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ON A CLASS OF BARIC ALGEBRAS SATISFYING A POLYNOMIAL IDENTITY OF DEGREE FIVE

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Abstract. In this paper, we are interested in a class of algebras satisfying the identity $x^5 + \lambda x^3x^2 - (1 + \lambda)x(x^2x^2) = 0$, with $\lambda \notin \{0; 1; \frac{1}{2}; \frac{3}{2}\}$ a scalar. After giving weightability conditions for these algebras, we study algebras of this class which are Bernstein or principal train algebras of rank 3.

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1. Introduction

Great advances in the study of Jordan algebras have been achieved through the work of several authors including A. A. Albert ([1]) and R. Schafer ([14]). These algebras are used in many areas of mathematics, including the algebra of population genetics. The class of Jordan algebras and their generalization have been the subject of studies by authors like Hentzel, Carini ([5]), Labra, M. Flores, Guzzo Jr. ([3,7,9]) and then recently by Dembega and Ouattara ([6]). In ([9]), Guzzo and Labra showed that any generalized almost-Jordan algebra satisfying the identity $x(x^2y)+2x(x(xy))-3x^3y = 0$, satisfies more generally an identity of degree greater than 4, one of which is given by $2x^4x^k - 2x^2x^{k+2} + (x^2)^2x^k - x^2(x^2x^k) = 0$, $\forall k \geq 1$. Later, Guiro et al.([8]) studied this last class of algebras for the case $k = 1$ and its generalization which is the identity $x^5+\lambda x^3x^2-(1+\lambda)x(x^2x^2) = 0$ where $\lambda \in K-\{0; 1; \frac{1}{2}\}$. They show that these two classes of algebras strictly contain that of generalized almost-Jordan algebras. They study structure and other notions such as derivations and representations of such algebras. In this paper we were interested in baric algebras that satisfies a generalization of the identity studied by Guiro et al., i.e. $x^5 + \lambda x^3x^2 - (1 + \lambda)x(x^2x^2) = 0$, where $\lambda \in K - \{0; 1; \frac{1}{2}; \frac{3}{2}\}$. After having given

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structural properties of these algebras, we give the conditions of such algebras to be Bernstein algebras (see [11], [15]) or principal algebra.

2. Peirce decomposition

Definition 2.1. *Let K be a commutative field and A a commutative K-algebra, not necessarily associative. We said A is:*

i) *power-associative algebra if $x^i x^j = x^{i+j}$ for any $x \in A$ and for all integers $i, j \geq 1$;*
ii) *a baric algebra if there exists a non-zero morphism of algebras $\omega : A \to K$. The morphism ω is then called the weight function of the algebra A. The weight of an element x of A is the scalar $\omega(x)$.*

Definition 2.2. *Let K be a commutative field. A baric K-algebra (A, ω) is a:*

(1) *Bernstein algebra if $(x^2)^2 - \omega(x)^2 x^2 = 0$ for all x in A;*
(2) *principal train algebra of rank $n \geq 2$ if there are $\alpha_1, \cdots, \alpha_{n-1} \in K$, such that $x^n + \alpha_1 \omega(x) x^{n-1} + \cdots + \alpha_{n-1} \omega(x)^{n-1} x = 0$ for any $x \in A$, where n is the smallest such integer.*

For the following result on Bernstein algebras, see [16].

Proposition 2.3. *Let K be an infinite commutative field of characteristic different from 2 and (A, ω) a Bernstein K-algebra. The algebra A admits a Peirce decomposition $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}}$, where $A_0 = \{x \in \ker\omega | ex = 0\}$ and $A_{\frac{1}{2}} = \{x \in \ker\omega | ex = \frac{1}{2}x\}$. For all $x_0 \in A_0,\ x_{\frac{1}{2}} \in A_{\frac{1}{2}}$, we have: $x_{\frac{1}{2}}^3 = 0$, $x_{\frac{1}{2}}(x_{\frac{1}{2}} x_0) = 0$, $x_{\frac{1}{2}} x_0^2 = 0$, $x_{\frac{1}{2}} x_0^2 = 0$.*

Theorem 2.4. [14] *Let K be a commutative field of characteristic different from $2, 3, 5$ and A a commutative K-algebra. Then, A is power-associative if and only if $(x^2)^2 = x^4$ for any $x \in A$.*

In the following, K is an infinite commutative field of characteristic different from $2, 3, 5$ and 7 and A be a K-algebra that satisfies the identity

$$x^5 + \lambda x^3 x^2 - (1+\lambda) x(x^2 x^2) = 0, \tag{1}$$

with $\lambda \in K - \{0; 1; \frac{1}{2}; \frac{3}{2}\}$.

