\bigvee}_{c \in \mathfrak{h}(aba)} \left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c \in \mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)} \left(\acute{\mathcal{F}}(c)\right)$$
$$= \widetilde{\bigvee}_{\alpha\beta\alpha \in \mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha\beta\alpha)\right) \widetilde{\leq} \widetilde{\bigvee}_{\alpha \in \mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(a)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(aba) = \widetilde{\bigwedge}_{c \in \mathfrak{h}(aba)} \left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c \in \mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)} \left(\grave{\mathcal{F}}(c)\right)$$
$$= \widetilde{\bigwedge}_{\alpha\beta\alpha \in \mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha\beta\alpha)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\alpha \in \mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(a).$$

(3) Let $a, b \in S$ be given. Suppose that $a \leq_S b$. Then $\mathfrak{h}(a) \subseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha \in \mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha)\right) \widetilde{\leq} \widetilde{\bigvee}_{\alpha \in \mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha \in \mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\alpha \in \mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}^{\mathfrak{h}} \in \mathscr{BF}(U \sim S)$. □

The following example demonstrates consistency with Theorem 3.5, confirming that the upper rough approximation remains a fuzzy semibipolar soft bi-filter.

Example 3.5. *Let $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ be an ordered semigroup, where the binary operation $\cdot$ on S is given by Table 2 and the order relation $\leq_S$ on S is defined by*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ by

$$\mathfrak{h}(\alpha) = \begin{cases} \{a\}, & \alpha = a, \\ \{a\}, & \alpha = b, \\ \{a, b\}, & \alpha = c, \\ \{a, b, c\}, & \alpha = d, \\ \{a, b, c, d, e\}, & \alpha = e. \end{cases}$$

We first verify that $\mathfrak{h}$ *satisfies conditions* (P1) *and* (P2) *of Definition 3.5.*

(P1) *From Table 2, the semigroup operation on S satisfies $\alpha\beta = \min\{\alpha, \beta\}$ with respect to $\leq_S$. Thus, for all $\alpha, \beta \in S$, one can check directly that $\mathfrak{h}(\alpha)\mathfrak{h}(\beta) = \mathfrak{h}(\alpha\beta)$.*

(P2) *If $\alpha \leq_S \beta$, then by the definition of $\mathfrak{h}$ we have $\mathfrak{h}(\alpha) \subseteq \mathfrak{h}(\beta)$.*

It follows that $(\mathscr{S}(U \sim S), \mathfrak{h})$ forms a $(\mathfrak{h}, =, \subseteq)$-approximation space. Now let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$, where

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.2_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.6_U, & \alpha = d, \\ 0.7_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.8_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.4_U, & \alpha = d, \\ 0.3_U, & \alpha = e. \end{cases}$$

We define the upper rough approximation $\mathcal{F}^{\mathfrak{h}} = (\acute{\mathcal{F}}^{\mathfrak{h}}, \grave{\mathcal{F}}^{\mathfrak{h}}, S)$ by

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta \in \mathfrak{h}(\alpha)} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta \in \mathfrak{h}(\alpha)} \grave{\mathcal{F}}(\beta), \quad \alpha \in S.$$

By direct computation, we obtain

$$\acute{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.1_U, & \alpha = b, \\ 0.2_U, & \alpha = c, \\ 0.4_U, & \alpha = d, \\ 0.7_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}^{\mathfrak{h}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.9_U, & \alpha = b, \\ 0.8_U, & \alpha = c, \\ 0.6_U, & \alpha = d, \\ 0.3_U, & \alpha = e. \end{cases}$$

