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CHARACTERIZATIONS OF PRODUCTS OF IVIFSS-SUBALGEBRAS AND IDEALS IN SHEFFER STROKE HILBERT ALGEBRAS

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Abstract. This paper develops a comprehensive framework for interval-valued intuitionistic fuzzy Sheffer stroke subsets (IVIFSS) within Sheffer stroke Hilbert algebras (SSHA). We introduce and characterize IVIFSS-subalgebras and IVIFSS-ideals, proving that these structures are closed under Cartesian products, intersections, and the operators $\oplus$ and $\otimes$. It is further shown that the product of two IVIFSS-subalgebras (or IVIFSS-ideals) preserves the corresponding algebraic properties in the product algebra. By employing level subset characterizations, we derive necessary and sufficient conditions for IVIFSS-filters and IVIFSS-deductive systems, demonstrating that both notions are equivalent. These results generalize classical theories of subalgebras, ideals, and filters in Hilbert algebras to the interval-valued intuitionistic fuzzy context, thereby enriching the study of fuzzy algebraic logic and providing a unified algebraic foundation for approximate reasoning frameworks.

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1. Introduction

Sheffer stroke Hilbert algebras (SSHAs) form a fundamental class of algebraic structures that underpin the formalization of logical operations. Characterized by the Sheffer stroke (NAND) operation, these algebras provide a versatile framework for modeling logical inference and have applications ranging from formal logic and automated reasoning to artificial intelligence and quantum computation.

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The Sheffer stroke is a minimal operation that can generate complex logical expressions, making it central to the study and construction of advanced algebraic-logical systems.

The exploration of Sheffer stroke operations across different algebraic settings has revealed significant structural insights. Notable investigations have examined Sheffer stroke reducts in basic algebras [17,20], their extension to MTL-algebras [19], and their interaction with ortholattices [7]. In parallel, the incorporation of fuzzy sets into Sheffer stroke BE-algebras [8] has demonstrated the potential of combining fuzzy uncertainty with non-classical algebraic frameworks.

Since Zadeh's introduction of fuzzy sets [22], which provided a systematic approach to handling imprecision and vagueness, numerous extensions and generalizations have emerged. These include integration with other uncertainty paradigms, such as soft sets and rough sets [1,3,6]. A particularly influential development has been the notion of intuitionistic fuzzy sets, introduced by Atanassov [2], which extend fuzzy sets by accounting for both membership and non-membership degrees. This dual characterization enhances their suitability for modeling incomplete or uncertain information, and they have been effectively applied in areas such as medical decision-making, optimization, and multi-criteria evaluation [11,12].

Hilbert algebras, first formalized in the mid-20th century by Henkin [13], provide an algebraic foundation for the study of implications in intuitionistic and related logics. Subsequent contributions by Horn, Diego, and others demonstrated important properties, including the local finiteness of these varieties [9]. Later work focused on the role of filters in Hilbert algebras as a mechanism for constructing deductive systems [4,5,15], with additional investigations addressing the fuzzification of subalgebras and deductive structures [10], further expanding the applicability of these algebras in the context of uncertainty and fuzzy reasoning.

Interval-valued intuitionistic fuzzy (IVIF) subsets of SSHAs have been studied with respect to their algebraic properties and structural preservation [14]. Building on this, the concepts of IVIF Sheffer stroke subsets and IVIFSS-ideals were introduced, together with their characterizations and extensions to neutrosophic settings [18].

In this paper, we present a comprehensive study of interval-valued intuitionistic fuzzy Sheffer stroke subsets (IVIFSS) within the framework of SSHAs. We introduce and characterize IVIFSS-subalgebras and IVIFSS-ideals, and establish their closure properties under Cartesian products, intersections, and the operators $\oplus$ and $\otimes$. It is shown that the product of two IVIFSS-subalgebras (or IVIFSS-ideals) is again an IVIFSS-subalgebra (or IVIFSS-ideal) in the product algebra, and equivalent conditions are derived through level subsets to describe subalgebraic and ideal structures. Furthermore, the relationships between IVIFSS-deductive systems and IVIFSS-filters are examined, where necessary and sufficient conditions for an interval-valued intuitionistic fuzzy set (IVIFS) to constitute a filter are obtained in terms of order-preserving properties and operator-based inequalities. These results not only extend the classical concepts of subalgebras, ideals, and filters in Hilbert algebras to the interval-valued intuitionistic fuzzy context, but also provide a unified algebraic foundation for the further development of fuzzy algebraic logic and its applications in approximate reasoning and computational intelligence.

The list of acronyms is given in Table 1.

Table 1. List of acronyms

Acronyms	Representation
SSHAs	Sheffer stroke Hilbert algebras
IFSs	Intuitionistic fuzzy sets
IVIFSs	Interval-valued intuitionistic fuzzy sets
IVIFSS-subalgebras	Interval-valued intuitionistic fuzzy Sheffer stroke subalgebras
IVIFSS-ideals	Interval-valued intuitionistic fuzzy Sheffer stroke ideals
IVIFSS-deductive systems	Interval-valued intuitionistic fuzzy Sheffer stroke deductive systems
IVIFSS-filters	Interval-valued intuitionistic fuzzy Sheffer stroke filters

2. Preliminaries

In this section, we provide the background needed for the results that follow by outlining the key properties and relationships among the algebraic and fuzzy constructs introduced earlier. We specify the main operations and structures and highlight their interactions. This preparation clarifies the setting and provides a solid theoretical basis for the subsequent sections.

Definition 2.1. [21] *Let $\mathcal{A} := (A, |)$ be a groupoid. The binary operation $|$ is called the Sheffer stroke (or Sheffer operation) if it satisfies the following axioms:*

(s1) $(\forall \mathfrak{z}, \mathfrak{x} \in A)\ (\mathfrak{z} \mid \mathfrak{x} = \mathfrak{x} \mid \mathfrak{z})$,

(s2) $(\forall \mathfrak{z}, \mathfrak{x} \in A)\ ((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{z} \mid \mathfrak{x}) = \mathfrak{z})$,

(s3) $(\forall \mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in A)\ (\mathfrak{z} \mid ((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y})) = ((\mathfrak{z} \mid \mathfrak{x}) \mid (\mathfrak{z} \mid \mathfrak{x})) \mid \mathfrak{y})$,

(s4) $(\forall \mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in A)\ ((\mathfrak{z} \mid ((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{x} \mid \mathfrak{x}))) \mid (\mathfrak{z} \mid ((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{x} \mid \mathfrak{x}))) = \mathfrak{z})$.

For later use, we introduce the following notation, which will be used throughout:

$$\mathfrak{y} \mid (\mathfrak{x} \mid \mathfrak{x}) := \mathfrak{y}^{\mathfrak{x}}.$$

Definition 2.2. [16] *A Sheffer stroke Hilbert algebra (SSHA) is a groupoid $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ equipped with a Sheffer stroke operation satisfying the following conditions:*

(sH1) $(\forall \mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in S_H)\ ((\mathfrak{z} \mid (\mathfrak{x}^{\mathfrak{y}} \mid \mathfrak{x}^{\mathfrak{y}})) \mid ((\mathfrak{z}^{\mathfrak{x}} \mid (\mathfrak{z}^{\mathfrak{y}} \mid \mathfrak{z}^{\mathfrak{y}})) \mid (\mathfrak{z}^{\mathfrak{x}} \mid (\mathfrak{z}^{\mathfrak{y}} \mid \mathfrak{z}^{\mathfrak{y}}))) = \mathfrak{z}^{\mathfrak{z}})$,

(sH2) $(\forall \mathfrak{z}, \mathfrak{x} \in S_H)\ (\mathfrak{z}^{\mathfrak{x}} = \mathfrak{x}^{\mathfrak{z}} = \mathfrak{z}^{\mathfrak{z}} \ \Rightarrow\ \mathfrak{z} = \mathfrak{x})$.

