

ELLIPTIC PROBLEMS WITH ROBIN BOUNDARY CONDITIONS WITH OPERATOR COEFFICIENTS STUDIED IN HÖLDER SPACES AND IN THE NON COMMUTATIVE FRAMEWORK

MOHAMMED RABAH, RABAH HAOUA*, MAAMAR ANDASMAS

Laboratory of pure and applied mathematics, university of Mostaganem, Algeria

*Corresponding author: rabah.haoua@univ-mosta.dz

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ABSTRACT. This article presents new results concerning second-order elliptic operational differential equations with boundary conditions where the coefficients are closed linear operators, in a noncommutative setting. The study is conducted when the second term f belongs to $C^\theta([0, 1]; X)$, with $\theta \in]0, 1[$ arbitrary, X being a complex Banach space. We prove new results on the existence, uniqueness, and maximum regularity of strict solutions, using semigroups and interpolation theory. This article can be considered a valuable addition to the work already by Haoua et al.

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1. INTRODUCTION AND HYPOTHESES

Let X be a complex Banach space. We consider the following abstract second-order differential equation of elliptic type

$$(1.1) \quad u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in]0, 1[,$$

together with the following Robin's type abstract boundary conditions:

$$(1.2) \quad \begin{cases} u'(0) - Hu(0) = d_0 \\ u(1) = d_1. \end{cases}$$

where A and H are closed linear operators in X , ω is a complex spectral parameter, $d_0, d_1 \in X$; and the second term f is in the space $C([0, 1]; X)$. Our main objective is to find a strict solution u to problem (1.1)-(1.2), that is a function u such that:

$$\begin{cases} u \in C^2([0, 1]; X) \cap C([0, 1]; D(A)) \\ u(0) \in D(H), \end{cases}$$

and which satisfies (1.1)-(1.2).

Since $f \in C([0, 1]; X)$ is not sufficient to solve problem (1.1)-(1.2), this is why we assume, throughout the work that the function f is in an interpolation space:

$$C^\theta([0, 1]; X) = \left\{ f \in C([0, 1]; X) : \sup_{x, y' \in [0, 1], x-y' \neq 0} \frac{\|f(x) - f(y')\|}{|x - y'|^\theta} < \infty \right\}.$$

The techniques used here are based on interpolation theory, Dunford's functional calculus and methods applied in [4] and [13]. Here are some works related to ours.

In [3], the authors considered, in a complex Banach space X , the problem (1.1)-(1.2), where $\lambda = \omega$ is a positive spectral parameter. For ω sufficiently large, under certain geometric assumptions on the space X and hypotheses on operators $A - \omega I$ and H including the fact that they commute in the resolvent sense, the authors have furnished necessary and sufficient conditions on the data d_0, d_1 to obtain the existence and the uniqueness of a solution to (1.1)-(1.2) verifying $u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A))$. In [4], the authors studied the problem (1.1)-(1.2) in a Hölder space but without the spectral parameter ω and in a commutative framework.

In [5], they developed an interesting new approach in a noncommutative setting, where they studied the problem (1.1)-(1.2), when $f \in L^p(0, 1; X)$ with $1 < p < \infty$ and when X is an UMD space with the same Robin boundary condition in 0.

Recently, in [13], the authors studied the same problem (1.1)-(1.2) in a noncommutative setting, when $f \in C^\theta([0, 1]; X)$, with $0 < \theta < 1$ and X is a Banach space. They considered the case where $D(\sqrt{-A}) \subset D(H)$, when $\sqrt{-A}$ is defined. Following on from this recent study, regarding hypotheses, we address the problem (1.1)-(1.2) in the case where $D(H) \subset D(A)$.

So, for $\varphi \in (0, \pi)$, we set

$$S_\varphi = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \varphi\} \cup \{0\}.$$

Our main assumptions are the following:

$$(1.3) \quad \begin{cases} \exists \varphi_0 \in (0, \pi) : S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \forall \omega \in S_{\varphi_0}, \|(A - \omega I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_A}{1 + |\omega|}, \end{cases}$$

($\rho(A)$ denotes the resolvent set of A ; this assumption means exactly the ellipticity of our problem as in Krein [15]).

We will assume, moreover that

$$(1.4) \quad D(H) \subset D(A),$$

and H has to satisfy

$$(1.5) \quad 0 \in \rho(H).$$

In all this work, we set for $\omega \in S_{\varphi_0}$

$$A_\omega = A - \omega I$$

Remark 1.1. It is well known that assumption (1.3) implies that the square roots $Q = -(-A)^{\frac{1}{2}}$ and $Q_\omega = -(-A + \omega I)^{\frac{1}{2}}$ are well defined and generate analytic semigroups $(e^{xQ})_{x \geq 0}$ and $(e^{xQ_\omega})_{x \geq 0}$ in X , not strongly continuous at zero, see Balakrishnan [2] for dense domains and Martinez [18] for non dense domains. Note that, $\overline{D(A)} = \overline{D(Q)}$, see Haase [12], Corollary 3.1.11. Page 59.

Remark 1.2. The space of Hölder space, $C^\theta([0, 1]; X)$, defined by

$$C^\theta([0, 1]; X) = \left\{ f \in C([0, 1]; X) : \sup_{x, y' \in [0, 1], x - y' \neq 0} \frac{\|f(x) - f(y')\|}{|x - y'|^\theta} < \infty \right\}.$$

is an interpolation space.

Remark 1.3. We have the following proposition, Let $\theta \in]0, 1[$. So

$$C^1([0, 1]; X) \subset C^\theta([0, 1]; X) \subset C([0, 1]; X),$$

and $C^\theta([0, 1]; X)$ with $\theta \in]0, 1[$ furnished with standard

$$\|f\|_{C^\theta([0, 1]; X)} = \|f\|_{C([0, 1]; X)} + \sup_{x, y' \in [0, 1], x - y' \neq 0} \frac{\|f(x) - f(y')\|}{|x - y'|^\theta},$$

is a Banach space, see [19], page 12.

Due to (1.3), we recall that $D_Q(\theta; +\infty) = (D(Q); X)_{1-\theta, +\infty}$ is the real interpolation space described see [7].

We will also suppose that

$$(1.6) \quad (Q - H)^{-1}(D(Q)) \subset D(Q^2),$$

and

$$(1.7) \quad Q^2(Q - H)^{-1}(D_Q(1 + \theta; +\infty)) \subset \overline{D(Q)};$$

here, for the moment, $(Q - H)^{-1}$ is understood in the term's specific context. Later on, we'll observe that this

$$(Q - H)^{-1} \in \mathcal{L}(X).$$

We recall the notation

$$D_Q(1 + \theta; +\infty) = \{\varphi \in D(Q) : Q\varphi \in D_Q(\theta; +\infty)\}.$$

To achieve the highest level of regularity, we replace (1.7) by

$$(1.8) \quad Q^2(Q - H)^{-1}(D_Q(1 + \theta; +\infty)) \subset D_Q(\theta; +\infty).$$

This paper is the valuable addition of [13], where the authors have studied problem (1.1)-(1.2) under (1.3), (1.5)~(1.8) and also

$$D(Q) \subset D(H),$$

instead of (1.4).