Example 2.5. *Let $\{e_0, e_1, e_2\}$ be a basis of a commutative K-algebra A with the following multiplication table: $e_0^2 = e_0, e_1^2 = 2e_1, e_1 e_2 = e_2$, the products not mentioned being zero. The algebra A satisfies the identity (1) and the set of non-zero idempotents of this algebra is $\{e_0, \alpha e_0 + \frac{1}{2} e_1 + \gamma e_2, \alpha \in \{0, 1\}, \gamma \in K\}$.*

In this paper, A is a commutative non-associative algebra satisfying the identity (1). We also assume that A admits an idempotent $e \neq 0$.

We begin by giving general results of this algebra class.

Theorem 2.6. *The Peirce decomposition of A relative to e is : $A = A_0 \oplus A_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ where $A_\rho = \{x \in A | ex = \rho x\}$, $\rho \in \{0, 1, \frac{1}{2}, \lambda\}$, and $L_e : A \to A, x \mapsto ex$.*

Proof. By partial linearization of (1) we have

$$x^4 y + x(x^3 y) + x(x(x^2 y)) + 2x(x(x(xy))) + 2\lambda x^3(xy) + \lambda x^2(x^2 y) + 2\lambda x^2(x(xy)) \\ -(1+\lambda)(x^2)^2 y - 4(1+\lambda)x(x^2(xy)) = 0. \tag{2}$$

For $x = e$ in (2), we have $P(L_e)(y) = 0$ where e is a non-zero idempotent of A, $L_e : A \to A, y \mapsto ey$ and $P(X) = 2X^4 - (2\lambda + 3)X^3 + (1 + 3\lambda)X^2 - \lambda X$. We have $P(X) = 2X(X-1)(X-\frac{1}{2})(X-\lambda)$ and according to a well-known theorem in linear algebra, $A = \ker P(L_e) = \ker L_e \oplus \ker(L_e - id_A) \oplus \ker(L_e - \frac{1}{2} id_A) \oplus \ker(L_e - \lambda id_A)$. So the theorem is established. □

Proposition 2.7. *Consider the Peirce decomposition of A given by $A = A_0 \oplus A_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$. We have the following relations:*

$$A_{\frac{1}{2}} A_s \subseteq A_j \oplus A_k \oplus A_l,$$

with $s, j, k, l \in \{0, 1, \frac{1}{2}, \lambda\}$ pairwise distinct.

Proof. By taking $x = e$, $y \in A_\mu$ and $z \in A_\nu$ in one of the partial linearisation of (1), we define a polynomial $Q_{\mu,\nu}$ by:

$$Q_{\mu,\nu}(X) = 2X^3 + (-4 - 2\lambda + 2(\mu+\nu))X^2 + (2(\mu^2+\nu^2) - 8(1+\lambda)\mu\nu + (1+2\lambda)(\mu+\nu) + 2\lambda)X \\ +2(\mu^3+\nu^3) - (3+4\lambda)(\mu^2+\nu^2) + 4\lambda\mu\nu(\mu+\nu+1) + (\mu+\nu). \tag{3}$$

We will check if $0, 1, \frac{1}{2}$ or λ are roots of the polynomial defined in (3) for $\mu = \frac{1}{2}$ and $\nu = s \in \{0, 1, \frac{1}{2}, \lambda\}$. This gives us:

$$Q_{\frac{1}{2},s}(X) = 2(X-j)(X-k)(X-l),$$

therefore $A_{\frac{1}{2}} A_s \subseteq A_j \oplus A_k \oplus A_l$, with $s, j, k, l \in \{0, 1, \frac{1}{2}, \lambda\}$ pairwise distinct. □

Proposition 2.8. *Let $A = A_0 \oplus A_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$, we have:*

i) $A_0^2 \subseteq A_0$;
ii) $A_1^2 \subseteq A_1$;
iii) $A_\lambda^2 = \{0\}$;
iv) $A_0 A_1 = \{0\}$;
v) $A_0 A_\lambda = \{0\}$;
vi) $A_1 A_\lambda \subseteq A_\lambda$.