Proposition 2.1. [16] *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. Then the binary relation*

$$\mathfrak{z} \leq \mathfrak{x} \quad \textit{if and only if} \quad \mathfrak{z}^{\mathfrak{x}} = 0$$

is a partial order on S_H.

Definition 2.3. [16] *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. A nonempty subset A of S_H is called a subalgebra of S_H if $\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}} \in A$ for all $\mathfrak{z}, \mathfrak{x} \in A$.*

Definition 2.4. [2] *Let X be a nonempty set. The intuitionistic fuzzy set (IFS) on X is defined as a structure*

$$A := \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z}) \rangle \mid \mathfrak{z} \in X\}, \tag{2.1}$$

where $\mu_A : X \to [0,1]$ represents the degree of membership of $\mathfrak{z}$ in X, and $\gamma_A : X \to [0,1]$ represents the degree of non-membership of $\mathfrak{z}$ in X such that

$$0 \leq \mu_A(\mathfrak{z}) + \gamma_A(\mathfrak{z}) \leq 1.$$

The IFS in (2.1) is simply denoted by $A = (\mu_A, \gamma_A)$.

Let $\mathcal{D}[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. Consider $I_1, I_2 \in \mathcal{D}[0,1]$. If $I_1 = [\mathfrak{n}_1, \mathfrak{m}_1]$ and $I_2 = [\mathfrak{n}_2, \mathfrak{m}_2]$, then

$$\text{rmin}\{I_1, I_2\} = [\min\{\mathfrak{n}_1, \mathfrak{n}_2\}, \min\{\mathfrak{m}_1, \mathfrak{m}_2\}]$$

and

$$\text{rmax}\{I_1, I_2\} = [\max\{\mathfrak{n}_1, \mathfrak{n}_2\}, \max\{\mathfrak{m}_1, \mathfrak{m}_2\}].$$

Thus, if $I_\mathfrak{i} = [\mathfrak{n}_\mathfrak{i}, \mathfrak{m}_\mathfrak{i}] \in \mathcal{D}[0,1]$ for $\mathfrak{i} = 1, 2, \ldots$, then we define

$$\text{rsup}_\mathfrak{i}\{I_\mathfrak{i}\} = [\sup_\mathfrak{i}\{\mathfrak{n}_\mathfrak{i}\}, \sup_\mathfrak{i}\{\mathfrak{m}_\mathfrak{i}\}],$$

and

$$\text{rinf}_\mathfrak{i}\{I_\mathfrak{i}\} = [\inf_\mathfrak{i}\{\mathfrak{n}_\mathfrak{i}\}, \inf_\mathfrak{i}\{\mathfrak{m}_\mathfrak{i}\}].$$

We write $I_1 \geq I_2$ if and only if $\mathfrak{n}_1 \geq \mathfrak{n}_2$ and $\mathfrak{m}_1 \leq \mathfrak{m}_2$. Similarly, the relations $I_1 \leq I_2$ and $I_1 = I_2$ are defined.

Definition 2.5. *An interval-valued intuitionistic fuzzy set (IVIFS) A over a universe X is defined as an object of the form*

$$A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid \mathfrak{z} \in X\},$$

where $\mu_A(\mathfrak{z}) : X \to \mathcal{D}[0,1]$ and $\gamma_A(\mathfrak{z}) : X \to \mathcal{D}[0,1]$. The functions $\mu_A(\mathfrak{z})$ and $\gamma_A(\mathfrak{z})$ represent the intervals of the degree of membership and non-membership of the element $\mathfrak{z}$ in the set A, respectively, where $\mu_A(\mathfrak{z}) = [\mu_A^{\mathfrak{g}}(\mathfrak{z}), \mu_A^{\mathfrak{d}}(\mathfrak{z})]$ and $\gamma_A(\mathfrak{z}) = [\gamma_A^{\mathfrak{g}}(\mathfrak{z}), \gamma_A^{\mathfrak{d}}(\mathfrak{z})]$ for all $\mathfrak{z} \in X$, subject to the condition $0 \leq \mu_A^{\mathfrak{g}}(\mathfrak{z}) + \gamma_A^{\mathfrak{d}}(\mathfrak{z}) \leq 1$. For simplicity, we denote the IVIFS A as $A = (\mu_A, \gamma_A)$, where $A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid \mathfrak{z} \in X\}$. Additionally, the complements of μ_A and γ_A are given by

$$\overline{\mu_A}(\mathfrak{z}) = [1 - \mu_A^{\mathfrak{d}}(\mathfrak{z}), 1 - \mu_A^{\mathfrak{g}}(\mathfrak{z})]$$

and

$$\overline{\gamma_A}(\mathfrak{z}) = [1 - \gamma_A^{\mathfrak{d}}(\mathfrak{z}), 1 - \gamma_A^{\mathfrak{g}}(\mathfrak{z})],$$

where $[\overline{\mu_A}(\mathfrak{z}), \overline{\gamma_A}(\mathfrak{z})]$ represents the complement of $\mathfrak{z}$ in A.

Definition 2.6. [14] *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. An IVIFS $A = (\mu_A, \gamma_A)$ in S_H is called an IVIFSS-subalgebra of S_H if*

$$(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \begin{pmatrix} \mu_A(\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}) \geq \mathrm{rmin}\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{x})\} \\ \gamma_A(\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}) \leq \mathrm{rmax}\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{x})\} \end{pmatrix}. \tag{2.2}$$

Proposition 2.2. [14] *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. Every IVIFSS-subalgebra $A = (\mu_A, \gamma_A)$ of S_H satisfies*

$$(\forall \mathfrak{z} \in S_H) \begin{pmatrix} \mu_A(0) \geq \mu_A(\mathfrak{z}) \\ \gamma_A(0) \leq \gamma_A(\mathfrak{z}) \end{pmatrix}. \tag{2.3}$$

Definition 2.7. [18] *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. An IVIFS $A = (\mu_A, \gamma_A)$ in S_H is called an IVIFSS-ideal of S_H if the following conditions hold:*

$$(\forall \mathfrak{z} \in S_H) \begin{pmatrix} \mu_A(0) \geq \mu_A(\mathfrak{z}) \\ \gamma_A(0) \leq \gamma_A(\mathfrak{z}) \end{pmatrix}, \tag{2.4}$$

$$(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \begin{pmatrix} \mu_A(\mathfrak{z}) \geq \mathrm{rmin}\{\mu_A(\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}), \mu_A(\mathfrak{x})\} \\ \gamma_A(\mathfrak{z}) \leq \mathrm{rmax}\{\gamma_A(\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}), \gamma_A(\mathfrak{x})\} \end{pmatrix}. \tag{2.5}$$

3. Product of IVIFSS-Subalgebras/Ideals in SSHAs

In this section, we study the behavior of interval-valued intuitionistic fuzzy Sheffer stroke subalgebras (IVIFSS-subalgebras) and IVIFSS-ideals under product constructions in SSHAs. We examine whether these structures are closed under Cartesian products and the operators $\oplus$ and $\otimes$, and we establish equivalent characterizations using level subsets. These results clarify how the fundamental algebraic properties of IVIFSS structures are preserved and extended in product algebras.