Remark 1.4. assume (1.3) we deduce that, there exists $\theta_0 \in]0, \pi/2[$ and $r_0 > 0$ such that the resolvent operator of A satisfies:

$$\rho(A) \supset S_{\varphi_0, r_0} = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \theta_0\} \cup \overline{B(0, r_0)},$$

where $\overline{B(0, r_0)}$ is the closed ball of radius r_0 and centered in 0.

Remark 1.5. Assume (1.3) and (1.4). Let $\omega > 0$, we know that there exists $P_\omega \in \mathcal{L}(X)$, such that

$$(1.9) \quad Q = Q_\omega + P_\omega \text{ and } P_\omega Q^{-1} = Q^{-1} P_\omega.$$

See [12], Proposition 3.1.7, p. 65. From (1.9) we deduce that P_ω is continuous from $D(Q)$ into $D(Q)$ by writing for all $\varkappa \in D(Q)$

$$\begin{aligned} P_\omega \varkappa &= P_\omega Q^{-1} Q \varkappa \\ &= Q^{-1} P_\omega Q \varkappa, \end{aligned}$$

which gives

$$\begin{aligned} \|P_\omega \varkappa\|_{D(Q)} &= \|P_\omega \varkappa\|_X + \|Q P_\omega \varkappa\|_X \\ &= \|P_\omega \varkappa\|_X + \|Q Q^{-1} P_\omega Q \varkappa\|_X \\ &= \|P_\omega \varkappa\|_X + \|P_\omega Q \varkappa\|_X \\ &\leq \|P_\omega\|_X \|\varkappa\|_X + \|P_\omega\|_X \|Q \varkappa\|_X \\ &\leq \|P_\omega\|_X [\|\varkappa\|_X + \|Q \varkappa\|_X] \end{aligned}$$

$$\leq \|P_\omega\|_X \|\mathcal{A}\|_{D(Q)},$$

therefore P_ω is continuous from any interpolation space $D_Q(\theta; +\infty)$ into itself.

Remark 1.6. Let $\omega > 0$, we have

$$D_{Q_\omega}(\theta; +\infty) = D_Q(\theta; +\infty), \quad D_{Q_\omega}(1 + \theta; +\infty) = D_Q(1 + \theta; +\infty).$$

Moreover $D(Q_\omega) = D(Q)$ and $D(Q_\omega^2) = D(Q^2)$ with

$$Q^2 = Q_\omega^2 + 2P_\omega Q_\omega + P_\omega^2,$$

However, we have

$$\begin{aligned} Q_\omega^{-2} &= Q^{-2} - (-A)^{-1} + (-A + \omega I)^{-1} \\ &= Q^{-2} - (-A)^{-1}(-A + \omega I)(-A + \omega I)^{-1} + (-A)^{-1}(-A)(-A + \omega I)^{-1} \\ &= Q^{-2} - \omega(-A)^{-1}(-A + \omega I)^{-1} \\ &= Q^{-2} - \omega Q^{-2} Q_\omega^{-2}. \end{aligned}$$

Remark 1.7. Let $\omega > 0$, let us show that if $(Q - H)^{-1}$ exists and verifies

$$(1.10) \quad (Q - H)^{-1}(D(Q)) \subset D(Q^2),$$

then

$$(Q_\omega - H)^{-1}(D(Q_\omega)) \subset D(Q_\omega^2),$$

when $(Q_\omega - H)^{-1}$ exists.

Proof. Assume (1.10) and set

$$(Q_\omega - H)^{-1}(Q^{-1}(\xi)) = \eta,$$

then

$$Q^{-1}(\xi) = (Q_\omega - H)(\eta) = (Q_\omega - Q)(\eta) + (Q - H)(\eta);$$

We have seen that from (1.9)

$$Q = Q_\omega + P_\omega \text{ and } Q^{-1}P_\omega = P_\omega Q^{-1};$$

Since $\eta \in D(Q)$, then there exists $\zeta \in X$ such that $\eta = Q^{-1}(\zeta)$, therefore

$$\begin{aligned} Q^{-1}(\xi) &= (Q_\omega - Q)Q^{-1}(\zeta) + (Q - H)\eta \\ &= -Q^{-1}P_\omega(\zeta) + (Q - H)\eta, \end{aligned}$$

and

$$\eta = (Q - H)^{-1} [Q^{-1}(\xi) + Q^{-1}P_\omega(\zeta)] \in D(Q^2).$$

□

Remark 1.8. Let $\omega > 0$. Now, prove that if $(Q - H)^{-1}$ exists and verifies the following properties

$$(1.11) \quad \begin{cases} (Q - H)^{-1}(D(Q)) \subset D(Q^2) \\ \text{For some } \xi \in X, (Q - H)^{-1}(\xi) \in D(Q^2), \end{cases}$$

then, when $(Q_\omega - H)^{-1}$ exists, we have also

$$(Q_\omega - H)^{-1}(\xi) \in D(Q^2).$$

Proof. In fact, set

$$(Q_\omega - H)^{-1}(\xi) = \eta,$$

so

$$\xi = (Q_\omega - H)(\eta) = (Q_\omega - Q)(\eta) + (Q - H)(\eta),$$

where $\eta = Q^{-1}(\varphi)$, $\varphi \in X$. As above, we have

$$\xi = -Q^{-1}P_\omega(\varphi) + (Q - H)(\eta),$$

from which it follows

$$(Q - H)^{-1}(\xi) = -(Q - H)^{-1}Q^{-1}P_\omega(\varphi) + \eta,$$

in virtue of (1.11), we obtain

$$\eta = (Q - H)^{-1}(\xi) + (Q - H)^{-1}Q^{-1}P_\omega(\varphi) \in D(Q^2).$$

□

Remark 1.9. Let $\omega > 0$. Assume that $(Q - H)^{-1}$ exists and verifies the four following properties

$$(1.12) \quad \begin{cases} \text{i) } (Q - H)^{-1}(D(Q)) \subset D(Q^2) \\ \text{ii) } Q^2(Q - H)^{-1}(D(Q)) \subset \overline{D(Q)} \\ \text{iii) For some } \xi \in X, (Q - H)^{-1}(\xi) \in D(Q^2) \\ \text{iv) } Q^2(Q - H)^{-1}(\xi) + f(0) \in \overline{D(Q)}, \end{cases}$$

then, when $(Q_\omega - H)^{-1}$ exists, we have also

$$\begin{cases} (Q_\omega - H)^{-1}(\xi) \in D(Q^2) \text{ and} \\ Q^2(Q_\omega - H)^{-1}(\xi) + f(0) \in \overline{D(Q)}. \end{cases}$$

Proof. In fact, by setting

$$(Q_\omega - H)^{-1}(\xi) = \eta,$$

and using the last calculus, we get

$$\eta = (Q - H)^{-1}(\xi) + (Q - H)^{-1}Q^{-1}P_\omega(\varphi),$$

so

$$\begin{aligned} & Q^2(Q_\omega - H)^{-1}(\xi) + f(0) \\ = & Q^2(\eta) + f(0) = Q^2(Q - H)^{-1}(\xi) + Q^2(Q - H)^{-1}Q^{-1}P_\omega(\varphi) \in \overline{D(Q)}. \end{aligned}$$

□

Regarding the necessary conditions

Let us assume (1.3) and that there exists a classical solution to the problem (1.1)-(1.2)

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x), & x \in]0, 1[, \\ u'(0) = Hu(0)d_0, & u(1) = d_1. \end{cases}$$

This is a mixed-type, second-order abstract elliptic problem; therefore, we are informed that necessary we have

$$\begin{cases} f(1) - A_\omega d_1 \in \overline{D(A)} = \overline{D(Q)} \\ u'(0) \in D(Q) \\ Qu'(0) \in \overline{D(A)} = \overline{D(Q)}. \end{cases}$$

The proof of these results can be found in [16], theorem 8, on page 732.