Next, we verify that $\mathcal{F}$ satisfies conditions (P1)–(P3) *of Definition 3.4. Indeed, this verification follows directly from the fact that $\alpha\beta = \min\{\alpha, \beta\}$ with respect to $\leq_S$ on S, together with the order-theoretic definitions of $\acute{\mathcal{F}}$ and $\grave{\mathcal{F}}$. Consequently,*

$$\acute{\mathcal{F}}(\alpha\beta) = \acute{\mathcal{F}}(\alpha) \widetilde{\wedge} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}(\alpha\beta) = \grave{\mathcal{F}}(\alpha) \widetilde{\vee} \grave{\mathcal{F}}(\beta),$$

for all $\alpha, \beta \in S$*, and the monotonicity conditions with respect to* $\leq_S$ *are immediately satisfied. Therefore* $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BF}(U \sim S)$*. Finally, by a similar argument,* $\mathcal{F}^{\mathfrak{h}}$ *also satisfies conditions* (P1)–(P3) *of Definition 3.4, and hence* $(\acute{\mathcal{F}}^{\mathfrak{h}}, \grave{\mathcal{F}}^{\mathfrak{h}}, S) \in \mathscr{BF}(U \sim S)$*.*

Theorem 3.6. *Let* $(\mathscr{S}(U \sim S), \mathfrak{h})$ *be a given* $(\mathfrak{h}, =, \subseteq)$*-approximation space, and let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$*. If* $\mathcal{F} \in \mathscr{I}(U \sim S)$*, then* $\mathcal{F}_{\mathfrak{h}} \in \mathscr{I}(U \sim S)$*.*

Proof. Suppose $\mathcal{F} \in \mathscr{I}(U \sim S)$. Then, we consider the following three arguments.

(1) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(a)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \widetilde{\leq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(a).\end{aligned}$$

(2) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\beta)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right) \widetilde{\leq} \widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\beta)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(b).\end{aligned}$$

(3) Let $a, b \in G$ be given. Suppose that $a \leq_G b$. Then $\mathfrak{h}(a) \subseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) \widetilde{\geq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) \widetilde{\leq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}_{\mathfrak{h}} \in \mathscr{I}(U \sim G)$. □

Theorem 3.7. *Let* $(\mathscr{S}(U \sim S), \mathfrak{h})$ *be a given* $(\mathfrak{h}, =, \cong)$*-approximation space, and let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$*. If* $\mathcal{F} \in \mathscr{G}(U \sim S)$*, then* $\mathcal{F}_{\mathfrak{h}} \in \mathscr{G}(U \sim S)$*.*

Proof. Suppose that $\mathcal{F} \in \mathscr{G}(U \sim S)$. Let $a, b \in S$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(c)\right) = \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha\beta)\right)\\ &\tilde{\geq}\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a),\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\tilde{\wedge}\acute{\mathcal{F}}(\beta)\right) = \left(\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right)\right)\tilde{\wedge}\left(\widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\beta)\right)\right)\\ &= \acute{\mathcal{F}}_{\mathfrak{h}}(a)\tilde{\wedge}\acute{\mathcal{F}}_{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(c)\right) = \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha\beta)\right)\\ &\tilde{\leq}\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a),\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\beta)\right) = \left(\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right)\right)\tilde{\vee}\left(\widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\beta)\right)\right)\\ &= \grave{\mathcal{F}}_{\mathfrak{h}}(a)\tilde{\vee}\grave{\mathcal{F}}_{\mathfrak{h}}(b).\end{aligned}$$

Therefore $\mathcal{F}_{\mathfrak{h}} \in \mathscr{G}(U \sim S)$. □

Theorem 3.8. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, =, \subseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{B}\mathscr{I}(U \sim S)$, then $\mathcal{F}_{\mathfrak{h}} \in \mathscr{B}\mathscr{I}(U \sim S)$.*

Proof. Assume $\mathcal{F} \in \mathscr{I}\mathscr{F}(U \sim S)$. Then, we consider the following three arguments.