Definition 3.1. *Let* $A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid k \in \mathcal{X}\}$ *and* $B = \{\langle \mathfrak{x}, \mu_B(\mathfrak{x}), \gamma_B(\mathfrak{x})\rangle \mid \mathfrak{x} \in \mathcal{Y}\}$ *be IVIFSs of* $\mathcal{X}$ *and* $\mathcal{Y}$*, respectively. The Cartesian product* $A \times B = \{\langle (\mathfrak{z}, \mathfrak{x}), \mu_{A\times B}(\mathfrak{z}, \mathfrak{x}), \gamma_{A\times B}(\mathfrak{z}, \mathfrak{x})\rangle \mid \mathfrak{z} \in \mathcal{X}, \mathfrak{x} \in \mathcal{Y}\}$ *defined by*

$$\mu_{A\times B}(\mathfrak{z}, y) = \mathrm{rmin}\{\mu_A(\mathfrak{z}), \mu_B(\mathfrak{x})\}$$

and

$$\gamma_{A\times B}(\mathfrak{z}, \mathfrak{x}) = \mathrm{rmax}\{\gamma_A(\mathfrak{z}), \gamma_B(\mathfrak{x})\},$$

where $\mu_{A\times B} : \mathcal{X} \times \mathcal{Y} \to \mathcal{D}[0, 1]$ *and* $\gamma_{A\times B} : \mathcal{X} \times \mathcal{Y} \to \mathcal{D}[0, 1]$ *for all* $\mathfrak{z} \in \mathcal{X}$ *and* $\mathfrak{x} \in \mathcal{Y}$.

Remark 3.1. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be SSHAs. We define* $|_{\mathcal{X}\times\mathcal{Y}}$ *on* $\mathcal{X} \times \mathcal{Y}$ *by*

$$(\mathfrak{z}, \mathfrak{x}) \; |_{\mathcal{X}\times\mathcal{Y}} \; (\mathfrak{n}, \mathfrak{m}) = (\mathfrak{z}|_{\mathcal{X}}\mathfrak{n}, \mathfrak{x}|_{\mathcal{Y}}\mathfrak{m})$$

for every $(\mathfrak{z}, \mathfrak{x}), (\mathfrak{n}, \mathfrak{m}) \in \mathcal{X} \times \mathcal{Y}$*. The identity element of this product algebra is defined by*

$$0_{\mathcal{X}\times\mathcal{Y}} := (0_{\mathcal{X}}, 0_{\mathcal{Y}}).$$

and $(\mathcal{X} \times \mathcal{Y}, |_{\mathcal{X}\times\mathcal{Y}}, 0_{\mathcal{X}\times\mathcal{Y}})$ *is an SSHA.*

To improve the clarity of this manuscript, we introduce the following notation, which will be used consistently throughout the text:

$$(\mathfrak{n}_1, \mathfrak{m}_1) \; |_{\mathcal{X}\times\mathcal{Y}} \; ((\mathfrak{n}_2, \mathfrak{m}_2) \; |_{\mathcal{X}\times\mathcal{Y}} \; (\mathfrak{n}_2, \mathfrak{m}_2)) := (\mathfrak{n}_{12}, \mathfrak{m}_{12})_{A\times B}$$

and

$$\mathfrak{n}_1|_{\mathcal{X}}(\mathfrak{n}_2|_{\mathcal{X}}\mathfrak{n}_2) := (\mathfrak{n}_1{}^{\mathfrak{n}_2})_A.$$

Proposition 3.1. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be SSHAs. If* $A = (\mu_A, \gamma_A)$ *and* $B = (\mu_B, \gamma_B)$ *are IVIFSS-subalgebras of* $\mathcal{X}$ *and* $\mathcal{Y}$*, respectively, then* $A \times B$ *is also an IVIFSS-subalgebra of* $\mathcal{X} \times \mathcal{Y}$.

Proof. Let $(\mathfrak{z}_1, \mathfrak{x}_1), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned}
&\mu_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \; |_{\mathcal{X}\times\mathcal{Y}} \; (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}) \\
&= \mu_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{z}_2})_A, (\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B) \\
&= \mathrm{rmin}\{\mu_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A, \mu_B(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\} \\
&\geq \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_A(\mathfrak{z}_2)\}, \mathrm{rmin}\{\mu_B(\mathfrak{x}_1), \mu_B(\mathfrak{x}_2)\}\} \\
&= \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_B(\mathfrak{x}_1)\}, \mathrm{rmin}\{\mu_A(\mathfrak{z}_2), \mu_B(\mathfrak{x}_2)\}\} \\
&= \mathrm{rmin}\{\mu_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1), \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}
\end{aligned}$$

and

$$\gamma_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \; |_{\mathcal{X}\times\mathcal{Y}} \; (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}))$$

$$= \gamma_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{z}_2})_A, (\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)$$

$$= \mathrm{rmin}\{\gamma_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A, \gamma_B(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\}$$

$$\leq \mathrm{rmin}\{\mathrm{rmax}\{\gamma_A(\mathfrak{z}_1), \gamma_A(\mathfrak{z}_2)\}, \mathrm{rmax}\{\gamma_B(\mathfrak{x}_1), \gamma_B(\mathfrak{x}_2)\}\}$$

$$= \mathrm{rmin}\{\mathrm{rmax}\{\gamma_A(\mathfrak{z}_1), \gamma_B(\mathfrak{x}_1)\}, \mathrm{rmax}\{\gamma_A(\mathfrak{z}_2), \gamma_B(\mathfrak{x}_2)\}\}$$

$$= \mathrm{rmax}\{\mathrm{rmin}\{\gamma_A(\mathfrak{z}_1), \gamma_B(\mathfrak{x}_1)\}, \mathrm{rmin}\{\gamma_A(\mathfrak{z}_2), \gamma_B(\mathfrak{x}_2)\}\}$$

$$= \mathrm{rmax}\{\gamma_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1), \gamma_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}.$$

Hence, $A \times B$ is an IVIFSS-subalgebra of $\mathcal{X} \times \mathcal{Y}$. □

Let $(\mathcal{X}, |_{\mathcal{X}}, 0_{\mathcal{X}})$ and $(\mathcal{Y}, |_{\mathcal{Y}}, 0_{\mathcal{Y}})$ be SSHAs, and let A and B be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively. For the product IVIFSS $A \times B$ on $\mathcal{X} \times \mathcal{Y}$, the *complementary membership* and *complementary non-membership* functions $\overline{\mu}_{A\times B}$ and $\overline{\gamma}_{A\times B}$ are defined by

$$\overline{\mu}_{A\times B}(\mathfrak{z}, \mathfrak{x}) := 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(\mathfrak{z}, \mathfrak{x}) = [\,1 - \mu^U_{A\times B}(\mathfrak{z}, \mathfrak{x}),\ 1 - \mu^L_{A\times B}(\mathfrak{z}, \mathfrak{x})\,]$$

and

$$\overline{\gamma}_{A\times B}(\mathfrak{z}, \mathfrak{x}) := 1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}(\mathfrak{z}, \mathfrak{x}) = [\,1 - \gamma^U_{A\times B}(\mathfrak{z}, \mathfrak{x}),\ 1 - \gamma^L_{A\times B}(\mathfrak{z}, \mathfrak{x})\,]$$

for all $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{X} \times \mathcal{Y}$.

Lemma 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. If $A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid \mathfrak{z} \in \mathcal{X}\}$ and $B = \{\langle \mathfrak{x}, \mu_B(\mathfrak{x}), \gamma_B(\mathfrak{x})\rangle \mid \mathfrak{x} \in \mathcal{Y}\}$ are IVIFSS-subalgebras of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then $\oplus(A \times B) = \{\langle (\mathfrak{z}, \mathfrak{x}), \mu_{A\times B}(\mathfrak{z}, \mathfrak{x}), \overline{\mu}_{A\times B}(\mathfrak{z}, \mathfrak{x})\rangle \mid (\mathfrak{z}, \mathfrak{x}) \in \mathcal{X} \times \mathcal{Y}\}$ is an IVIFSS-subalgebra of $\mathcal{X} \times \mathcal{Y}$.*

