

(m, σ^*) -CONTINUITY FOR MULTIFUNCTIONSMONTRI THONGMOON¹, AREEYUTH SAMA-AE², CHAWALIT BOONPOK^{1,*}

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ABSTRACT. A new class of continuous multifunctions between an m -space and an ideal topological space, called upper (lower) (m, σ^*) -continuous multifunctions, has been defined and studied. Furthermore, several characterizations and some properties concerning upper (m, σ^*) -continuous multifunctions and lower (m, σ^*) -continuous multifunctions are considered.

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1. INTRODUCTION

In 2000, Noiri and Popa [20] introduced and investigated the notions of upper M -continuous multifunctions and lower M -continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. Moreover, Noiri and Popa [19] introduced the notions of upper and lower continuity for multifunctions from a set satisfying certain minimal condition into a topological space, namely upper m -continuous multifunctions and lower m -continuous multifunctions. Furthermore, several characterizations and some properties of upper m -continuous multifunctions and lower m -continuous multifunctions were presented in [18] and [19]. Semi- \mathcal{I} -open sets, pre- \mathcal{I} -open sets, α - \mathcal{I} -open sets, β - \mathcal{I} -open sets and δ - \mathcal{I} -open sets play an important role in the research of generalizations of continuity in ideal topological spaces. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [12] introduced and investigated the notions of weakly pre- \mathcal{I} -open sets and weakly pre- \mathcal{I} -continuous functions. Hatir and Noiri [13] investigated some characterizations of semi- \mathcal{I} -continuous functions. On the other hand, the present authors introduced and investigated the concepts of upper

\star -continuous multifunctions [10], lower \star -continuous multifunctions [10], upper $\beta(\star)$ -continuous multifunctions [7], lower $\beta(\star)$ -continuous multifunctions [7], upper $s\beta(\star)$ -continuous multifunctions [5], lower $s\beta(\star)$ -continuous multifunctions [5], upper α - \star -continuous multifunctions [2], lower α - \star -continuous multifunctions [2], upper almost α - \star -continuous multifunctions [3], lower almost α - \star -continuous multifunctions [3], upper weakly α - \star -continuous multifunctions [4], lower weakly α - \star -continuous multifunctions [4], upper $\theta(\star)$ -quasi continuous multifunctions [6], lower $\theta(\star)$ -quasi continuous multifunctions [6], p_i -continuous multifunctions [1] and weakly p_i -continuous multifunctions [1]. In this paper, we introduce the concepts of upper (m, σ^*) -continuous multifunctions and lower (m, σ^*) -continuous multifunctions. We also investigate some characterizations of upper (m, σ^*) -continuous multifunctions and lower (m, σ^*) -continuous multifunctions.

2. PRELIMINARIES

A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called an m -structure [11] on X if m satisfies the following properties: (1) $\emptyset \in m$ and $X \in m$; (2) $\cup_{\alpha \in \nabla} A_\alpha \in m$ whenever $A_\alpha \in m$ for each $\alpha \in \nabla$. We call the pair (X, m) an m -space. Each member of m is said to be m -open and the complement of an m -open set is said to be m -closed. Let (X, m) be an m -space and A a subset of X . The m -closure $mCl(A)$ and m -interior $mInt(A)$ of A are defined in [16] as follows:

$$mCl = \cap \{F \mid A \subseteq F \text{ and } X - F \in m\}$$

$$\text{and } mInt = \cup \{U \mid U \subseteq A \text{ and } U \in m\}.$$

Lemma 1. [11] *Let (X, m) be an m -space. Then, for a subset A of X , the following properties hold:*

- (1) $A \in m$ if and only if $A = mInt(A)$;
- (2) A is m -closed if and only if $A = mCl(A)$;
- (3) $mCl(A)$ is m -closed and $mInt(A)$ is m -open.