(1) By Theorem 3.7, we have $\mathcal{F}_{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b, c \in S$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}^{\mathfrak{h}}(abc) &= \widetilde{\bigwedge}_{d\in\mathfrak{h}(abc)}\left(\acute{\mathcal{F}}(d)\right) = \widetilde{\bigwedge}_{d\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(d)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\gamma\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\alpha\beta\gamma)\right)\tilde{\geq}\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a),\gamma\in\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\alpha)\tilde{\wedge}\acute{\mathcal{F}}(\gamma)\right)\\ &= \left(\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right)\right)\tilde{\wedge}\left(\widetilde{\bigwedge}_{\gamma\in\mathfrak{h}(c)}\left(\acute{\mathcal{F}}(\gamma)\right)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(a)\tilde{\wedge}\acute{\mathcal{F}}^{\mathfrak{h}}(c)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}^{\mathfrak{h}}(abc) &= \widetilde{\bigvee}_{d\in\mathfrak{h}(abc)}\left(\grave{\mathcal{F}}(d)\right) = \widetilde{\bigvee}_{d\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(d)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\gamma\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\alpha\beta\gamma)\right)\tilde{\leq}\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a),\gamma\in\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\alpha)\tilde{\vee}\grave{\mathcal{F}}(\gamma)\right)\\ &= \left(\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right)\right)\tilde{\vee}\left(\widetilde{\bigvee}_{\gamma\in\mathfrak{h}(c)}\left(\grave{\mathcal{F}}(\gamma)\right)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(a)\tilde{\vee}\grave{\mathcal{F}}^{\mathfrak{h}}(c)\end{aligned}$$

(3) Let $a, b \in S$ be given. Suppose that $a \leq_S b$. Then $\mathfrak{h}(a) \subseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right)\tilde{\geq}\widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}^{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}^{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right)\tilde{\leq}\widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}^{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}_{\mathfrak{h}} \in \mathscr{B}\mathscr{I}(U \sim S)$. □

The following example verifies Theorem 3.8, in the sense that the lower rough approximation is still a fuzzy semibipolar soft bi-ideal.

Example 3.6. *Let* $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ *be an ordered semigroup, where the binary operation* $\cdot$ *on* S *is given by Table 2 and the order relation* $\leq_S$ *on* S *is defined by*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ *by*

$$\mathfrak{h}(\alpha) = \begin{cases} \{a\}, & \alpha = a, \\ \{a\}, & \alpha = b, \\ \{a, b\}, & \alpha = c, \\ \{a, b, c\}, & \alpha = d, \\ \{a, b, c, d, e\}, & \alpha = e. \end{cases}$$

We claim that $\mathfrak{h}$ *satisfies conditions* (P1)–(P2) *of the* $\mathfrak{h}$*-approximation space.*

(P1) *Table 2 shows that the binary operation on* S *is compatible with the order* $\leq_S$*, namely* $\alpha\beta = \min\{\alpha, \beta\}$*. Consequently, the equality* $\mathfrak{h}(\alpha)\mathfrak{h}(\beta) = \mathfrak{h}(\alpha\beta)$ *holds for all* $\alpha, \beta \in S$*.*

(P2) *If* $\alpha \leq_S \beta$*, then* $\mathfrak{h}(\alpha) \subseteq \mathfrak{h}(\beta)$ *holds immediately from the above definition of* $\mathfrak{h}$*.*

Therefore $(\mathscr{S}(U \sim S), \mathfrak{h})$ *is a* $(\mathfrak{h}, =, \subseteq)$*-approximation space in the sense of Definition 3.5. Now define* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.7_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.3_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.3_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.7_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

Define $(\acute{\mathcal{F}}_{\mathfrak{h}}, \grave{\mathcal{F}}_{\mathfrak{h}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta \in \mathfrak{h}(\alpha)} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta \in \mathfrak{h}(\alpha)} \grave{\mathcal{F}}(\beta), \quad \alpha \in S.$$

Then, by direct computation, we obtain

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.9_U, & \alpha = b, \\ 0.7_U, & \alpha = c, \\ 0.4_U, & \alpha = d, \\ 0.1_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.1_U, & \alpha = b, \\ 0.3_U, & \alpha = c, \\ 0.6_U, & \alpha = d, \\ 0.9_U, & \alpha = e. \end{cases}$$