Proof. Let $(\mathfrak{z}_1, \mathfrak{x}_1), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\overline{\mu}_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B})$$

$$= \overline{\mu}_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{z}_2})_A, (\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A, (\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A, \mu_B(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\}$$

$$\leq 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_A(\mathfrak{z}_2)\}, \mathrm{rmin}\{\mu_B(\mathfrak{x}_1), \mu_B(\mathfrak{x}_2)\}\}$$

$$= \mathrm{rmax}\{1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_B(\mathfrak{x}_1)\}, 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A(\mathfrak{z}_2), \mu_B(\mathfrak{x}_2)\}\}$$

$$= \mathrm{rmax}\{1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1), 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}$$

$$= \mathrm{rmax}\{\overline{\mu}_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1), \overline{\mu}_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}.$$

Hence, $\oplus(A \times B)$ is an IVIFSS-subalgebra of $\mathcal{X} \times \mathcal{Y}$. □

Lemma 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. If $A=\{\langle \mathfrak{z},\mu_A(\mathfrak{z}),\gamma_A(\mathfrak{z})\rangle \mid x\in\mathcal{X}\}$ and $B=\{\langle\mathfrak{x},\mu_B(\mathfrak{x}),\gamma_B(\mathfrak{x})\rangle\mid \mathfrak{x}\in\mathcal{Y}\}$ are IVIFSS-subalgebras of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then $\otimes(A\times B)=\{\langle(\mathfrak{z},\mathfrak{x}),\overline{\gamma}_{A\times B}(\mathfrak{z},\mathfrak{x}),\gamma_{A\times B}(\mathfrak{z},\mathfrak{x})\rangle\mid(\mathfrak{z},\mathfrak{x})\in \mathcal{X}\times\mathcal{Y}\}$ is an IVIFSS-subalgebra of $\mathcal{X}\times\mathcal{Y}$.*

Proof. Let $(\mathfrak{z}_1,\mathfrak{x}_1),(\mathfrak{z}_2,\mathfrak{x}_2)\in\mathcal{X}\times\mathcal{Y}$. Then

$$
\begin{aligned}
&\overline{\mu}_{A\times B}((\mathfrak{z}12,\mathfrak{z}12)_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}12,\mathfrak{z}12)_{A\times B})\\
&=\overline{\gamma}_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A,(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\\
&=1_{\mathcal{X}\times\mathcal{Y}}-\gamma_{A\times B}((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A,(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\\
&=1_{\mathcal{X}\times\mathcal{Y}}-\mathrm{rmin}\{\gamma_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A,\gamma_B((\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B))\}\\
&\geq 1_{\mathcal{X}\times\mathcal{Y}}-\mathrm{rmin}\{\mathrm{rmax}\{\gamma_A(\mathfrak{z}_1),\gamma_A(\mathfrak{z}_2)\},\mathrm{rmax}\{\gamma_B(\mathfrak{x}_1),\gamma_B(\mathfrak{x}_2)\}\}\\
&=1_{\mathcal{X}\times\mathcal{Y}}-\mathrm{rmax}\{\mathrm{rmin}\{\gamma_A(\mathfrak{z}_1),\gamma_B(\mathfrak{x}_1)\},\mathrm{rmin}\{\gamma_A(\mathfrak{z}_2),\gamma_B(\mathfrak{x}_2)\}\}\\
&=\mathrm{rmin}\{1_{\mathcal{X}\times\mathcal{Y}}-\gamma_{A\times B}(\mathfrak{z}_1,\mathfrak{x}_1),1_{\mathcal{X}\times\mathcal{Y}}-\gamma_{A\times B}(\mathfrak{z}_2,\mathfrak{x}_2)\}\\
&=\mathrm{rmin}\{\overline{\gamma}_{A\times B}(\mathfrak{z}_1,\mathfrak{x}_1),\overline{\gamma}_{A\times B}(\mathfrak{z}_2,\mathfrak{x}_2)\}.
\end{aligned}
$$

Hence, $\otimes(A\times B)$ is an IVIFSS-subalgebra of $\mathcal{X}\times\mathcal{Y}$. □

Theorem 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. The IVIFSs $A=(\mu_A,\gamma_A)$ and $B=\{\langle\mathfrak{x},\mu_B(\mathfrak{x}),\gamma_B(\mathfrak{x})\rangle\mid\mathfrak{x}\in\mathcal{T}\}$ are IVIFSS-subalgebras of $\mathcal{X}$ and $\mathcal{Y}$, respectively if and only if $\oplus(A\times B)$ and $\otimes(A\times B)$ are IVIFSS-subalgebras of $\mathcal{X}\times\mathcal{Y}$.*

Proof. It follows from the Lemmas 3.1 and 3.2. □

Proposition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. If $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ are two IVIFSS-ideals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then the Cartesian product $A\times B$ is also an IVIFSS-ideal of $\mathcal{X}\times\mathcal{Y}$.*

Proof. Let $(\mathfrak{z},\mathfrak{x})\in\mathcal{X}\times\mathcal{Y}$. Then

$$
\begin{aligned}
\mu_{A\times B}(0_{\mathcal{X}},0_{\mathcal{Y}})&=\mathrm{rmin}\{\mu_A(0_{\mathcal{X}}),\mu_B(0_{\mathcal{Y}})\}\\
&\geq\mathrm{rmin}\{\mu_A(\mathfrak{z}),\mu_B(\mathfrak{x})\}\\
&=\mu_{A\times B}(\mathfrak{z},\mathfrak{x})
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_{A\times B}(0_{\mathcal{X}},0_{\mathcal{Y}})&=\mathrm{rmax}\{\gamma_A(0_{\mathcal{X}}),\gamma_B(0_{\mathcal{Y}})\}\\
&\leq\mathrm{rmax}\{\gamma_A(\mathfrak{z}),\gamma_B(\mathfrak{x})\}\\
&=\gamma_{A\times B}(\mathfrak{z},\mathfrak{x}).
\end{aligned}
$$

Let $(\mathfrak{z}_1, \mathfrak{x}_1), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned}
&\mu_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)\\
&= \mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_B(\mathfrak{x}_1)\}\\
&\geq \mathrm{rmin}\{\mathrm{rmin}\{\mu_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A)\}, \mathrm{rmin}\{\mu_B((\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\}\}\\
&= \mathrm{rmin}\{\mathrm{rmin}\{\mu_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A), \mu_B((\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\},\\
&\quad \mathrm{rmin}\{\mu_A(\mathfrak{z}_2), \mu_B(\mathfrak{x}_2)\}\}\\
&= \mathrm{rmin}\{\mu_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \;|_{\mathcal{X}\times\mathcal{Y}}\; (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B})), \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)\\
&= \mathrm{rmin}\{\gamma_A(\mathfrak{z}_1), \gamma_B(\mathfrak{x}_1)\}\\
&\geq \mathrm{rmin}\{\mathrm{rmax}\{\gamma_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{z}_2})_A), \gamma_A(\mathfrak{z}_2)\},\\
&\quad \mathrm{rmax}\{\gamma_B((\mathfrak{x}_1{}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}(\mathfrak{x}_1{}^{\mathfrak{x}_2})_B), \gamma_B(\mathfrak{x}_2)\}\}\\
&= \mathrm{rmax}\{\mathrm{rmin}\{\gamma_A((\mathfrak{z}_1{}^{\mathfrak{z}_2})_A|_{\mathcal{X}}(\mathfrak{z}_1{}^{\mathfrak{x}_2})_A), \gamma_B((\mathfrak{x}_1{}^{\mathfrak{x}_2})_B)\},\\
&\quad \mathrm{rmin}\{\mu_A(\mathfrak{z}_2), \mu_B(\mathfrak{x}_2)\}, \mathrm{rmin}\{\gamma_A(\mathfrak{z}_2), \gamma_B(\mathfrak{x}_2)\}\}\\
&= \mathrm{rmax}\{\gamma_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \;|_{\mathcal{X}\times\mathcal{Y}}\; (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2),\\
&\quad \gamma_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVIFSS-ideal of $\mathcal{X} \times \mathcal{Y}$. □