The plan of the paper is as follows:

We introduced our hypothesis (1.3)~(1.8). in the first part, following the presentation of our problem (1.1)-(1.2) and a few writers' works on the same theme. Next, we presented all of the implications of these theories in the form of notes. We provide some useful fundamental lemmas in the next section. The representation formula for strict solution u of problem (1.1)-(1.2) is provided in Section 3. The primary result's evidence is covered in Section 4. Section 5 concludes by providing some specific instances of partial differential equations that our theory can be applied to.

2. TECHNICAL LEMMAS

In this section, we recall some well-known facts, useful in our proofs.

Lemma 2.1. *Note that*

$$\varepsilon(\phi) = \begin{cases} \phi & \text{if } \phi \in (0, \frac{\pi}{2}] \\ \pi - \phi & \text{if } \phi \in [\frac{\pi}{2}, \pi). \end{cases}$$

1): Let $\theta \in (0, \varepsilon(\phi))$, Then A_ω is an operator of type θ with bound $C_\theta := \frac{C_A}{\cos\left(\frac{\phi+\theta}{2}\right)}$. Moreover for

any $\varkappa \in S_{\varphi_0}$, we have

$$\|(A_\omega + \varkappa I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_\theta}{1 + |\omega| + |\varkappa|}.$$

2): Let $\varpi \in \mathbb{R}$ and $\varkappa \in \mathbb{C}$ such that $\theta = |\varpi| \in (0, \varepsilon(\phi))$ and $\operatorname{Re}(\varkappa e^{-i\varpi/2}) > 0$. Then loperator $Q_\omega + \varkappa I$ is boundedly invertible and

$$\|(Q_\omega + \varkappa I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_{\varkappa, \varpi}}{\sqrt{1 + |\omega| + |\varkappa|}},$$

$$\text{with } C_{\varkappa, \varpi} := \frac{C_A}{\cos(\arg(\varkappa) - \varpi/2) \cos\left(\frac{\phi + \theta}{2}\right)}.$$

3): Let $\psi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\varepsilon(\phi)}{2}\right)$, then Q_ω is a linear operator of type ψ with bound $C_\psi := \frac{C_A}{\cos^2(\beta_\psi)}$ where

$$\beta_\psi = \frac{\pi}{4} + \frac{\psi - \varepsilon(\phi)}{2} \in \left(0, \frac{\pi}{2}\right).$$

Moreover for $\varkappa \in S_\psi$, we have

$$\|(Q_\omega + \varkappa I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_\varkappa}{\sqrt{1 + |\omega| + |\varkappa|}}.$$

Proof. 1): See [8], Lemma 2.4, p. 99.

2): The idea is to use the calculus given in [8], in Lemma2.4, at the end of p. 99:

$$\|(Q_\omega + \varkappa I)^{-1}\|_{\mathcal{L}(X)} = \left\| \frac{1}{\pi} \int_0^{+\infty} \frac{\sqrt{r} e^{i\varpi/2}}{r e^{i\varpi} + \varkappa^2} (A_\omega + r e^{i\varpi} I)^{-1} e^{i\varpi} dr \right\|_{\mathcal{L}(X)},$$

since

$$\|(A_\omega + r e^{i\varpi} I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C_\theta}{1 + r + |\omega|},$$

then

$$\begin{aligned} \|(Q_\omega + \varkappa I)^{-1}\|_{\mathcal{L}(X)} &\leq \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{\sqrt{r}}{|r e^{i\varpi} + \varkappa^2|} \frac{1}{1 + |\omega| + |\varkappa|} dr \\ &\leq \frac{C_\theta}{\pi} \int_0^{+\infty} \frac{\sqrt{r}}{\left| \cos\left(\frac{\arg(\varkappa^2) - \varpi}{2}\right) (r + |\varkappa|^2) \right|} \frac{1}{1 + |\omega| + |\varkappa|} dr \\ &\leq \frac{C_\theta}{\pi \cos(\arg(\varkappa) - \varpi/2)} \int_0^{+\infty} \frac{\sqrt{r}}{(r + |\varkappa|^2)(1 + |\omega| + r)} dr \\ &\leq \frac{C_A}{\pi \cos(\arg(\varkappa) - \varpi/2) \cos[(\phi + \theta)/2]} \frac{1}{\sqrt{1 + |\omega| + |\varkappa|}} \\ &\leq \frac{C_{\varkappa, \varpi}}{\sqrt{1 + |\omega| + |\varkappa|}}. \end{aligned}$$

3): See [8], Lemma 2.4, p. 100.

□

From the previous Lemma, we deduce that

$$(2.1) \quad \forall \varkappa \geq 0, \quad \left\| (A_\omega + \varkappa I)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{C_0}{1 + |\omega| + |\varkappa|},$$

and

$$(2.2) \quad \forall \varkappa \in S_{\pi/2}, \quad \left\| (Q_\omega + \varkappa I)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{C_1}{\sqrt{1 + |\omega| + |\varkappa|}},$$

where $C_0 := C_A / \cos(\phi/2)$ and $C_1 := C_A / \cos^2(\phi/2)$.

Lemma 2.2. For all $\omega \in S_{\varphi_0}$ there exists a constant $C_A > 0$ independent of such that:

- 1): $\|Q_\omega^{-1}\|_{\mathcal{L}(E)} \leq \frac{C_A}{\sqrt{1 + |\omega|}}$.
 2): For $z \in B\left(0, \frac{\sqrt{1 + |\omega|}}{2C_A}\right)$, we have $z \in \rho(Q_\omega)$ and

$$\left\| (Q_\omega - zI)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{2C_A}{\sqrt{1 + |\omega|}}.$$

Proof. For statement 1, it is enough to consider (2.2) with $\varkappa = 0$.