An ideal \mathcal{I} on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal \mathcal{I} on X is called an *ideal topological space* and is denoted by (X, τ, \mathcal{I}) . For an ideal topological space (X, τ, \mathcal{I}) and a subset A of X , $A^*(\mathcal{I})$ is defined as follows:

$$A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^*(\mathcal{I})$ is simply written as A^* . In [15], A^* is called the local function of A with respect to \mathcal{I} and τ and $Cl^*(A) = A^* \cup A$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ finer than τ . A subset A is said to be τ^* -closed [14] if $A^* \subseteq A$. The interior of a subset A in $(X, \tau^*(\mathcal{I}))$ is denoted by $Int^*(A)$. A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be τ^* -semi-open [17] (*semi- \mathcal{I}^* -open* [9]) if $A \subseteq Cl^*(Int^*(A))$. The complement of a τ^* -semi-open set

is called τ^* -semi-closed. For a subset A of an ideal topological space (X, τ, \mathcal{I}) , the intersection of all τ^* -semi-closed sets containing A is called the τ^* -semi-closure [9] of A and is denoted by $sCl^*(A)$. The union of all τ^* -semi-open sets contained in A is called the τ^* -semi-interior [9] of A and is denoted by $sInt^*(A)$.

Lemma 2. [9] For a subset A of an ideal topological space (X, τ, \mathcal{I}) , the following properties hold:

- (1) $sCl^*(A) = A \cup Int^*(Cl^*(A))$.
- (2) $sInt^*(A) = A \cap Cl^*(Int^*(A))$.

By a multifunction $F : X \rightarrow Y$, we mean a point-to-set correspondence from X into Y , and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \rightarrow Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. UPPER AND LOWER (m, σ^*) -CONTINUOUS MULTIFUNCTIONS

In this section, we introduce the notions of upper (m, σ^*) -continuous multifunctions and lower (m, σ^*) -continuous multifunctions. Moreover, several characterizations of upper (m, σ^*) -continuous multifunctions and lower (m, σ^*) -continuous multifunctions are discussed.

Definition 1. A multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$ is said to be upper (m, σ^*) -continuous at a point $x \in X$ if for each σ^* -open set V of Y such that $F(x) \subseteq V$, there exists an m -open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$ is said to be upper (m, σ^*) -continuous if F is upper (m, σ^*) -continuous at each point x of X .

Theorem 1. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$, the following properties are equivalent:

- (1) F is upper (m, σ^*) -continuous;
- (2) $F^+(V)$ is m -open in X for every σ^* -open set V of Y ;
- (3) $F^-(K)$ is m -closed in X for every σ^* -closed set K of Y ;
- (4) $mCl(F^-(B)) \subseteq F^-(Cl^*(B))$ for every subset B of Y ;
- (5) $F^+(Int^*(B)) \subseteq mInt(F^+(B))$ for every subset B of Y .

Proof. (1) \Rightarrow (2): Let V be any σ^* -open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), there exists an m -open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and so $x \in mInt(F^+(V))$. Therefore, $F^+(V) \subseteq mInt(F^+(V))$. This shows that $F^+(V)$ is m -open in X .

(2) \Rightarrow (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y .

(3) \Rightarrow (4): Let B be any subset of Y . Then, $Cl^*(B)$ is σ^* -closed in Y and by (3), $mCl(F^-(B)) \subseteq mCl(F^-(Cl^*(B))) = F^-(Cl^*(B))$.

(4) \Rightarrow (5): Let B be any subset of Y . Thus by (4), we have

$$\begin{aligned} X - \text{mInt}(F^+(B)) &= \text{mCl}(X - F^+(B)) \\ &= \text{mCl}(F^-(Y - B)) \\ &\subseteq F^-(\text{Cl}^*(Y - B)) \\ &= F^-(Y - \text{Int}^*(B)) \\ &= X - F^+(\text{Int}^*(B)) \end{aligned}$$

and hence $F^+(\text{Int}^*(B)) \subseteq \text{mInt}(F^+(B))$.

(5) \Rightarrow (1): Let $x \in X$ and V be any σ^* -open set of Y such that $F(x) \subseteq V$. By (5), $x \in F^+(V) = \text{mInt}(F^+(V))$. There exists an m -open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper (m, σ^*) -continuous. \square

Definition 2. A multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be lower (m, σ^*) -continuous at a point $x \in X$ if for each σ^* -open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an m -open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be lower (m, σ^*) -continuous if F is lower (m, σ^*) -continuous at each point x of X .