Lemma 3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. If $A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid \mathfrak{z} \in \mathcal{X}\}$ and $B = \{\langle \mathfrak{x}, \mu_B(\mathfrak{x}), \gamma_B(\mathfrak{x})\rangle \mid \mathfrak{x} \in \mathcal{Y}\}$ are IVIFSS-ideals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then $\oplus(A\times B) = \{\langle (\mathfrak{z}, \mathfrak{x}), \mu_{A\times B}(\mathfrak{z}, \mathfrak{x}), \overline{\mu}_{A\times B}(\mathfrak{z}, \mathfrak{x})\rangle \mid (\mathfrak{z}, \mathfrak{x}) \in \mathcal{X}\times\mathcal{Y}\}$ is an IVIFSS-ideal of $\mathcal{X} \times \mathcal{Y}$.*

Proof. Let $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned}
\overline{\mu}_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}}) &= 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}})\\
&= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A(0_{\mathcal{X}}), \mu_B(0_{\mathcal{Y}})\}\\
&\leq 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A(\mathfrak{z}), \mu_B(\mathfrak{x})\}\\
&= 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(\mathfrak{z}, \mathfrak{x})\\
&= \overline{\mu}_{A\times B}(\mathfrak{z}, \mathfrak{x}).
\end{aligned}$$

Let $(\mathfrak{z}_1, \mathfrak{x}_1), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\overline{\mu}_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_A(\mathfrak{z}_1), \mu_B(\mathfrak{x}_1)\}$$

$$\leq 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(({\mathfrak{z}_1}^{\mathfrak{z}_2})_A|_{\mathcal{X}}({\mathfrak{z}_1}^{\mathfrak{z}_2})_A), \mu_A(\mathfrak{z}_2)\},$$

$$\mathrm{rmin}\{\mu_B(({\mathfrak{x}_1}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}({\mathfrak{x}_1}^{\mathfrak{x}_2})_B), \mu_B(\mathfrak{x}_2)\}\}$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mathrm{rmin}\{\mu_A(({\mathfrak{z}_1}^{\mathfrak{z}_2})_A|_{\mathcal{X}}({\mathfrak{z}_1}^{\mathfrak{x}_2})_A),$$

$$\mu_B(({\mathfrak{x}_1}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}({\mathfrak{x}_1}^{\mathfrak{x}_2})_B)\}, \mathrm{rmin}\{\mu_A(\mathfrak{z}_2), \mu_B(\mathfrak{x}_2)\}\}$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mu_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \mid_{\mathcal{X}\times\mathcal{Y}} (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}$$

$$= \mathrm{rmax}\{1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \mid_{\mathcal{X}\times\mathcal{Y}} (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), 1_{\mathcal{X}\times\mathcal{Y}} - \mu_{A\times B}((\mathfrak{z}_2, \mathfrak{x}_2))\}$$

$$= \mathrm{rmax}\{\overline{\mu}_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \mid_{\mathcal{X}\times\mathcal{Y}} (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \overline{\mu}_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\}.$$

Hence, $\oplus(A \times B)$ is an IVIFSS-ideal of $\mathcal{X} \times \mathcal{Y}$. □

Lemma 3.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. If $A = \{\langle \mathfrak{z}, \mu_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\rangle \mid \mathfrak{z} \in \mathcal{X}\}$ and $B = \{\langle \mathfrak{x}, \gamma_B(\mathfrak{x}), \gamma_B(\mathfrak{x})\rangle \mid \mathfrak{x} \in \mathcal{Y}\}$ are IVIFSS-ideals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then $\otimes(A\times B) = \{\langle(\mathfrak{z}, \mathfrak{x}), \overline{\gamma}_{A\times B}(\mathfrak{z}, \mathfrak{x}), \gamma_{A\times B}(\mathfrak{z}, \mathfrak{x})\rangle \mid (\mathfrak{z}, \mathfrak{x}) \in \mathcal{X}\times\mathcal{Y}\}$ is an IVIFSS-ideal of $\mathcal{X} \times \mathcal{Y}$.*

Proof. Let $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned}
\overline{\gamma}_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}}) &= 1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}}) \\
&= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\gamma_A(0_{\mathcal{X}}), \gamma_B(0_{\mathcal{Y}})\} \\
&\geq 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\gamma_A(\mathfrak{z}), \gamma_B(\mathfrak{x})\} \\
&= 1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}(\mathfrak{z}, \mathfrak{x}) \\
&= \overline{\gamma}_{A\times B}(\mathfrak{z}, \mathfrak{x}).
\end{aligned}$$

Let $(\mathfrak{z}_1, \mathfrak{x}_1), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\overline{\gamma}_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1)$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\gamma_A(\mathfrak{z}_1), \gamma_B(\mathfrak{x}_1)\}$$

$$\geq 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmin}\{\mathrm{rmax}\{\gamma_A(({\mathfrak{z}_1}^{\mathfrak{z}_2})_A|_{\mathcal{X}}({\mathfrak{z}_1}^{\mathfrak{x}_2})_A), \gamma_A(\mathfrak{z}_2)\},$$

$$\mathrm{rmax}\{\gamma_B(({\mathfrak{x}_1}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}({\mathfrak{x}_1}^{\mathfrak{x}_2})_B), \gamma_B(\mathfrak{x}_2)\}\}$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmax}\{\mathrm{rmin}\{\gamma_A(({\mathfrak{z}_1}^{\mathfrak{z}_2})_A|_{\mathcal{X}}({\mathfrak{z}_1}^{\mathfrak{x}_2})_A),$$

$$\gamma_B(({\mathfrak{x}_1}^{\mathfrak{x}_2})_B|_{\mathcal{Y}}({\mathfrak{x}_1}^{\mathfrak{x}_2})_B)\}, \mathrm{rmin}\{\gamma_A(\mathfrak{z}_2), \gamma_B(\mathfrak{x}_2)\}\}$$

$$= 1_{\mathcal{X}\times\mathcal{Y}} - \mathrm{rmax}\{\gamma_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \mid_{\mathcal{X}\times\mathcal{Y}} (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \gamma_{A\times B}((\mathfrak{z}_2, \mathfrak{x}_2))\}$$

$$= \text{rmin}\{1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}),$$

$$1_{\mathcal{X}\times\mathcal{Y}} - \gamma_{A\times B}(\mathfrak{z}_2,\mathfrak{x}_2)\}$$

$$= \text{rmin}\{\overline{\gamma}_{A\times B}((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}),\overline{\gamma}_{A\times B}((\mathfrak{z}_2,\mathfrak{x}_2))\}.$$

Hence, $\otimes(A\times B)$ is an IVIFSS-subalgebra of $\mathcal{X}\times\mathcal{Y}$. □

Theorem 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. The IVIFSs $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ are IVIFSS-ideals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, if and only if $\oplus(A\times B)$ and $\otimes(A\times B)$ are IVIFSS-ideals of $\mathcal{X}\times\mathcal{Y}$.*

Proof. It follows from the Lemmas 3.3 and 3.4. □

Definition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively. For $[\mathfrak{a}_1,\mathfrak{a}_2],[\mathfrak{b}_1,\mathfrak{b}_2]\in\mathcal{D}[0,1]$, the set*

$$\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2]) = \{(\mathfrak{z},\mathfrak{x})\in\mathcal{X}\times\mathcal{Y}\mid \mu_{A\times B}(\mathfrak{z},\mathfrak{x})\geq[\mathfrak{a}_1,\mathfrak{a}_2]\}$$

is called upper $[\mathfrak{a}_1,\mathfrak{a}_2]$-level of $A\times B$ and

$$\mathcal{L}(\gamma_{A\times B}:[\mathfrak{b}_1,\mathfrak{b}_2]) = \{(\mathfrak{z},\mathfrak{x})\in\mathcal{X}\times\mathcal{Y}\mid \gamma_{A\times B}(\mathfrak{z},\mathfrak{x})\leq[\mathfrak{b}_1,\mathfrak{b}_2]\}$$

is called lower $[\mathfrak{b}_1,\mathfrak{b}_2]$-level of $A\times B$.