For statement 2, we consider $z \in B\left(0, \frac{\sqrt{1 + |\omega|}}{2C_A}\right)$. Then

$$\|zQ_\omega^{-1}\|_{\mathcal{L}(X)} = |z| \|Q_\omega^{-1}\|_{\mathcal{L}(X)} \leq \frac{\sqrt{1 + |\omega|}}{2C_A} \times \frac{C_A}{\sqrt{1 + |\omega|}} = \frac{1}{2} < 1,$$

hence $Q_\omega - zI = Q_\omega(I - zQ_\omega^{-1})$ is boundedly invertible with

$$\begin{aligned} \left\| (Q_\omega - zI)^{-1} \right\|_{\mathcal{L}(X)} &\leq \|Q_\omega^{-1}\|_{\mathcal{L}(X)} \left\| (I - zQ_\omega^{-1})^{-1} \right\|_{\mathcal{L}(X)} \\ &\leq \frac{2C_A}{\sqrt{1 + |\omega|}}. \end{aligned}$$

□

Lemma 2.3. Let $s > 0$ and $\omega \in S_{\varphi_0}$, we have

- 1): $I + e^{-sQ_\omega}$ is boundedly invertible and $(I + e^{-sQ_\omega})^{-1} \in \mathcal{L}(D(X))$.
 2): $\exists M > 0 : \forall s > 0$, we have

$$\left\| (I + e^{-sQ_\omega})^{-1} \right\|_{\mathcal{L}(X)} \leq M \text{ and } \left\| (I + e^{-sQ_\omega})^{-1} \right\|_{\mathcal{L}(D(Q))} \leq M.$$

3): We have

$$(I + e^{-sQ_\omega})^{-1} \in L(D(Q), X)_{\theta, p},$$

with $0 < \theta < 1$ and $p \geq 1$.

Proof. **1):** Since Q_ω is boundedly invertible there exist $M > 0$ and $\nu > 0$ such that for all $s > 0$

$$\|e^{-sQ_\omega}\|_{\mathcal{L}(E)} \leq Me^{-s\nu},$$

which implies that

$$\lim_{n \rightarrow +\infty} \sup \|e^{-snQ_\omega}\|^{1/n} \leq e^{-s\nu},$$

then $(I + e^{-sQ_\omega})^{-1}$ exists and belongs to $\mathcal{L}(X)$. It is easy to show that Q_ω^{-1} and e^{-sQ_ω} commute.

Then for all $\xi \in D(Q)$, we have

$$\begin{aligned} (I + e^{-sQ_\omega})^{-1} \xi &= (I + e^{-sQ_\omega})^{-1} Q_\omega^{-1} Q_\omega \xi \\ &= Q_\omega^{-1} (I + e^{-sQ_\omega})^{-1} Q_\omega \xi, \end{aligned}$$

and thus $(I + e^{-sQ_\omega})^{-1} \xi \in D(Q)$ and

$$\|(I + e^{-sQ_\omega})^{-1}\|_{\mathcal{L}(D(Q))} \leq \|Q_\omega^{-1}\| \|(I + e^{-sQ_\omega})^{-1}\|_{\mathcal{L}(X)}.$$

2): We have

$$(I + e^{-sQ_\omega})^{-1} = I + \frac{1}{2\pi i} \int_\gamma \frac{e^{sz}}{1 + e^{sz}} (-Q_\omega - z)^{-1} dz,$$

where γ is the boundary of S_{φ_0} oriented from $\infty e^{i(\frac{\pi}{2} + \varphi_0)}$ to $\infty e^{-i(\frac{\pi}{2} + \varphi_0)}$. It follows by analyticity that

$$\int_\gamma \frac{e^{sz}}{1 + e^{sz}} (-Q_\omega - z)^{-1} dz = \int_\gamma \frac{e^z}{1 + e^z} \left(-Q_\omega - \frac{z}{s}\right)^{-1} \frac{dz}{s},$$

from which we deduce

$$\begin{aligned} \left\| \int_\gamma \frac{e^z}{1 + e^z} \left(-Q_\omega - \frac{z}{s}\right)^{-1} \frac{dz}{s} \right\| &\leq \int_\gamma \left| \frac{e^z}{1 + e^z} \right| \left\| \left(-Q_\omega - \frac{z}{s}\right)^{-1} \right\| \frac{d|z|}{s} \\ &\leq M \int_\gamma \left| \frac{e^z}{1 + e^z} \right| \frac{1}{\left(\frac{z}{s}\right)} \frac{d|z|}{s} \\ &\leq M \int_\gamma \frac{e^{\operatorname{Re}(z)}}{|z|} d|z|. \end{aligned}$$

Then

$$\|(I + e^{-sQ_\omega})^{-1}\|_{\mathcal{L}(E)} \leq M.$$

3): This last result is obtained by interpolation. □

Lemma 2.4. Let $\omega \in S_{\varphi_0}$, we have $Q_\omega^2 H^{-1} \in \mathcal{L}(X)$; moreover there exists a constant $M > 0$ independent of ω such that:

$$(2.3) \quad \|AH^{-1}\|_{\mathcal{L}(X)} \leq M,$$

$$(2.4) \quad \|Q_\omega^2 H^{-1}\|_{\mathcal{L}(X)} \leq M(1 + |\omega|),$$

and

$$(2.5) \quad \|Q_\omega H^{-1}\|_{\mathcal{L}(X)} \leq M(\sqrt{1 + |\omega|}).$$

Proof. We have

$$\begin{aligned} \|Q_\omega^2 H^{-1}\|_{\mathcal{L}(X)} &= \|(-A + \omega I)H^{-1}\|_{\mathcal{L}(X)} \\ &\leq \|AH^{-1}\|_{\mathcal{L}(X)} + \|\omega H^{-1}\|_{\mathcal{L}(X)} \\ &\leq M(1 + |\omega|), \end{aligned}$$

and

$$\begin{aligned} \|Q_\omega H^{-1}\|_{\mathcal{L}(X)} &= \|Q_\omega^{-1} Q_\omega^2 H^{-1}\|_{\mathcal{L}(X)} \\ &\leq \|Q_\omega^{-1}\|_{\mathcal{L}(X)} \|Q_\omega^2 H^{-1}\|_{\mathcal{L}(X)} \\ &\leq M(\sqrt{1 + |\omega|}). \end{aligned}$$

□

Lemma 2.5. Assume that (1.3)~(1.5) hold. Then there exists $\rho_0 > 0$ and a constant $C > 0$ independent of $\omega \in S_{\varphi_0, \rho_0}$ such that the operator $Q_\omega \pm H_\mu$ is boundedly invertible, moreover

$$(2.6) \quad \|(Q_\omega \pm H)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{\sqrt{1 + |\omega|}}.$$

Proof. From (43), there exists $\rho_0 > 0$ such that for all $\omega \in S_{\varphi_0, \rho_0}$, we have

$$\|Q_\omega H^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{2},$$

Then from (1.5), that operator $Q_\omega \pm H$ is invertible and we have

$$\begin{aligned} \|(Q_\omega \pm H)^{-1}\|_{\mathcal{L}(X)} &= \|H^{-1}(Q_\omega H^{-1} \pm I)^{-1}\|_{\mathcal{L}(X)} \\ &\leq \frac{1}{1 - \|Q_\omega H^{-1}\|_{\mathcal{L}(X)}} \|H^{-1}\|_{\mathcal{L}(X)} \end{aligned}$$

$$\leq \frac{C}{\sqrt{1+|\omega|}}.$$

□

Now, for $\omega \in S_{\varphi_0, \rho_0}$, we define operator Λ_ω by:

$$\Lambda_\omega = Q_\omega - H + e^{2Q_\omega} (Q_\omega + H).$$

Note that, since $D(H) \subset D(Q^2)$, we have $D(\Lambda_\omega) = D(H)$ and furnish results on Λ_ω .