Theorem 2. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

- (1) F is lower (m, σ^*) -continuous;
- (2) $F^-(V)$ is m -open in X for every σ^* -open set V of Y ;
- (3) $F^+(K)$ is m -closed in X for every σ^* -closed set K of Y ;
- (4) $\text{mCl}(F^+(B)) \subseteq F^+(\text{Cl}^*(B))$ for every subset B of Y ;
- (5) $F(\text{mCl}(A)) \subseteq \text{Cl}^*(F(A))$ for every subset A of X ;
- (6) $F^-(\text{Int}^*(B)) \subseteq \text{mInt}(F^-(B))$ for every subset B of Y .

Proof. We prove only the implications (4) \Rightarrow (5) and (5) \Rightarrow (6) being the proofs of the other similar to those of Theorem 1.

(4) \Rightarrow (5): Let A be any subset of X . By (4), we have $\text{mCl}(A) \subseteq \text{mCl}(F^+(F(A))) \subseteq F^+(\text{Cl}^*(F(A)))$ and so $F(\text{mCl}(A)) \subseteq \text{Cl}^*(F(A))$.

(5) \Rightarrow (6): Let B be any subset of Y . Thus by (5),

$$\begin{aligned} F(\text{mCl}(F^+(Y - B))) &\subseteq \text{Cl}^*(F(F^+(Y - B))) \\ &\subseteq \text{Cl}^*(Y - B) = Y - \text{Int}^*(B). \end{aligned}$$

Since

$$F(\text{mCl}(F^+(Y - B))) = F(\text{mCl}(X - F^-(B)))$$

$$= F(X - \text{mInt}(F^-(B))),$$

we have

$$X - \text{mInt}(F^-(B)) \subseteq F^+(Y - \text{Int}^*(B)) = X - F^-(\text{Int}^*(B))$$

and hence $F^-(\text{Int}^*(B)) \subseteq \text{mInt}(F^-(B))$. \square

A point x in an ideal topological space (X, τ, \mathcal{I}) is called a \star_δ -cluster point of A if $\text{Int}^*(\text{Cl}^*(U)) \cap A \neq \emptyset$ for every τ^* -open set U of X containing x . The set of all \star_δ -cluster points of A is called the \star_δ -closure of A and is denoted by $\star_\delta\text{Cl}(A)$. A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be \star_δ -closed if $\star_\delta\text{Cl}(A) = A$. The complement of a \star_δ -closed set is called \star_δ -open.

Definition 3. An ideal topological space (X, τ, \mathcal{I}) is said to be τ^* -semi-regular if for each τ^* -semi-closed set F and each $x \notin F$, there exist disjoint τ^* -semi-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3. An ideal topological space (X, τ, \mathcal{I}) is τ^* -semi-regular if and only if for each $x \in X$ and each τ^* -semi-open set U of X containing x , there exists a τ^* -semi-open set V such that $x \in V \subseteq s\text{Cl}^*(V) \subseteq U$.

Lemma 4. Let (X, τ, \mathcal{I}) be a τ^* -semi-regular space. Then, the following properties hold:

- (1) $\text{Cl}^*(A) = \star_\delta\text{Cl}(A)$ for every subset A of X .
- (2) Every τ^* -open set is \star_δ -open.

Theorem 3. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$, where (Y, σ, \mathcal{I}) is σ^* -semi-regular, the following properties are equivalent:

- (1) F is upper (m, σ^*) -continuous;
- (2) $F^-(\star_\delta\text{Cl}(B))$ is m -closed in X for every subset B of Y ;
- (3) $F^-(K)$ is m -closed in X for every \star_δ -closed set K of Y ;
- (4) $F^+(V)$ is m -open in X for every \star_δ -open set V of Y .

Proof. (1) \Rightarrow (2): Let B be any subset of Y . By Lemma 4, $\star_\delta\text{Cl}(B)$ is σ^* -closed in Y . Since F is upper (m, σ^*) -continuous, by Theorem 1 $F^-(\star_\delta\text{Cl}(B))$ is m -closed in X .

(2) \Rightarrow (3): Let K be any \star_δ -closed set of Y . Then, $\star_\delta\text{Cl}(K) = K$ and by (2), we have $F^-(K)$ is m -closed in X .

(3) \Rightarrow (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y .