Hence $\mathcal{F}_{\mathfrak{h}} \neq \mathcal{F}$, since for instance $\acute{\mathcal{F}}_{\mathfrak{h}}(b) = 0.9_U \neq 0.7_U = \acute{\mathcal{F}}(b)$. Next, we show that $\mathcal{F} \in \mathscr{BI}(U \sim S)$. Indeed, by Table 2 the semigroup operation on S is given by $\alpha\beta = \min\{\alpha, \beta\}$ with respect to $\leq_S$. Together with the fact that $\acute{\mathcal{F}}$ is order-reversing and $\grave{\mathcal{F}}$ is order-preserving on $(S, \leq_S)$, it can be verified directly that $\mathcal{F}$ satisfies conditions (P1)–(P3) of Definition 3.3. Therefore $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BI}(U \sim S)$. Moreover, the same argument applies to the rough approximation $\mathcal{F}_{\mathfrak{h}}$. Since $\acute{\mathcal{F}}_{\mathfrak{h}}$ remains order-reversing and $\grave{\mathcal{F}}_{\mathfrak{h}}$ remains order-preserving on $(S, \leq_S)$, and the product in S is still determined by $\min$, it follows that $(\acute{\mathcal{F}}_{\mathfrak{h}}, \grave{\mathcal{F}}_{\mathfrak{h}}, S) \in \mathscr{BI}(U \sim S)$.

Theorem 3.9. *Let $(\mathscr{S}(U \sim S), \mathfrak{h})$ be a given $(\mathfrak{h}, =, \supseteq)$-approximation space, and let $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. If $\mathcal{F} \in \mathscr{F}(U \sim S)$, then $\mathcal{F}_{\mathfrak{h}} \in \mathscr{F}(U \sim S)$.*

Proof. Suppose $\mathcal{F} \in \mathscr{F}(U \sim S)$. Then, we consider the following four arguments.

(1) By Theorem 3.7, we have $\mathcal{F}_{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)} \left(\acute{\mathcal{F}}(c)\right) \tilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\alpha\beta)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(a)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)} \left(\grave{\mathcal{F}}(c)\right) \tilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\alpha\beta)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(a).\end{aligned}$$

(3) Let $a, b \in G$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(ab)} \left(\acute{\mathcal{F}}(c)\right) \tilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigwedge}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\alpha\beta)\right) \tilde{\leq} \widetilde{\bigwedge}_{\beta\in\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\beta)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(b)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(ab) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(ab)} \left(\grave{\mathcal{F}}(c)\right) \tilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(c)\right) \\ &= \widetilde{\bigvee}_{\alpha\beta\in\mathfrak{h}(a)\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\alpha\beta)\right) \tilde{\geq} \widetilde{\bigvee}_{\beta\in\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\beta)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(b).\end{aligned}$$

(4) Let $a, b \in G$ be given. Suppose that $a \leq_G b$. Then $\mathfrak{h}(a) \supseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)} \left(\acute{\mathcal{F}}(\alpha)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)} \left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)} \left(\grave{\mathcal{F}}(\alpha)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)} \left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(b).$$

From (1)-(4) above, it follows that $\mathcal{F}_{\mathfrak{h}} \in \mathscr{F}(U \sim G)$. □

Theorem 3.10. *Let* $(\mathscr{S}(U \sim S), \mathfrak{h})$ *be a given* $(\mathfrak{h}, =, \supseteq)$*-approximation space, and let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$. *If* $\mathcal{F} \in \mathscr{BF}(U \sim S)$, *then* $\mathcal{F}_{\mathfrak{h}} \in \mathscr{BF}(U \sim S)$.

Proof. Suppose $\mathcal{F} \in \mathscr{BF}(U \sim S)$. Then, we consider the following three arguments.

(1) By Theorem 3.7, we have $\mathcal{F}_{\mathfrak{h}} \in \mathscr{G}(U \sim S)$.

(2) Let $a, b \in S$ be given. Then

$$\begin{aligned}\acute{\mathcal{F}}_{\mathfrak{h}}(aba) &= \widetilde{\bigwedge}_{c\in\mathfrak{h}(aba)}\left(\acute{\mathcal{F}}(c)\right) \tilde{\leq} \widetilde{\bigwedge}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigwedge}_{\alpha\beta\alpha\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha\beta\alpha)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(a)\end{aligned}$$

and

$$\begin{aligned}\grave{\mathcal{F}}_{\mathfrak{h}}(aba) &= \widetilde{\bigvee}_{c\in\mathfrak{h}(aba)}\left(\grave{\mathcal{F}}(c)\right) \tilde{\geq} \widetilde{\bigvee}_{c\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(c)\right)\\ &= \widetilde{\bigvee}_{\alpha\beta\alpha\in\mathfrak{h}(a)\mathfrak{h}(b)\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha\beta\alpha)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(a).\end{aligned}$$