Lemma 2.6. (1) *There exist $r_0 > 0$ and $M > 0$ such that for all S_{φ_0, r_0} , we have*

$$(2.7) \quad \begin{cases} 0 \in \rho \left((I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} - I \right) \\ \left\| \left[(I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} - I \right]^{-1} \right\|_{\mathcal{L}(X)} \leq 2. \end{cases}$$

(2) *The operator Λ_ω is boundedly invertible and*

$$(2.8) \quad \|\Lambda_\omega^{-1}\|_{\mathcal{L}(X)} \leq M,$$

and

$$(2.9) \quad \|Q_\omega^2 \Lambda_\omega^{-1}\|_{\mathcal{L}(X)} \leq M(1 + |\omega|),$$

Note that $Q_\omega^2 \Lambda_\omega^{-1}$ has the same behavior as $Q_\omega H^{-1}$.

Proof. Let $\omega \in S_{\varphi_0, r_0}$, for some $r > 0$. From (1.5), we have $Q_\omega H^{-1} \in \mathcal{L}(X)$ so

$$(I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} - I \in \mathcal{L}(X),$$

and we have

$$\begin{aligned} & \left\| (I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} \right\|_{\mathcal{L}(X)} \\ & \leq M \|Q_\omega H^{-1}\|_{\mathcal{L}(X)} \\ & \leq M \left(\frac{1}{\sqrt{1+|\omega|}} + \frac{|\omega|}{\sqrt{1+|\omega|}} \right) \\ & \leq M \left(\frac{1}{|\omega|^{1/2}} + |\omega|^{1/2} \right) \leq \frac{2M}{r^{1.2}}. \end{aligned}$$

So there exists $r_0 > 0$ such that for all $\omega \in S_{\varphi_0, r_0}$, we have

$$\left\| (I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} \right\| \leq \frac{1}{2}.$$

We deduce that

$$L_\omega = (I - e^{2Q_\omega}) \left[(I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} - I \right] \in \mathcal{L}(X)$$

is boundedly invertible. Moreover

$$L_\omega^{-1} = \left[(I - e^{2Q_\omega})^{-1} (I + e^{2Q_\omega}) Q_\omega H^{-1} - I \right]^{-1} (I - e^{2Q_\omega})^{-1},$$

satisfies

$$\|L_\omega^{-1}\|_{\mathcal{L}(X)} \leq 2M.$$

Now, we write

$$\Lambda_\omega = (I + e^{2Q_\omega}) Q_\omega - (I - e^{2Q_\omega})^{-1} H = L_\omega H,$$

is boundedly invertible with

$$\Lambda_\omega^{-1} = H^{-1} L_\omega^{-1}.$$

Then

$$\|Q_\omega^2 \Lambda_\omega^{-1}\|_{\mathcal{L}(X)} = \|Q_\omega^2 H^{-1}\|_{\mathcal{L}(X)} \|L_\omega^{-1}\|_{\mathcal{L}(X)} \leq M(1 + |\omega|).$$

□

Lemma 2.7. Assume (1.3)~(1.6). Fix $\omega_1 \in S_{\varphi_0, r_0}$. Then there exists $C > 0$ such that for all $\omega \in S_{\varphi_0, r_0}$, we have

(1) $Q_\omega^2 (Q_\omega - H)^{-1} Q_\omega^{-1} \in \mathcal{L}(X)$, with

$$\left\| Q_\omega^2 (Q_\omega - H)^{-1} Q_\omega^{-1} \right\|_{\mathcal{L}(X)} \leq C.$$

(2) There exists $W_\omega \in \mathcal{L}(X)$ such that

$$\Lambda_\omega^{-1} = (Q_\omega - H)^{-1} (I - e^{2Q_\omega} W_\omega),$$

with

$$\|W_\omega\|_{\mathcal{L}(X)} \leq C \text{ and } \|Q_\omega^2 \Lambda_\omega^{-1} Q_\omega^{-1}\|_{\mathcal{L}(X)} \leq C.$$

Proof. Let $\omega \in S_{\varphi_0, r_0}$.

(1) Due to Lemma 2.4 and Lemma 2.5 and by using the same techniques as in [9], Lemma 5.3, we obtain that

$$\left\| Q_\omega^2 (Q_\omega - H)^{-1} Q_\omega^{-1} \right\|_{\mathcal{L}(X)} = \left\| Q_\omega^2 H^{-1} (Q_\omega H^{-1} - I)^{-1} Q_\omega^{-1} \right\|_{\mathcal{L}(X)} \leq C.$$

(2) For the second assertion, see [9], Lemma 6.4. Statement 3.

□

3. REPRESENTATION FORMULA OF THE SOLUTION

In this section, we assume that (1.3) is satisfied and that problem (1.1)-(1.2) has a strict solution. To find this solution, we will use the classical method of reducing the order of the equation. This method uses fractional powers of the operators and the Dunford (or Dunford-Riesz) method on the explicit representation of the solution, obtained through the assumptions and heuristic reasoning. The method of reducing the order is also called Kerin's S.G. reduction method [15]. The advantage of this method is that if problem (1.1)-(1.2) has a solution, then this solution is necessarily unique.

Proposition 3.1. *We assume (1.3)~(1.5) and we consider $\omega \in S_{\varphi_0, r_0}$, then if the problem (1.1)-(1.2) has a strict solution, that is to say $u \in C^2([0, 1], X) \cap C([0, 1], D(A))$ such that $u(0) \in D(H)$, and u check the problem (1.1)-(1.2), u is given by the following formula*

$$u(x) = u_s(x) + u_r(x), \quad x \in [0, 1],$$

with $u_s(x)$ a singular part of the solution u , and $u_r(x)$ a regular part defined for $x \in [0, 1]$, by

$$\begin{aligned} u_s(x) &= e^{xQ_\omega} \left[(Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0) \right] + I(x) \\ &\quad + e^{(1-x)Q_\omega} \left(d_1 + \frac{1}{2} Q_\omega^{-2} f(1) \right) + J(x) \text{ and} \\ u_r(x) &= R_1(x) + R_2(x) + R_3(x), \end{aligned}$$

Proof. Under (1.3)~(1.5), we consider $\omega \in S_{\varphi_0, r_0}$, if u is a strict solution of (1.1)-(1.2) then there exist $z_0, z_1 \in X$ such that

$$(3.1) \quad u(x) = e^{xQ_\omega} z_0 + e^{(1-x)Q_\omega} z_1 + I(x) + J(x), \quad x \in [0, 1],$$

Note here the framework is non-commutative, unlike [3], we take into account the commutativity between A and H . Now, taking into account the fact that $I - e^{2Q_\omega}$ is invertible, we set $T_\omega = (I - e^{2Q_\omega})^{-1} = T_\omega = I + e^{2Q_\omega} T_\omega$ and $S(x) = T_\omega (e^{xQ_\omega} - e^{(1-x)Q_\omega} e^{Q_\omega}) \in \mathcal{L}(X)$, $x \in [0, 1]$, then there exists μ_0 and μ_1 in X such as formula (3.1) writes in the following form

$$(3.2) \quad u(x) = S(x) \mu_0 + S(1-x) \mu_1 + I(x) + J(x), \quad x \in [0, 1],$$

where

$$(3.3) \quad \begin{cases} I(x) = \frac{1}{2} Q_\omega^{-1} \int_0^x e^{(x-s)Q_\omega} f(s) ds, \\ J(x) = \frac{1}{2} Q_\omega^{-1} \int_x^1 e^{(s-x)Q_\omega} f(s) ds, \\ S(x) = T_\omega (e^{xQ_\omega} - e^{(1-x)Q_\omega} e^{Q_\omega}). \end{cases}$$