Proof. This result can be demonstrated in the same way as that of Proposition 2.7. □

3. Baric algebras satisfying identity (1)

If A is a commutative K-algebra, then we define a new algebra $A^\sharp = K \oplus A$, whose multiplication is given by $(\alpha + u)(\mu + v) = \alpha\mu + \alpha v + \mu u + uv$, where $\alpha, \mu \in K$ and $u, v \in A$, $A^\sharp$ admits the unit element $1 + 0$.

Proposition 3.1. *Suppose that A is power-associative. Let $\omega : A^\sharp \to K$ given by $\omega(\alpha + u) = \alpha$, then $(A^\sharp, \omega)$ is a baric algebra satisfying the identity (1).*

Proof. Let $\alpha \in K, u \in A$, $x = \alpha + u$, thus $x^2 = \alpha^2 + 2\alpha u + u^2$, $x^3 = xx^2 = \alpha^3 + 3\alpha u^2 + 3\alpha^2 u + u^3$, $x^5 = \alpha^5 + 5\alpha^4 u + 10\alpha^3 u^2 + 10\alpha^2 u^3 + 5\alpha u^4 + u^5$ and. Therefore

$$x^5 + \lambda x^2 x^3 - (1 + \lambda)x(x^2 x^2) = \alpha(1 - 2\lambda)(u^4 - u^2 u^2) = 0.$$

We show now that ω is a non-zero morphism. Let $x, y \in A^\sharp$ be such that $x = \alpha + u$ and $y = \mu + v$ with $u, v \in A$, $\alpha, \mu \in K$, we have $xy = \alpha\mu + \alpha v + \mu u + uv$ and $x + y = (\alpha + \mu) + u + v$.
We have for all α non-zero of $K, \omega(\alpha + u) = \alpha \neq 0$, so ω is non-zero.
We have: $\omega(x + y) = \omega((\alpha + \mu) + u + v) = \alpha + \mu = \omega(\alpha + u) + \omega(\mu + v)$,
$\omega(xy) = \omega(\alpha\mu + \alpha v + \mu u + uv) = \alpha\mu = \omega(\alpha + u)\omega(\mu + v)$ et $\omega(1_K + u) = 1_K$. ☐

This result given in this section show that algebras of this class can be weighted, that is to say provided with weight morphism, under certain conditions.

In [10, Proposition 2.1], the authors gives through the following proposition, the conditions for a commutative non-associative algebra to be equipped with a weight morphism on a commutative field of characteristic different from 2.

Proposition 3.2. *Let A be a commutative non-associative K-algebra. The following assertions are equivalent:*

i) *A has a weight morphism;*
ii) *A admits a basis $(e_i)_{i \in I}$ such that*
$e_i e_j = \sum_{k \in I} \gamma_{i,j,k} e_k$ *and* $\sum_{k \in I} \gamma_{i,j,k} = 1$;
iii) *There exists an ideal E of codimension 1 such that $A^2 \not\subseteq E$;*
iv) *There exists an ideal E of codimension 1, and $e \notin E$ such that $e^2 - e \in E$.*

In the rest of the document we assume that A is a commutative non-associative algebra satisfying the identity (1) that admits an idempotent e such that $\omega(e) = 1$.