(3) Let $a, b \in S$ be given. Suppose that $a \leq_S b$. Then $\mathfrak{h}(a) \supseteq \mathfrak{h}(b)$. Thus

$$\acute{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(a)}\left(\acute{\mathcal{F}}(\alpha)\right) \tilde{\leq} \widetilde{\bigwedge}_{\alpha\in\mathfrak{h}(b)}\left(\acute{\mathcal{F}}(\alpha)\right) = \acute{\mathcal{F}}_{\mathfrak{h}}(b)$$

and

$$\grave{\mathcal{F}}_{\mathfrak{h}}(a) = \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(a)}\left(\grave{\mathcal{F}}(\alpha)\right) \tilde{\geq} \widetilde{\bigvee}_{\alpha\in\mathfrak{h}(b)}\left(\grave{\mathcal{F}}(\alpha)\right) = \grave{\mathcal{F}}_{\mathfrak{h}}(b).$$

From (1)-(3) above, it follows that $\mathcal{F}_{\mathfrak{h}} \in \mathscr{BF}(U \sim S)$. □

The following example confirms Theorem 3.10 by showing that the lower rough approximation of a fuzzy semibipolar soft bi-filter remains a fuzzy semibipolar soft bi-filter.

Example 3.7. *Let* $(S := \{a, b, c, d, e\}, \cdot, \leq_S)$ *be an ordered semigroup, where the binary operation* $\cdot$ *on* S *is given by Table 2 and the order relation* $\leq_S$ *on* S *is defined by*

$$a \leq_S b \leq_S c \leq_S d \leq_S e.$$

Define $\mathfrak{h} : S \to \mathscr{P}(S) \setminus \{\emptyset\}$ *by*

$$\mathfrak{h}(\alpha) = \{\beta \in S \mid \alpha \leq_S \beta\} = \begin{cases}\{a, b, c, d, e\}, & \alpha = a,\\ \{b, c, d, e\}, & \alpha = b,\\ \{c, d, e\}, & \alpha = c,\\ \{d, e\}, & \alpha = d,\\ \{e\}, & \alpha = e.\end{cases}$$

We show that $\mathfrak{h}$ *satisfies conditions* (P1)–(P2) *of Definition 3.5.*

(P1) *For all* $\alpha, \beta \in S$, *we have* $\mathfrak{h}(\alpha)\mathfrak{h}(\beta) = \mathfrak{h}(\alpha\beta)$. *Indeed, since the multiplication in Table 2 satisfies* $\alpha\beta = \min\{\alpha, \beta\}$ *(with respect to* $\leq_S$*), it follows that* $\mathfrak{h}(\alpha\beta) = \mathfrak{h}(\min\{\alpha, \beta\})$, *which coincides with the set product* $\mathfrak{h}(\alpha)\mathfrak{h}(\beta)$.

(P2) *If* $\alpha \leq_S \beta$, *then* $\mathfrak{h}(\alpha) \supseteq \mathfrak{h}(\beta)$ *holds immediately from the definition of* $\mathfrak{h}$.

Hence $(\mathscr{S}(U \sim S), \mathfrak{h})$ *is a* $(\mathfrak{h}, =, \supseteq)$*-approximation space. Now let* $\mathcal{F} := (\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{S}(U \sim S)$, *where*

$$\acute{\mathcal{F}}(\alpha) = \begin{cases} 0.1_U, & \alpha = a, \\ 0.2_U, & \alpha = b, \\ 0.4_U, & \alpha = c, \\ 0.6_U, & \alpha = d, \\ 0.7_U, & \alpha = e, \end{cases} \qquad \grave{\mathcal{F}}(\alpha) = \begin{cases} 0.9_U, & \alpha = a, \\ 0.8_U, & \alpha = b, \\ 0.6_U, & \alpha = c, \\ 0.4_U, & \alpha = d, \\ 0.3_U, & \alpha = e. \end{cases}$$

Define $(\acute{\mathcal{F}}_{\mathfrak{h}}, \grave{\mathcal{F}}_{\mathfrak{h}}, S) \in \mathscr{S}(U \sim S)$ *by*