Now to obtain the final representation of u , it is enough to find μ_0 and μ_1 , and for this we use the boundary conditions of the problem (1.1)-(1.2), we have

$$\begin{cases} u(0) = \mu_0 + J(0), u'(0) = Q_\omega T_\omega (I + e^{2Q_\omega}) \mu_0 - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ Hu(0) = T_\omega Q_\omega (I + e^{2Q_\omega}) \Lambda_\omega^{-1} \beta_0 - T_\omega \beta_0, \end{cases}$$

so

$$\begin{aligned} u'(0) - Hu(0) &= T_\omega \beta_0 - Q_\omega T_\omega (I + e^{2Q_\omega}) J(0) - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ &= d_0 + 2T_\omega Q_\omega e^{Q_\omega} \mu_1 + 2T_\omega Q_\omega J(0) - Q_\omega T_\omega (I + e^{2Q_\omega}) J(0) \\ &\quad - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ &= d_0. \end{aligned}$$

and

$$u(1) = \mu_1 + I(1) = d_1,$$

hence

$$(3.4) \quad \begin{cases} \mu_0 = \Lambda_\omega^{-1} \beta_0 - J(0) \text{ with} \\ \beta_0 = (I - e^{2Q_\omega}) d_0 + 2Q_\omega e_1^{Q_\omega} + 2Q_\omega J(0) \\ \mu_1 = d_1 - I(1). \end{cases}$$

Finally, if the problem (1.1)-(1.2) admits a strict solution u , then this solution can be defined by (3.2), (3.3) and (3.4). \square

Remark 3.2. To apply the hölderianity assumption on f , we need to change the previous representation. Noting that they exist α_0, α_1 , and $\alpha_2 \in X$ (obviously calculable) such that

$$\begin{aligned} u(x) &= (I + e^{2Q_\omega} T_\omega) \left(e^{xQ_\omega} - e^{(1-x)Q_\omega} e^{Q_\omega} \right) \mu_0 \\ &\quad + (I + e^{2Q_\omega} T_\omega) \left(e^{(1-x)Q_\omega} - e^{xQ_\omega} e^{Q_\omega} \right) (d_1 - I(1)) + I(x) + J(x) \\ &= e^{xQ_\omega} \mu_0 + e^{(1-x)Q_\omega} (d_1 - I(1)) + I(x) + J(x) + e^{xQ_\omega} e^{Q_\omega} \alpha_1 + e^{(1-x)Q_\omega} e^{Q_\omega} \alpha_2 \\ &= e^{xQ_\omega} (\Lambda_\omega^{-1} [d_0 + 2Q_\omega J(0)] - J(0)) + I(x) \\ &\quad + e^{(1-x)Q_\omega} (d_1 - I(1)) + J(x) \\ &\quad + e^{xQ_\omega} \Lambda_\omega^{-1} e^{Q_\omega} \alpha_0 + e^{xQ_\omega} e^{Q_\omega} \alpha_1 + e^{(1-x)Q_\omega} e^{Q_\omega} \alpha_2. \end{aligned}$$

Thus $\Lambda_\omega^{-1} = (Q_\omega - H)^{-1} - (Q_\omega - H)^{-1} e^{2Q_\omega} W_\omega$, so there exist ξ_1, ξ_2, ξ_3 in X (clearly computable) such that

$$(3.5) \quad u(x) = e^{xQ_\omega} \left[(Q_\omega - H)^{-1} (d_0 + 2Q_\omega J(0)) - J(0) \right] + I(x)$$

$$\begin{aligned}
& +e^{(1-x)Q_\omega} (d_1 - I(1)) + J(x) \\
& +e^{xQ_\omega} (Q_\omega - H)^{-1} e^{Q_\omega} \xi_1 \\
& +e^{xQ_\omega} e^{Q_\omega} \xi_2 + e^{(1-x)Q_\omega} e^{Q_\omega} \xi_3.
\end{aligned}$$

Now, we write

$$\begin{aligned}
J(0) &= \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{sQ_\omega} (f(s) - f(0)) ds + \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(0) ds \\
&= \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{sQ_\omega} (f(s) - f(0)) ds + \frac{1}{2}Q_\omega^{-2} e^{Q_\omega} f(0) - \frac{1}{2}Q_\omega^{-2} f(0),
\end{aligned}$$

and

$$\begin{aligned}
I(1) &= \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} (f(s) - f(1)) ds + \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(1) ds \\
&= \frac{1}{2}Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} (f(s) - f(1)) ds - \frac{1}{2}Q_\omega^{-2} f(1) + \frac{1}{2}Q_\omega^{-2} e^{Q_\omega} f(1),
\end{aligned}$$

so from (3.5), we can define $\eta_1, \eta_2, \eta_3 \in X$ such that

$$\begin{aligned}
(3.6) \quad u(x) &= e^{xQ_\omega} \left[(Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2}Q_\omega^{-2} f(0) \right] + I(x) \\
&+ e^{(1-x)Q_\omega} \left(d_1 + \frac{1}{2}Q_\omega^{-2} f(1) \right) + J(x) \\
&+ R_1(x) + R_2(x) + R_3(x),
\end{aligned}$$

with

$$(3.7) \quad \begin{cases} R_1(x) = e^{xQ_\omega} (Q_\omega - H)^{-1} \left(e^{Q_\omega} \eta_1 + \int_0^1 e^{sQ_\omega} (f(s) - f(0)) ds \right) \\ R_2(x) = -\frac{1}{2}e^{xQ_\omega} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} (f(s) - f(0)) ds - \frac{1}{2}e^{(1-x)Q_\omega} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} (f(s) - f(1)) ds \\ R_3(x) = e^{xQ_\omega} e^{Q_\omega} \eta_2 + e^{(1-x)Q_\omega} e^{Q_\omega} \eta_3. \end{cases}$$

The study above shows that if problem (1.1)-(1.2) has a strict solution, then it is uniquely determined by (3.6).

Remark 3.3. The behavior of u near 0 is given by the first term

$$e^{xQ_\omega} \left[(Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2}Q_\omega^{-2} f(0) \right] + I(x),$$

in (3.6), the behavior of u near one 1 is covered by the second term,

$$e^{(1-x)Q_\omega} \left(d_1 + \frac{1}{2}Q_\omega^{-2} f(1) \right) + J(x),$$

and while terms $R_1(x)$, $R_2(x)$, and $R_3(x)$ are regular (see Propositions 4.5 and 4.6 below). Note that we did not need a commutativity hypothesis between H and Q_ω in the above calculus.

- According to the commutativity assumption made in ([4]), which provides the term $R_1(x)$ is regular. We are unable to employ in this study; instead, we use (1.6) and (1.7) to determine the regularity of $R_1(x)$.

4. REGULARITY RESULTS

In order to analyze the terms of the solution, we begin this part by proving a few technical lemmas.