Lemma 3.3. *The algebra A admits the Peirce decomposition $A = Ke \oplus A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ where $U_1 = A_1 \cap \ker(\omega)$.*

Proof. Since $\omega(e) = 1$, we have $A = Ke \oplus \ker(\omega)$. Let $x \in A_1$, thus $x = \alpha e + a$ where $a \in \ker(\omega)$ and $\alpha \in K$. Since $ex = x$, then $x = \alpha e + ea = \alpha e + a$. So $ea = a$ and then $a \in A_1 \cap \ker(\omega)$. Therefore

$A_1 \subseteq Ke \oplus (A_1 \cap \ker(\omega))$. The other inclusion is obvious. As a result $A_1 = Ke \oplus (A_1 \cap \ker(\omega))$. Setting $U_1 = A_1 \cap \ker(\omega)$, we have $\ker(\omega) = A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ and $A = Ke \oplus A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$. □

Proposition 3.4. *The following assertions are equivalent:*

i) *A has a weight morphism;*

ii) $A_1 = Ke \oplus U_1$, $U_1 \subseteq A_1$, $U_1^2 \subseteq U_1$, $U_1 A_{\frac{1}{2}} \subseteq A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda$, $U_1 A_0 = \{0\}$, $U_1 A_\lambda \subseteq A_\lambda$, $A_{\frac{1}{2}} A_s \subseteq U_1 \oplus A_j \oplus A_k$ *with* $s, j, k \in \{0, \frac{1}{2}, \lambda\}$ *pairwise distinct.*

Proof. Let be $A = A_0 \oplus A_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$, a Peirce decomposition of A.

$(i) \to (ii)$ Since A admits an idempotent e with $\omega(e) = 1$, according to Lemma 3.3, we have $A = Ke \oplus A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ where $U_1 = A_1 \cap \ker(\omega)$. We have $\ker(\omega) = A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ and since $ker(\omega)$ is an un ideal of A, then

$U_1^2 \subseteq U_1 A_1 \subseteq A_1^2 \subseteq A_1 \cap ker(\omega) = U_1$,

$U_1 A_{\frac{1}{2}} \subseteq A_1 A_{\frac{1}{2}} \subseteq (A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda) \cap ker(\omega) = A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda$,

$U_1 A_0 \subseteq A_1 A_0 = \{0\}$, $U_1 A_\lambda \subseteq A_1 A_\lambda \subseteq A_\lambda$,

$A_{\frac{1}{2}} A_s \subseteq (A_1 \oplus A_j \oplus A_k) \cap ker(\omega) = U_1 \oplus A_j \oplus A_k$ with $s, j, k \in \{0, \frac{1}{2}, \lambda\}$ pairwise distinct thus A is weightable which gives (ii).

$(ii) \to (i)$ Conversely (ii) leads to $A = Ke \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda$. We have $U_1^2 \subseteq U_1$, $U_1 A_{\frac{1}{2}} \subseteq A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda$, $U_1 A_0 = \{0\}$, $U_1 A_\lambda \subseteq A_\lambda$, $A_{\frac{1}{2}} A_s \subseteq U_1 \oplus A_j \oplus A_k$ with $s, j, k \in \{0, \frac{1}{2}, \lambda\}$ two by two distinct therefore $I = A_0 + U_1 + A_{\frac{1}{2}} + A_\lambda$ is an ideal of A of codimension 1. Moreover $e \notin I$ and $e^2 - e \in I$ therefore, according to Proposition 3.2, A is weightable. □

4. Algebras satisfying the identity (1) which are Bernstein

In the following, we will consider (A, ω) a baric K-algebra satisfying the identity (1) and e a non-zero idempotent such that $\omega(e) = 1$.

Lemma 4.1. *Let* $A = Ke \oplus A_0 \oplus A_{\frac{1}{2}}$ *Peirce decomposition of A. Suppose that* $A_0^2 = \{0\}$, $A_{\frac{1}{2}}^2 \subseteq A_0$, $A_0 A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$. *For any* $x_\alpha \in A_\alpha$ *where* $\alpha \in \{0, \frac{1}{2}\}$, *the following identities hold.*

(i) $x_{\frac{1}{2}}^3 = 0$,

(ii) $x_{\frac{1}{2}}(x_{\frac{1}{2}} x_0) = 0$,

(iii) $x_0(x_0 x_{\frac{1}{2}}) = 0$,

(iv) $(x_0 x_{\frac{1}{2}})^2 = 0$,

(v) $x_{\frac{1}{2}}^2 (x_{\frac{1}{2}} x_0) = 0$.