Theorem 3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $A\times B$ is an IVIFSS-subalgebra of $\mathcal{X}\times\mathcal{Y}$, then non-empty upper $[s_1,s_2]$-level cut $\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$ and non-empty lower $[\mathfrak{b}_1,\mathfrak{b}_2]$-level cut $\mathcal{L}(\gamma_{A\times B}:[\mathfrak{b}_1,\mathfrak{b}_2])$ are subalgebras of $\mathcal{X}\times\mathcal{Y}$ for all $[\mathfrak{a}_1,\mathfrak{a}_2],[\mathfrak{b}_1,\mathfrak{b}_2]\in\mathcal{D}[0,1]$.*

Proof. Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively be such that $A\times B$ is an IVIFSS-subalgebra of $\mathcal{X}\times\mathcal{Y}$. Then

$$\mu_{A\times B}((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}) \geq \text{rmin}\{\mu_{A\times B}(\mathfrak{z}_1,\mathfrak{x}_1),\mu_{A\times B}(\mathfrak{z}_2,\mathfrak{x}_2)\}$$

and

$$\gamma_{A\times B}((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}) \leq \text{rmax}\{\gamma_{A\times B}((\mathfrak{z}_1,\mathfrak{x}_1)),\gamma_{A\times B}(\mathfrak{z}_2,\mathfrak{x}_2)\}$$

for all $(\mathfrak{z}_1,\mathfrak{x}_1),(\mathfrak{z}_2,\mathfrak{x}_2)\in\mathcal{X}\times\mathcal{Y}$. Let $[\mathfrak{a}_1,\mathfrak{a}_2],[\mathfrak{b}_1,\mathfrak{b}_2]\in\mathcal{D}[0,1]$ be such that $\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$ and $\mathcal{L}(\gamma_{A\times B}:[\mathfrak{b}_1,\mathfrak{b}_2])$ are non-empty sets of $\mathcal{X}\times\mathcal{Y}$. Let $(\mathfrak{z},\mathfrak{x})\in\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$. Again, let $(\mathfrak{z}_1,\mathfrak{x}_1),(\mathfrak{z}_2,\mathfrak{x}_2)\in\mathcal{X}\times\mathcal{Y}$ be such that $(\mathfrak{z}_1,\mathfrak{x}_1),(\mathfrak{z}_2,\mathfrak{x}_2)\in\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$. Then

$$\mu_{A\times B}((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}) \geq \text{rmin}\{[\mathfrak{a}_1,\mathfrak{a}_2],[\mathfrak{a}_1,\mathfrak{a}_2]\}$$

$$= [\mathfrak{a}_1,\mathfrak{a}_2].$$

This implies, $((\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B}\ |_{\mathcal{X}\times\mathcal{Y}}\ (\mathfrak{z}_{12},\mathfrak{z}_{12})_{A\times B})\in\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$. Thus, $\mathcal{U}(\mu_{A\times B}:[\mathfrak{a}_1,\mathfrak{a}_2])$ is a subalgebra of $\mathcal{X}\times\mathcal{Y}$.

Let $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned}\gamma_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}) &\leq \text{rmax}\{\gamma_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1), \gamma_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\} \\ &\leq \text{rmax}\{[\mathfrak{b}_1, \mathfrak{b}_2], [\mathfrak{b}_1, \mathfrak{b}_2]\} \\ &= [\mathfrak{b}_1, \mathfrak{b}_2].\end{aligned}$$

This implies, $((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}) \in \mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$. Thus, $\mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$ is a subalgebra of $\mathcal{X} \times \mathcal{Y}$. □

Theorem 3.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be SSHAs. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $A\times B$ is an IVIFSS-ideal of $\mathcal{X}\times\mathcal{Y}$, then non-empty upper $[\mathfrak{a}_1, \mathfrak{a}_2]$-level cut $\mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$ and non-empty lower $[\mathfrak{b}_1, \mathfrak{b}_2]$-level cut $\mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$ are ideals of $\mathcal{X} \times \mathcal{Y}$ for all $[\mathfrak{a}_1, \mathfrak{a}_2], [\mathfrak{b}_1, \mathfrak{b}_2] \in \mathcal{D}[0, 1]$.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IVIFSs of $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $A \times B$ is an IVIFSS-ideal of $\mathcal{X}\times\mathcal{Y}$. Let $[\mathfrak{a}_1, \mathfrak{a}_2], [\mathfrak{b}_1, \mathfrak{b}_2] \in \mathcal{D}[0, 1]$ be such that $\mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$ and $\mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$ are non-empty sets of $\mathcal{X} \times \mathcal{Y}$. Let $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$. Since $\mu_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}}) \geq \mu_{A\times B}(\mathfrak{z}, \mathfrak{x}) \geq [\mathfrak{a}_1, \mathfrak{a}_2]$, we have $(0_{\mathcal{X}}, 0_{\mathcal{Y}}) \in \mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$. Let $((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$. Then

$$\begin{aligned}&\mu_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1) \\ &\geq \text{rmin}\{\mu_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \mu_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\} \\ &\geq \text{rmin}\{[\mathfrak{a}_1, \mathfrak{a}_2], [\mathfrak{a}_1, \mathfrak{a}_2]\} \\ &= [\mathfrak{a}_1, \mathfrak{a}_2].\end{aligned}$$

Hence, $(\mathfrak{z}_1, \mathfrak{x}_1) \in \mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$. Therefore, $\mathcal{U}(\mu_{A\times B} : [\mathfrak{a}_1, \mathfrak{a}_2])$ is an ideal of $\mathcal{X} \times \mathcal{Y}$. Let $(\mathfrak{z}, \mathfrak{x}) \in \mathcal{L}(\gamma_{A\times B} : [t_1, t_2])$. Since $\gamma_{A\times B}(0_{\mathcal{X}}, 0_{\mathcal{Y}}) \leq \gamma_{A\times B}(x, y) \leq [t_1, t_2]$, we have $(0_{\mathcal{X}}, 0_{\mathcal{Y}}) \in \mathcal{L}(\gamma_{A\times B} : [t_1, t_2])$. Let $((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), (\mathfrak{z}_2, \mathfrak{x}_2) \in \mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$. Then

$$\begin{aligned}&\gamma_{A\times B}(\mathfrak{z}_1, \mathfrak{x}_1) \\ &\leq \text{rmax}\{\gamma_{A\times B}((\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B} \,|_{\mathcal{X}\times\mathcal{Y}}\, (\mathfrak{z}_{12}, \mathfrak{z}_{12})_{A\times B}), \gamma_{A\times B}(\mathfrak{z}_2, \mathfrak{x}_2)\} \\ &\leq \text{rmax}\{[\mathfrak{b}_1, \mathfrak{b}_2], [\mathfrak{b}_1, \mathfrak{b}_2]\} \\ &= [\mathfrak{b}_1, \mathfrak{b}_2].\end{aligned}$$

Hence, $(\mathfrak{z}_1, \mathfrak{x}_1) \in \mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$. Therefore, $\mathcal{L}(\gamma_{A\times B} : [\mathfrak{b}_1, \mathfrak{b}_2])$ is an ideal of $\mathcal{X} \times \mathcal{Y}$. □

Definition 3.3. *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. An IVIFS $A = (\mu_A, \gamma_A)$ in S_H is called an IVIFSS-deductive system of S_H if it satisfies*

$$(\forall \mathfrak{z} \in S_H) \begin{pmatrix} \mu_A(0) \geq \mu_A(\mathfrak{z}) \\ \gamma_A(0) \leq \gamma_A(\mathfrak{z}) \end{pmatrix}, \tag{3.1}$$

$$(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \begin{pmatrix} \mu_A(\mathfrak{x}) \geq \min\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{z}^{\mathfrak{x}})\} \\ \gamma_A(\mathfrak{x}) \leq \max\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{z}^{\mathfrak{x}})\} \end{pmatrix}. \tag{3.2}$$