Proposition 4.1. (1) Let $\varphi \in X$ and $\theta \in]0, 1[$. Then $e^{xQ}\varphi \in C([0, 1]; X)$ if and only if $\varphi \in \overline{D(Q)}$.

(2) Let $\theta \in]0, 1[$, $f \in C^\theta([0, 1]; X)$ and $\varphi \in X$. Set

$$\gamma(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q} f(s) ds.$$

Then $\gamma \in C^1([0, 1]; X) \cap C([0, 1]; D(Q))$ if and only if $\varphi \in D(Q)$ and $f(0) + Q\varphi \in \overline{D(Q)}$.

Considering the well known real interpolation space

$$(D(Q), X)_{1-\theta, \infty} = (X, D(Q))_{\theta, \infty} = D_Q(\theta; +\infty),$$

(see H. Triebel see [21] p. 25 and 76), we have also:

Proposition 4.2. (1) Let $\psi \in X$ and $\theta \in]0, 1[$. Then $e^{xQ}\psi \in C^\theta([0, 1]; X)$ if and only if $\psi \in D_Q(\theta; +\infty)$.

(2) Let $g \in C^\theta([0, 1]; X)$. Set

$$\delta(x) = \int_0^x e^{(x-s)Q} (g(s) - g(0)) ds.$$

Then $\delta \in C^{1,\theta}([0, 1]; X) \cap C^\theta([0, 1]; D(Q))$.

(3) Let $h \in C([0, 1]; X)$ and $\psi \in X$. Set

$$w(x) = e^{xQ}\psi + \int_0^x e^{(x-s)Q} h(s) ds.$$

Then $w \in C^{1,\theta}([0, 1]; X) \cap C^\theta([0, 1]; D(Q))$ if and only if $h \in C^\theta([0, 1]; X)$, $\psi \in D(Q)$ and $h(0) + Q\psi \in D_Q(\theta; +\infty)$.

(4) Let $f \in C^\theta([0, 1]; X)$. Then

$$Q \int_0^1 e^{sQ} (f(s) - f(0)) ds \in (D(Q), X)_{1-\theta, \infty}.$$

Proof. Statement 2 is obtained by applying the Da Prato-Grisvard sum theory [7]. Statement 3 which improves Statement 2 is due to E. Sinestrari [19], see also G. Da Prato [6]. We can find in D. Guidetti [11] a simple proof of these results (see Corollary 2.1 and Theorem 2.4, p.136). \square

Notation 4.3. Let φ and h be two given X -valued functions defined on $[0, 1]$ and $\theta \in]0, 1[$. We write

$$\varphi \simeq_{\theta} h,$$

if

$$\varphi - h \in C^{\theta}([0, 1]; X).$$

As a consequence of Proposition 4.2 we get (see [10] Proposition 8, p. 976):

Proposition 4.4. Let $g \in C^{\theta}([0, 1]; X)$ with $\theta \in]0, 1[$ and $\varphi \in D(Q)$. Set

$$\gamma(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q}g(s) ds.$$

Then

$$Q\gamma(\cdot) \simeq_{\theta} e^{\cdot Q}(Q\varphi + g(0)).$$

Proof. It is an easy consequence of Proposition 4.2 and Proposition 1.2, statement (ii) in [19], that

$$\begin{aligned} \gamma(x) &= Qe^{xQ}\varphi + Q \int_0^x e^{(x-s)Q}(g(s) - g(0)) ds + Q \int_0^x e^{(x-s)Q}g(0) ds \\ &= Qe^{xQ}\varphi + Q \int_0^x e^{(x-s)Q}(g(s) - g(0)) ds - [g(0) - e^{xQ}g(0)] \\ &= e^{xQ}[Q\varphi + g(0)] + Q \int_0^x e^{(x-s)Q}(g(s) - g(0)) ds - g(0). \end{aligned}$$

□

Proposition 4.5. Assume (1.3) \sim (1.7). Let $f \in C^{\theta}([0; 1]; X)$, with $0 < \theta < 1$ and $d_0, d_1 \in X$. Then for $\omega \in S_{\varphi_0, r_0}$:

- (1) $Q_{\omega}^2 R_3 \in C^{\infty}([0; 1]; X)$,
- (2) $R_1 \in C^2([0; 1]; X)$ and $Q_{\omega}^2 R_1 \in C([0; 1]; X)$,
- (3) $R_2 \in C^2([0; 1]; X)$ and $Q_{\omega}^2 R_2 \in C([0; 1]; X)$,
- (4) $Q^2 R_1, Q^2 R_2, Q^2 R_3 \in C([0; 1]; X)$.

Proof. see [13], Proposition 5, p 15. □

Proposition 4.6. Assume (1.3) \sim (1.8). Let $f \in C^{\theta}([0; 1]; X)$, with $0 < \theta < 1$, and $d_0, d_1 \in X$. Then for $\omega \in S_{\varphi_0, r_0}$:

- (1) $Q_{\omega}^2 R_3 \in C^{\infty}([0, 1]; X)$,
- (2) $R_1 \in C^2([0, 1]; X)$ and $Q_{\omega}^2 R_1 \in C^{\theta}([0, 1]; X)$,
- (3) $R_2 \in C^2([0, 1]; X)$ and $Q_{\omega}^2 R_2 \in C^{\theta}([0, 1]; X)$,
- (4) $Q^2 R_1, Q^2 R_2, Q^2 R_3 \in C^{\theta}([0, 1]; X)$.

Proof. see [13], Proposition 8, p 19. □

5. MAIN RESULT

Now, based on this study of the regularity of the solution, we are ready to provide our main results concerning the existence and uniqueness of the regular solution of the problem (1.1)-(1.2).

Theorem 5.1. Assume (1.3)~(1.7). Let $d_0 \in X$, $d_1 \in D(A)$ and $f \in C^\theta([0, 1]; X)$, with $0 < \theta < 1$. Then, for any $\omega \in S_{\varphi_0, r_0}$: Problem (1.1)-(1.2). admits a unique strict solution u if and only if

$$(5.1) \quad \begin{cases} (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] \in D(A) \\ A_\omega (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + f(0) \in \overline{D(A)} \\ A_\omega d_1 + f(1) \in \overline{D(A)}. \end{cases}$$

Proof. We first suppose that (5.1) is satisfied and we have to prove that u given by (3.6) is the strict solution of problem (1.1)-(1.2). But to prove that

$$u \in C^2([0, 1], X) \cap C([0, 1], D(A)),$$

it is enough to show that $A_\omega u \in C([0, 1], X)$. Using the Proposition 4.5, we have

$$A_\omega R_i \in C([0, 1], X), \quad i = 1, 2 \text{ and } A_\omega R_3 \in C^\infty([0, 1], X),$$

so we deduce

$$A_\omega v = A_\omega u - A_\omega R_1 - A_\omega R_2 - A_\omega R_3 \in C([0, 1], X),$$

where

$$\begin{aligned} v(x) &= e^{xQ_\omega} \left[(Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0) \right] + I(x) \\ &\quad + e^{(1-x)Q_\omega} \left(d_1 + \frac{1}{2} Q_\omega^{-2} f(1) \right) + J(x). \end{aligned}$$