Proof. Let $x = e + \alpha_1 x_0 + \alpha_2 x_{\frac{1}{2}} \in A$ such that $A_0^2 = \{0\}$, $A_{\frac{1}{2}}^2 \subseteq A_0$, $A_0 A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$. By replacing x with its expression in the identity (1) and by identification according to the values of $\alpha_1^i \alpha_2^j$, with i and j integers such as $0 \leq i \leq 4$, $1 \leq i \leq 5$ and $i + j \leq 5$, we obtain the desired result. □

Proposition 4.2. *Let $A = Ke \oplus A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ the Peirce decomposition of baric algebra satisfying identity (1). Then, A is Bernstein if and only if $U_1 = A_\lambda = A_0^2 = \{0\}$.*

Proof. Since the set of elements of weight 1 is dense in A according to the Zariski topology, the proof will be done by considering the elements of weight 1. Let $A = Ke \oplus A_0 \oplus U_1 \oplus A_{\frac{1}{2}} \oplus A_\lambda$ the Peirce decomposition of a baric algebra satisfying identity (1). We have $A_0^2 \subseteq A_0$, $U_1^2 \subseteq U_1$, $U_1 A_{\frac{1}{2}} \subseteq A_0 \oplus A_{\frac{1}{2}} \oplus A_\lambda$, $A_\lambda^2 = A_0 A_\lambda = U_1 A_0 = \{0\}$, $U_1 A_\lambda \subseteq A_\lambda$, $A_{\frac{1}{2}} A_s \subseteq U_1 \oplus A_j \oplus A_k$ with $s, j, k \in \{0, \frac{1}{2}, \lambda\}$ pairwise distinct.

Suppose that A is Bernstein. Then, $A_0^2 \subseteq A_{\frac{1}{2}} \cap A_0 = 0$ and $U_1 = A_\lambda = 0$.

Conversely, if we suppose the conditions $U_1 = A_\lambda = A_0^2 = \{0\}$ hold, thus we have $A_{\frac{1}{2}} A_0 \subseteq A_{\frac{1}{2}}$ and $A_{\frac{1}{2}}^2 \subseteq A_0$. Let $x = e + x_0 + x_{\frac{1}{2}} \in A$, then $x^2 = e + x_{\frac{1}{2}} + x_{\frac{1}{2}}^2 + 2x_0 x_{\frac{1}{2}}$ and $(x^2)^2 = e + x_{\frac{1}{2}}^2 + x_{\frac{1}{2}} + 2x_0 x_{\frac{1}{2}} + 4(x_0 x_{\frac{1}{2}})^2 + 2x_{\frac{1}{2}}^3 + 4x_{\frac{1}{2}}^2 (x_0 x_{\frac{1}{2}}) + 4x_{\frac{1}{2}}(x_0 x_{\frac{1}{2}})$. Using identities of Lemma 4.1 we have: $(x^2)^2 = e + x_{\frac{1}{2}} + x_{\frac{1}{2}}^2 + 2x_0 x_{\frac{1}{2}}$. Therefore, $(x^2)^2 - x^2 = 0$ and according to Zariski's topology, then $(x^2)^2 - \omega(x)^2 x^2 = 0$ and A is Bernstein algebra. □

5. Algebras satisfying the identity (1) which are principal train of rank 3

In this section we characterize algebras satisfying the identity (1) that are principal train of rank 3.

Theorem 5.1. *Let (A, ω) a baric K-algebra satisfying the identity (1). Then, A is a principal train of rank 3 and equation $x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2 x = 0 \quad (\star)$ if and only if $\alpha \in \{0; 1; \lambda\}$*

Proof. A partial linearization of $(\star)$ gives:

$$x^2 y + 2x(xy) - (1+\alpha)\omega(y)x^2 - 2(1+\alpha)\omega(x)xy$$
$$+\alpha\omega(x)^2 y + 2\alpha\omega(xy)x = 0. \qquad (*)$$