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigvee}_{\beta \in \mathfrak{h}(\alpha)} \acute{\mathcal{F}}(\beta), \qquad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \widetilde{\bigwedge}_{\beta \in \mathfrak{h}(\alpha)} \grave{\mathcal{F}}(\beta), \quad \alpha \in S.$$

Then, by direct computation, we obtain

$$\acute{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \acute{\mathcal{F}}(\alpha) \quad \text{and} \quad \grave{\mathcal{F}}_{\mathfrak{h}}(\alpha) = \grave{\mathcal{F}}(\alpha) \qquad (\forall \alpha \in S).$$

For instance, $\acute{\mathcal{F}}_{\mathfrak{h}}(c) = \widetilde{\bigvee}\{\acute{\mathcal{F}}(c), \acute{\mathcal{F}}(d), \acute{\mathcal{F}}(e)\} = \widetilde{\bigvee}\{0.4_U, 0.6_U, 0.7_U\} = 0.7_U = \acute{\mathcal{F}}(c)$, *and similarly for the remaining cases. Next, one can verify that* $\mathcal{F}$ *satisfies conditions* (P1)–(P3) *of the definition of a fuzzy semibipolar soft bi-filter. Therefore,* $(\acute{\mathcal{F}}, \grave{\mathcal{F}}, S) \in \mathscr{BF}(U \sim S)$. *Moreover, since* $\mathcal{F}_{\mathfrak{h}} = \mathcal{F}$ *in this case, it follows that* $(\acute{\mathcal{F}}_{\mathfrak{h}}, \grave{\mathcal{F}}_{\mathfrak{h}}, S) \in \mathscr{BF}(U \sim S)$.

4. Conclusions

This paper establishes a rough approximation framework for fuzzy semibipolar soft structures on ordered semigroups by means of set-valued functions. The proposed approach unifies order-theoretic properties with bipolar fuzzy information and provides a systematic treatment of rough upper and lower approximations under various $\mathfrak{h}$-approximation spaces.

The main contribution lies in demonstrating that fuzzy semibipolar soft ideals, filters, bi-ideals, and bi-filters are stable under rough approximation operators when suitable compatibility conditions between $\mathfrak{h}$ and the underlying order are satisfied. The obtained counterexamples further show that fuzzy semibipolar soft bi-ideals and bi-filters constitute proper generalizations of classical fuzzy semibipolar soft ideals and filters, thereby justifying the necessity of the proposed extensions.

A global view of the preservation phenomena is summarized in Table 3, which highlights the precise relationship between approximation spaces, rough operators, and the corresponding preserved fuzzy semibipolar soft structures.

TABLE 3. Approximation spaces and preserved fuzzy semibipolar soft structures

Approximation Space	Rough Operator	Preserved Structure
$(\mathfrak{h}, \subseteq, \supseteq)$	Upper	Ideal, Bi-ideal
$(\mathfrak{h}, \subseteq, \cong)$	Upper	Subgroupoid
$(\mathfrak{h}, =, \subseteq)$	Upper	Filter, Bi-filter
$(\mathfrak{h}, =, \subseteq)$	Lower	Ideal, Bi-ideal
$(\mathfrak{h}, =, \cong)$	Lower	Subgroupoid
$(\mathfrak{h}, =, \supseteq)$	Lower	Filter, Bi-filter
General $\mathfrak{h}$-spaces	Upper / Lower	Closure of fuzzy semibipolar soft sets

Overall, the results confirm that the proposed rough framework is structurally consistent and sufficiently general to accommodate multiple classes of fuzzy semibipolar soft algebraic systems. The use of set-valued functions plays a crucial role in controlling the interaction between order relations and rough approximations, thereby providing a flexible mechanism for modeling uncertainty within ordered algebraic structures.

As future research directions, the present framework can be extended to other non-classical algebraic systems, including BE-algebras [17], pseudo-BCI algebras [18], UP-algebras [19], BCK-algebras [20], and hemirings [21]. Such extensions are expected to broaden the applicability of fuzzy semibipolar soft rough models and to support further developments in uncertainty-oriented algebra and decision-making theory.

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