Definition 3.4. *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. An IVIFS $A = (\mu_A, \gamma_A)$ in S_H is called an IVIFSS-filter of S_H if it satisfies (3.1) and*

$$(\forall \mathfrak{z} \in S_H) \left(\mu_A(\mathfrak{z}^{\mathfrak{x}}) \geq \mu_A(\mathfrak{x}), \gamma_A(\mathfrak{z}^{\mathfrak{x}}) \leq \gamma_A(\mathfrak{x}) \right), \tag{3.3}$$

$$(\forall \mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in S_H) \begin{pmatrix} \mu_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{y}}) \geq \min\{\mu_A(\mathfrak{x}), \mu_A(\mathfrak{y})\} \\ \gamma_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{y}}) \leq \max\{\gamma_A(\mathfrak{x}), \gamma_A(\mathfrak{y})\} \end{pmatrix}. \tag{3.4}$$

Theorem 3.5. *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. Given an IVIFS $A = (\mu_A, \gamma_A)$ in S_H, the following are equivalent to each other.*

(1) *$A = (\mu_A, \gamma_A)$ is an IVIFSS-deductive system of S_H.*

(2) *$A = (\mu_A, \gamma_A)$ is an IVIFSS-filter of S_H.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an IVIFSS-deductive system of S_H and let $\mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in S_H$. Note that $\mathfrak{x} \mid (\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}) = 0$. Then $\mu_A(\mathfrak{z}^{\mathfrak{x}}) \geq \min\{\mu_A(\mathfrak{x}), \mu_A(\mathfrak{x} \mid (\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}))\} = \min\{\mu_A(\mathfrak{x}), \mu_A(0)\} = \mu_A(\mathfrak{x})$ and $\gamma_A(\mathfrak{z}^{\mathfrak{x}}) \leq \max\{\gamma_A(\mathfrak{x}), \gamma_A(\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}})\} = \max\{\gamma_A(\mathfrak{x}), \gamma_A(0)\} = \gamma_A(\mathfrak{x})$. Note that

$$\begin{aligned} \mathfrak{x} \mid (\mathfrak{x}^{\mathfrak{y}} \mid \mathfrak{x}^{\mathfrak{y}}) &= \mathfrak{x} \mid ((\mathfrak{x} \mid \mathfrak{y})^{\mathfrak{y}|\mathfrak{y}} \mid (\mathfrak{x} \mid \mathfrak{y})^{\mathfrak{y}|\mathfrak{y}}) \\ &= (\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x}^{\mathfrak{y}|\mathfrak{y}} \mid \mathfrak{x}^{\mathfrak{y}|\mathfrak{y}}) \\ &= (\mathfrak{x} \mid \mathfrak{y})^{\mathfrak{x}|\mathfrak{y}} \\ &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} \mu_A(\mathfrak{x}^{\mathfrak{y}}) &\geq \min\{\mu_A(\mathfrak{x}), \mu_A(\mathfrak{x} \mid (\mathfrak{x}^{\mathfrak{y}} \mid \mathfrak{x}^{\mathfrak{y}}))\} \\ &= \min\{\mu_A(\mathfrak{x}), \mu_A(0)\} \\ &= \mu_A(\mathfrak{x}) \end{aligned}$$

and

$$\begin{aligned} \gamma_A(\mathfrak{x}^{\mathfrak{y}}) &\leq \max\{\gamma_A(\mathfrak{x}), \gamma_A(\mathfrak{x}^{\mathfrak{y}} \mid \mathfrak{x}^{\mathfrak{y}})\} \\ &= \max\{\gamma_A(\mathfrak{x}), \gamma_A(0)\} \\ &= \gamma_A(\mathfrak{x}). \end{aligned}$$

Since $\mathfrak{y} \mid (((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))) = \mathfrak{y} \mid (\mathfrak{x} \mid \mathfrak{y}) = \mathfrak{x}^{\mathfrak{y}}$, we obtain

$$\mu_A((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$\geq \min\{\mu_A(\mathfrak{y}), \mu_A(\mathfrak{y} \mid (((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))))\}$$

$$= \min\{\mu_A(\mathfrak{y}), \mu_A(\mathfrak{x}^{\mathfrak{y}})\}$$

$$\geq \min\{\mu_A(\mathfrak{y}), \mu_A(\mathfrak{x})\}$$

and

$$\gamma_A((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$\leq \max\{\gamma_A(\mathfrak{y}), \gamma_A(\mathfrak{y} \mid (((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))))\}$$

$$= \max\{\gamma_A(\mathfrak{y}), \gamma_A(\mathfrak{x}^{\mathfrak{y}})\}$$

$$\leq \max\{\gamma_A(\mathfrak{y}), \gamma_A(\mathfrak{x})\}.$$

Hence,

$$\mu_A((\mathfrak{z} \mid (\mathfrak{x} \mid \mathfrak{y})) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$= \mu_A((\mathfrak{z}|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y})))|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y})))$$

$$\geq \mu_A((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$\geq \min\{\mu_A(\mathfrak{y}), \mu_A(\mathfrak{x})\}$$

and

$$\gamma_A((\mathfrak{z} \mid (\mathfrak{x} \mid \mathfrak{y})) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$= \gamma_A((\mathfrak{z}|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y})))|((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y})))$$

$$\leq \gamma_A((\mathfrak{x} \mid \mathfrak{y}) \mid (\mathfrak{x} \mid \mathfrak{y}))$$

$$\leq \max\{\gamma_A(\mathfrak{y}), \gamma_A(\mathfrak{x})\}.$$

Therefore $A = (\mu_A, \gamma_A)$ is an IVIFSS-filter of S_H.

Conversely, assume that $A = (\mu_A, \gamma_A)$ is an IVIFSS-filter of S_H and let $\mathfrak{z}, \mathfrak{x}, \mathfrak{y} \in X$. If we replace $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{z}$ with $\mathfrak{z}$, $\mathfrak{z}^{\mathfrak{x}}$ and $\mathfrak{x}$, respectively, in (3.4), then

$$\begin{aligned}
\mu_A(\mathfrak{x}) &= \mu_A((\mathfrak{z} \mid \mathfrak{z})^0 \mid (\mathfrak{x} \mid \mathfrak{x})) \\
&= \mu_A(((\mathfrak{z} \mid \mathfrak{z})^{\mathfrak{x}^{\mathfrak{x}}|\mathfrak{x}^{\mathfrak{x}}}) \mid (\mathfrak{x} \mid \mathfrak{x})) \\
&= \mu_A((\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}) \mid (\mathfrak{x} \mid \mathfrak{x})^{\mathfrak{x}}) \\
&= \mu_A((\mathfrak{x} \mid \mathfrak{z}^{\mathfrak{x}}) \mid \mathfrak{z}^{\mathfrak{x}}) \\
&= \mu_A(((((\mathfrak{z} \mid \mathfrak{z}) \mid \mathfrak{x}) \mid \mathfrak{x}) \mid \mathfrak{x}) \mid (((\mathfrak{z} \mid \mathfrak{z}) \mid \mathfrak{x}) \mid \mathfrak{x})) \\
&= \mu_A((\mathfrak{x} \mid (\mathfrak{z} \mid \mathfrak{z}^{\mathfrak{x}})) \mid (\mathfrak{z} \mid \mathfrak{z}^{\mathfrak{x}})) \\
&\geq \min\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{z}^{\mathfrak{x}})\}
\end{aligned}$$