Setting $y = x^3$ in $(*)$, we obtain $x^2 x^3 = \omega(x)^3[(\alpha+1)^2 x^2 - (\alpha^2 + 2\alpha)\omega(x)x]$. For $y = x^2$ in $(*)$, we have, $L_x((x^2)^2) = \omega(x)^3[(2\alpha^2 + \alpha + 1)x^2 - (2\alpha^2 + \alpha)\omega(x)x]$. The equality $(\star)$ gives $L_x^2(x^3) = \omega(x)^3[(\alpha^3 + \alpha^2 + \alpha + 1)x^2 - (\alpha^3 + \alpha^2 + \alpha)\omega(x)x]$. Therefore, $0 = x^5 + \lambda x^2 x^3 - (1+\lambda)x(x^2)^2 = \omega(x)^3(\alpha^3 - (1+\lambda)\alpha^2 + \lambda\alpha)[x^2 - \omega(x)x]$ and $\alpha^3 - (1+\lambda)\alpha^2 + \lambda\alpha = 0$, so $\alpha \in \{0; 1; \lambda\}$. The converse is obvious. □

Theorem 5.2. *Let (A, ω) a baric K-algebra satisfying the identity (1) that is principal train of equation $x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2 x = 0$ with $\alpha \in \{1; \lambda\}$. Then, we have $A = Ke \oplus A_{\frac{1}{2}} \oplus A_\alpha$ and $A_{\frac{1}{2}}^2 \subseteq A_\alpha$, $A_\alpha^2 = \{0\}$, $A_{\frac{1}{2}} A_\alpha \subseteq A_{\frac{1}{2}}$.*

Remark 5.3. *Let (A, ω) a baric K-algebra satisfying the identity (1) that is principal train algebra of equation $x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2 x = 0$ with $\alpha \in \{0; 1; \lambda\}$*

(1) *If $\alpha = 0$, i.e $x^3 - \omega(x)x^2 = 0 \quad \forall x \in A$, then A is a Bernstein-Jordan algebra.*

(2) *If* $\alpha = 1$, *i.e* $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0 \quad \forall x \in A$, *then* $A = Ke \oplus A_{\frac{1}{2}} \oplus U_1$ *and* $A_{\frac{1}{2}}^2 \subseteq U_1$, $U_1^2 = \{0\}$, $U_1 A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$. *So* $A_0 = A_\lambda = 0$.

(3) *If* $\alpha = \lambda$, *i.e* $x^3 - (1+\lambda)\omega(x)x^2 + \lambda\omega(x)^2 x = 0 \quad \forall x \in A$, *then* $A = Ke \oplus A_{\frac{1}{2}} \oplus A_\lambda$ *and* $A_{\frac{1}{2}}^2 \subseteq A_\lambda$, $A_\lambda^2 = \{0\}$, $A_{\frac{1}{2}} A_\lambda \subseteq A_{\frac{1}{2}}$. *So* $U_1 = A_0 = 0$.

Lemma 5.4. *Let* $A = Ke \oplus A_{\frac{1}{2}} \oplus U_1$ *be a Peirce decomposition of a baric* K*-algebra satisfying the identity* (1) *that is principal train of equation* $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$. *For all* $x_1 \in U_1$ *and* $x_{\frac{1}{2}} \in A_{\frac{1}{2}}$, *we have:*

(1) $x_{\frac{1}{2}}^3 = 0$,
(2) $x_{\frac{1}{2}}(x_{\frac{1}{2}} x_1) = 0$,
(3) $x_{\frac{1}{2}}^2(x_{\frac{1}{2}} x_1) = 0$,
(4) $(x_{\frac{1}{2}} x_1)^2 = 0$,
(5) $x_1(x_1 x_{\frac{1}{2}}) = 0$.