(4) \Rightarrow (1): Let V be any σ^* -open set of Y . Since (Y, σ, \mathcal{I}) is σ^* -semi-regular, we have V is \star_δ -open in Y and by (4), $F^+(V)$ is m -open in X . Thus, F is upper (m, σ^*) -continuous by Theorem 1. \square

Theorem 4. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$, where (Y, σ, \mathcal{I}) is σ^* -semi-regular, the following properties are equivalent:

- (1) F is lower (m, σ^*) -continuous;
- (2) $F^+(\star_\delta Cl(B))$ is m -closed in X for every subset B of Y ;
- (3) $F^+(K)$ is m -closed in X for every \star_δ -closed set K of Y ;
- (4) $F^-(V)$ is m -open in X for every \star_δ -open set V of Y .

Proof. The proof is similar to that of Theorem 3. □

Recall that a point x in an ideal topological space (X, τ, \mathcal{I}) is called a \star_θ -cluster point [8] of A if $Cl^*(U) \cap A \neq \emptyset$ for every τ^* -open set U of X containing x . The set of all \star_θ -cluster points of A is called the \star_θ -closure [8] of A and is denoted by $\star_\theta Cl(A)$. A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be \star_θ -closed [8] if $\star_\theta Cl(A) = A$. The complement of a \star_θ -closed set is said to be \star_θ -open.

Lemma 5. [8] For a subset A of an ideal topological space (X, τ, \mathcal{I}) , the following properties hold:

- (1) If A is τ^* -open in X , then $Cl^*(A) = \star_\theta Cl(A)$.
- (2) $\star_\theta Cl(A)$ is τ^* -closed in X .

Definition 4. An ideal topological space (X, τ, \mathcal{I}) is called τ^* -regular if for each τ^* -closed set F and each $x \notin F$, there exist disjoint τ^* -open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 6. An ideal topological space (X, τ, \mathcal{I}) is τ^* -regular if and only if for each $x \in X$ and each τ^* -open set U of X containing x , there exists a τ^* -open set V such that $x \in V \subseteq Cl^*(V) \subseteq U$.

Lemma 7. Let (X, τ, \mathcal{I}) be a τ^* -regular space. Then, the following properties hold:

- (1) $Cl^*(A) = \star_\theta Cl(A)$ for every subset A of X .
- (2) Every τ^* -open set is \star_θ -open.

Theorem 5. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{I})$, where (Y, σ, \mathcal{I}) is σ^* -regular, the following properties are equivalent:

- (1) F is upper (m, σ^*) -continuous;
- (2) $F^-(\star_\theta Cl(B))$ is m -closed in X for every subset B of Y ;
- (3) $F^-(K)$ is m -closed in X for every \star_θ -closed set K of Y ;
- (4) $F^+(V)$ is m -open in X for every \star_θ -open set V of Y .

Proof. (1) \Rightarrow (2): Let B be any subset of Y . By Lemma 7, $\star_\theta Cl(B)$ is σ^* -closed in Y . Since F is upper (m, σ^*) -continuous, by Theorem 1 $F^-(\star_\theta Cl(B))$ is m -closed in X .

(2) \Rightarrow (3): Let K be any \star_θ -closed set of Y . Then, $\star_\theta Cl(K) = K$. Thus by (2), $F^-(K)$ is m -closed in X .

(3) \Rightarrow (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y .

(4) \Rightarrow (1): Let V be any σ^* -open set of Y . Since (Y, σ, \mathcal{I}) is σ^* -regular, we have V is \star_θ -open in Y and by (4), $F^+(V)$ is m -open in X . Thus, F is upper (m, σ^*) -continuous by Theorem 1. □

Theorem 6. For a multifunction $F : (X, m) \rightarrow (Y, \sigma, \mathcal{J})$, where (Y, σ, \mathcal{J}) is σ^* -regular, the following properties are equivalent:

- (1) F is lower (m, σ^*) -continuous;
- (2) $F^+(\star_\theta Cl(B))$ is m -closed in X for every subset B of Y ;
- (3) $F^+(K)$ is m -closed in X for every \star_θ -closed set K of Y ;
- (4) $F^-(V)$ is m -open in X for every \star_θ -open set V of Y .

Proof. The proof is similar to that of Theorem 5. □

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