and

$$
\begin{aligned}
\gamma_A(\mathfrak{x}) &= \gamma_A(((\mathfrak{z} \mid \mathfrak{z})^0 \mid (\mathfrak{x} \mid \mathfrak{x}))) \\
&= \gamma_A(((\mathfrak{z} \mid \mathfrak{z})^{\mathfrak{x}^{\mathfrak{x}} | \mathfrak{x}^{\mathfrak{x}}}) \mid (\mathfrak{x} \mid \mathfrak{x})) \\
&= \gamma_A((\mathfrak{z}^{\mathfrak{x}} \mid \mathfrak{z}^{\mathfrak{x}}) \mid (\mathfrak{x} \mid \mathfrak{x})^{\mathfrak{x}}) \\
&= \gamma_A((\mathfrak{x} \mid \mathfrak{z}^{\mathfrak{x}}) \mid \mathfrak{z}^{\mathfrak{x}}) \\
&= \gamma_A(((((\mathfrak{z} \mid \mathfrak{z}) \mid \mathfrak{x}) \mid \mathfrak{x}) \mid \mathfrak{x}) \mid (((\mathfrak{z} \mid \mathfrak{z}) \mid \mathfrak{x}) \mid \mathfrak{x})) \\
&= \gamma_A((\mathfrak{x} \mid (\mathfrak{z} \mid \mathfrak{z}^{\mathfrak{x}})) \mid (\mathfrak{z} \mid \mathfrak{z}^{\mathfrak{x}})) \\
&\leq \max\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{z}^{\mathfrak{x}})\}.
\end{aligned}
$$

Therefore, $A = (\mu_A, \gamma_A)$ is an IVIFSS-deductive system of S_H. □

Proposition 3.3. *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. Every IVIFSS-filter $A = (\mu_A, \gamma_A)$ of S_H satisfies*

$$
(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \left(\begin{array}{l} \mu_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) \geq \mu_A(\mathfrak{z}) \\ \gamma_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) \leq \gamma_A(\mathfrak{z}) \end{array} \right), \tag{3.5}
$$

$$
(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \left(\mathfrak{z} \leq \mathfrak{x} \Rightarrow \left\{ \begin{array}{l} \mu_A(\mathfrak{x}) \geq \mu_A(\mathfrak{z}) \\ \gamma_A(\mathfrak{x}) \leq \gamma_A(\mathfrak{z}) \end{array} \right. \right). \tag{3.6}
$$

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIFSS-filter of S_H. Then

$$
\begin{aligned}
\mu_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) &= \mu_A(\mathfrak{x}^{\mathfrak{z}|\mathfrak{z}}) \\
&\leq \min\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{z})\} \\
&= \mu_A(\mathfrak{z})
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) &= \gamma_A(\mathfrak{x}^{\mathfrak{z}|\mathfrak{z}}) \\
&\geq \max\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\} \\
&= \gamma_A(\mathfrak{z})
\end{aligned}
$$

for all $\mathfrak{z}, \mathfrak{x} \in S_H$. Therefore, (3.5) is valid. Let $\mathfrak{z}, \mathfrak{x} \in S_H$ be such that $\mathfrak{z} \leq \mathfrak{x}$. Then $\mathfrak{z}^{\mathfrak{x}} = 0$, and so

$$
\begin{aligned}
\mu_A(\mathfrak{x}) &= \mu_A(0^{\mathfrak{x}}) \\
&= \mu_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) \\
&\geq \mu_A(x)
\end{aligned}
$$

and

$$\begin{aligned}\gamma_A(\mathfrak{x}) &= \gamma_A(0^{\mathfrak{x}}) \\ &= \gamma_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{x}}) \\ &\leq \gamma_A(\mathfrak{z}).\end{aligned}$$

□

Theorem 3.6. *Let $\mathcal{S}_{\mathcal{H}} := (S_H, |)$ be an SSHA. An IVIFS $A = (\mu_A, \gamma_A)$ in S_H is an IVIFSS-filter of S_H if and only if it satisfies the condition (3.6) and*

$$(\forall \mathfrak{z}, \mathfrak{x} \in S_H) \begin{pmatrix} \mu_A((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{z} \mid \mathfrak{x})) \geq \min\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{x})\} \\ \gamma_A((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{z} \mid \mathfrak{x})) \leq \max\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{z})\} \end{pmatrix}. \tag{3.7}$$

Proof. Let $A = (\mu_A, \gamma_A)$ be an IVIFSS-filter of S_H. Then the condition (3.6) is valid. Then $\mu_A((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{z} \mid \mathfrak{x})) = \mu_A((0 \mid 0)^{\mathfrak{z}|\mathfrak{x}}) \geq \min\{\mu_A(\mathfrak{z}), \mu_A(\mathfrak{x})\}$ and $\gamma_A((\mathfrak{z} \mid \mathfrak{z}) \mid (\mathfrak{z} \mid \mathfrak{x})) = \gamma_A((0 \mid 0)^{\mathfrak{z}|\mathfrak{x}}) \leq \max\{\gamma_A(\mathfrak{z}), \gamma_A(\mathfrak{x})\}$ for all $\mathfrak{z}, \mathfrak{x} \in S_H$.

Conversely, assume that $A = (\mu_A, \gamma_A)$ satisfies (3.6) and (3.7). Since $\mathfrak{z} \leq 0$ and $\mathfrak{x} \leq \mathfrak{z}^{\mathfrak{x}}$ for all $\mathfrak{z}, \mathfrak{x} \in S_H$, we have $\mu_A(0) \leq \mu_A(\mathfrak{z})$, $\gamma_A(0) \leq \gamma_A(\mathfrak{z})$, $\mu_A(\mathfrak{z}^{\mathfrak{x}}) \leq \mu_A(\mathfrak{x})$, and $\gamma_A(\mathfrak{z}^{\mathfrak{x}}) \leq \gamma_A(\mathfrak{x})$. So,

$$\mu_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{y}}) \leq \mu_A((\mathfrak{x}|\mathfrak{y})|(\mathfrak{x}|\mathfrak{y})) \leq \min\{\mu_A(\mathfrak{x}), \mu_A(\mathfrak{y})\}$$

and

$$\gamma_A(\mathfrak{z}^{\mathfrak{x}|\mathfrak{y}}) \leq \gamma_A((\mathfrak{x}|\mathfrak{y})|(\mathfrak{x}|\mathfrak{y})) \leq \max\{\gamma_A(\mathfrak{x}), \gamma_A(\mathfrak{y})\}$$

for all $\mathfrak{z}, \mathfrak{x} \in S_H$. Hence, $A = (\mu_A, \gamma_A)$ is an IVIFSS-filter of S_H. □

4. Conclusion

In this paper, we have developed a unified framework for studying interval-valued intuitionistic fuzzy Sheffer stroke subsets (IVIFSS) in the setting of Sheffer stroke Hilbert algebras (SSHAs). We derived necessary and sufficient conditions for establishing deals and proved that these structures are closed under Cartesian products, intersections, and the operators $\oplus$ and $\otimes$. It was shown that the product of two IVIFSS-subalgebras (or IVIFSS-ideals) retains the corresponding algebraic properties within the product algebra, and level subset characterizations were derived to describe these structures in greater detail. Moreover, the relationships between IVIFSS-deductive systems and IVIFSS-filters were analyzed, yielding necessary and sufficient conditions establishing their equivalence. Overall, the results extend classical algebraic concepts of subalgebras, ideals, and filters to the interval-valued intuitionistic fuzzy framework, providing a consistent algebraic foundation for reasoning under uncertainty.

Future investigations could explore several directions based on the current findings. One natural extension is to apply the IVIFSS framework to other non-classical algebraic systems, such as BCK/BCI-algebras, BE-algebras, and MTL-algebras, to examine whether similar closure and equivalence properties hold. Another promising direction is to define and analyze morphisms between IVIFSS structures, yielding categorical formulations and homomorphism theorems in the fuzzy setting. Additionally, potential applications may include the design of fuzzy inference models and decision-making systems that rely on interval-valued intuitionistic fuzzy logic derived from the Sheffer stroke operation.

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