Theorem 5.5. *Let* A *be a baric* K*-algebra satisfying the identity* (1) *that is principal train of equation* $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$. *Then,* A *is power-associative.*

Proof. Let $A = Ke \oplus A_{\frac{1}{2}} \oplus U_1$ be a Peirce decomposition of a baric K-algebra satisfying the identity (1) that is principal train of equation $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$. According to the Zariski topology, the proof will be done by considering the elements of weight 1. Let be $x = e + x_1 + x_{\frac{1}{2}} \in A$ with $x_1 \in U_1$ and $x_{\frac{1}{2}} \in A_{\frac{1}{2}}$, we have, using Theorem 5.2 and identities of Lemma 5.4:

$$x^2 = e + 2x_1 + x_{\frac{1}{2}} + 2x_{\frac{1}{2}} x_1 + x_{\frac{1}{2}}^2,$$

$$\begin{aligned}
x^3 &= e + 3x_1 + x_{\frac{1}{2}} + 4x_{\frac{1}{2}} x_1 + 2x_{\frac{1}{2}}^2 + x_{\frac{1}{2}}^3 + 2x_1(x_1 x_{\frac{1}{2}}) \\
&\quad + 2x_{\frac{1}{2}}(x_{\frac{1}{2}} x_1) = e + 3x_1 + x_{\frac{1}{2}} + 4x_{\frac{1}{2}} x_1 + 2x_{\frac{1}{2}}^2, \\
x^4 &= e + 4x_1 + x_{\frac{1}{2}} + 6x_{\frac{1}{2}} x_1 + 3x_{\frac{1}{2}}^2 + 2x_{\frac{1}{2}}^3 + 4x_1(x_1 x_{\frac{1}{2}}) \\
&\quad + 4x_{\frac{1}{2}}(x_{\frac{1}{2}} x_1) = e + 4x_1 + x_{\frac{1}{2}} + 6x_{\frac{1}{2}} x_1 + 3x_{\frac{1}{2}}^2, \\
(x^2)^2 &= e + 4x_1 + x_{\frac{1}{2}} + 6x_{\frac{1}{2}} x_1 + 3x_{\frac{1}{2}}^2 + 2x_{\frac{1}{2}}^3 + 8x_1(x_1 x_{\frac{1}{2}}) \\
&\quad + 4x_{\frac{1}{2}}(x_{\frac{1}{2}} x_1) + 4(x_{\frac{1}{2}} x_1)^2 = e + 4x_1 + x_{\frac{1}{2}} + 6x_{\frac{1}{2}} x_1 + 3x_{\frac{1}{2}}^2
\end{aligned}$$

so $(x^2)^2 - x^4 = 0$ and A is power-associative. □

Lemma 5.6. *Let* $A = Ke \oplus A_{\frac{1}{2}} \oplus A_\lambda$ *be a Peirce decomposition of a baric* K*-algebra satisfying the identity* (1) *that is principal train of equation* $x^3 - (1+\lambda)\omega(x)x^2 + \lambda\omega(x)^2 x = 0$. *For all* $x_\lambda \in A_\lambda$ *and* $x_{\frac{1}{2}} \in A_{\frac{1}{2}}$, *we have:*

(1) $x_{\frac{1}{2}}^3 = 0$,
(2) $x_{\frac{1}{2}}(x_{\frac{1}{2}} x_\lambda) = 0$,
(3) $x_{\frac{1}{2}}^2(x_{\frac{1}{2}} x_\lambda)$,
(4) $(x_{\frac{1}{2}} x_\lambda)^2 = 0$,

(5) $x_\lambda(x_\lambda x_{\frac{1}{2}}) = 0$.

Corollary 5.7. *Let A be a baric K-algebra satisfying the identity (1) that is principal train of equation $x^3 - (1+\lambda)\omega(x)x^2 + \lambda\omega(x)^2 x = 0$. Then A is not power-associative.*

Proof. For $x = e + x_\lambda \in A$ with $x_\lambda \in A_\lambda$, we have:
$x^2 = e + 2\lambda x_\lambda$, $x^3 = e + (2\lambda^2 + \lambda)x_\lambda$, $x^4 = e + \lambda(2\lambda^2 + \lambda + 1)x_\lambda$, $(x^2)^2 = e + 4\lambda^2 x_\lambda$, so $x^4 - (x^2)^2 = \lambda(2\lambda^2 - 3\lambda + 1)x_\lambda = \lambda(2\lambda - 1)(\lambda - 1)x_\lambda \neq 0$ and A is not power-associative. □

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