Then it is now known from Proposition 4.1, statement 1), that this term is continuous on $[0, 1]$ if and only if

$$\begin{cases} A_\omega (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + f(0) \in \overline{D(A)} \\ A_\omega d_1 + f(1) \in \overline{D(A)}. \end{cases}$$

Conversely, we suppose that (5.1) is satisfied and show that the problem (1.1)-(1.2) has a strict solution u . From Proposition 4.1, we have

$$\begin{cases} e^{xQ_\omega} \left[A_\omega (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + f(0) \right] \in C([0, 1], X) \\ e^{(1-x)Q_\omega} [A_\omega d_1 + f(1)] \in C([0, 1], X). \end{cases}$$

Which implies that $A_\omega v$ in $C([0, 1], X)$. So

$$A_\omega u \in C([0, 1], X),$$

then we deduce

$$u \in C([0, 1], D(A)).$$

Since $f \in C^\theta([0, 1]; X)$, so

$$u'' = f - A_\omega u \in C([0, 1], X),$$

this means

$$u \in C^2([0, 1], X) \cap C([0, 1], D(A)).$$

Moreover, we have according to (3.4) $u(1) = \mu_1 + I(1) = d_1$ and taking into account the fact that

$$\begin{cases} u(0) = \mu_0 + J(0) = \Lambda_\omega^{-1} \beta_0 \in D(H) \\ Hu(0) = T_\omega Q_\omega (I + e^{2Q_\omega}) \Lambda_\omega^{-1} \beta_0 - T_\omega \beta_0, \end{cases}$$

where

$$\begin{cases} \mu_0 = \Lambda_\omega^{-1} \beta_0 - J(0) \\ \beta_0 = (I - e^{2Q_\omega}) d_0 + 2Q_\omega e^{Q_\omega} \mu_1 + 2Q_\omega J(0) \in X. \end{cases}$$

And also, we have

$$\begin{aligned} u'(0) &= Q_\omega T_\omega (I + e^{2Q_\omega}) \mu_0 - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ &= Q_\omega T_\omega (I + e^{2Q_\omega}) \Lambda_\omega^{-1} \beta_0 - Q_\omega T_\omega (I + e^{2Q_\omega}) J(0) - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0), \end{aligned}$$

so

$$\begin{aligned} u'(0) - Hu(0) &= T_\omega \beta_0 - Q_\omega T_\omega (I + e^{2Q_\omega}) J(0) - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ &= d_0 + 2T_\omega Q_\omega e^{Q_\omega} \mu_1 + 2T_\omega Q_\omega J(0) - Q_\omega T_\omega (I + e^{2Q_\omega}) J(0) \\ &\quad - 2Q_\omega T_\omega e^{Q_\omega} \mu_1 - Q_\omega J(0) \\ &= d_0. \end{aligned}$$

Finally (5.1) is satisfied. □

Theorem 5.2. Assume (1.3)~(1.8). Let $d_0 \in X$, $d_1 \in D(A)$ and $f \in C^\theta([0, 1]; X)$, with $0 < \theta < 1$. Then, for any $\omega \in S_{\varphi_0, r_0}$ Problem (1.1)-(1.2) has a unique strict solution u satisfying the maximal regularity property u'' , $A_\omega u \in C^\theta([0, 1]; X)$, if and only if

$$(5.2) \quad \begin{cases} (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] \in D(A) \\ A_\omega (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + f(0) \in D_Q(\theta, +\infty) \\ A_\omega d_1 + f(1) \in D_Q(\theta, +\infty). \end{cases}$$

Proof. Let $f \in C^\theta([0, 1]; X)$, with $0 < \theta < 1$, $d_0 \in X$, $d_1 \in D(A)$ and we consider $\omega \in S_{\varphi_0, r_0}$. Suppose there exists a unique strict solution u of the problem (1.1)–(1.2) defined by (3.2), (3.3) and (3.4),

which satisfied the maximum regularity property u'' , $A_\omega u \in C^\theta([0, 1]; X)$, this solution u must satisfy (1.1)–(1.2), moreover, according to the lemma 4.5,

$$u \in C^2([0, 1], X) \cap C([0, 1], D(Q^2)),$$

and u'' , $A_\omega u \in C^\theta([0, 1]; X)$, if and only if

$$\tilde{u} = u - R_1 - R_2 - R_3 \in C^2([0, 1], X) \cap C([0, 1], D(Q^2)) \text{ and } \tilde{u}'', A\tilde{u} \in C^\theta([0, 1]; X),$$

with

$$\begin{aligned} \tilde{u}(x) = & u(x) - R_1(x) - R_2(x) - R_3(x) = e^{xQ_\omega} \left[(Q_\omega - H_\mu)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0) \right] \\ & + I(x) + e^{(1-x)Q_\omega} \left(d_1 + \frac{1}{2} Q_\omega^{-2} f(1) \right) + J(x). \end{aligned}$$

Now, using Lemma 4.5 (second part) $\tilde{u} \in C^2([0, 1], X) \cap C([0, 1], D(A_\omega))$, and $\tilde{u}'', A_\omega \tilde{u} \in C^\theta([0, 1]; X)$ if and only if

$$\begin{cases} (Q_\omega - H_\mu)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0) \in D(Q_\omega^2) \\ \frac{1}{2} f(0) + Q_\omega^2 \left[(Q_\omega - H_\mu)^{-1} [d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0) \right] \in D_Q(\theta; +\infty) \\ d_1 + \frac{1}{2} Q_\omega^{-2} f(1) \in D(Q_\omega^2), \\ \frac{1}{2} f(1) + Q_\omega^2 [d_1 + \frac{1}{2} Q_\omega^{-2} f(1)] \in D_Q(\theta; +\infty). \end{cases}$$

After a calculation, as above, we find

$$\begin{cases} (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] \in D(A_\omega), \\ Q_\omega^2 (Q_\omega - H)^{-1} [d_0 - Q_\omega^{-1} f(0)] + f(0) \in D_Q(\theta; +\infty), \\ Q_\omega^2 d_1 + f(1) \in D_Q(\theta; +\infty). \end{cases}$$

□

6. APPLICATION

We present below the concrete problem of the heat equation in the square domain $\Omega = (0, 1) \times (0, 1)$ with a dynamic Wentzell condition on one of its lateral boundaries, in order to illustrate the results obtained in this work.

$$(P) \begin{cases} \frac{\partial u}{\partial t}(t, x, y) = \Delta_{x,y} u(t, x, y), & (t, x, y) \in (0, +\infty) \times \Omega \\ \frac{\partial u}{\partial t}(t, 0, y) = \frac{\partial u}{\partial x}(t, 0, y) + \frac{\partial^2 u}{\partial y^2}(t, 0, y), & (t, 0, y) \in (0, +\infty) \times \Gamma_0 \\ u(t, 1, y) = 0, & (t, 1, y) \in (0, +\infty) \times \Gamma_1 \\ u(t, x, 0) = u(t, x, 1) = 0, & x \in (0, 1), \\ u(0, x, y) = u_0(x, y), & (x, y) \in (0, 1) \times (0, 1), \end{cases}$$

where

$$\begin{cases} \Gamma_0 = \{0\} \times (0, 1), \Gamma_1 = \{1\} \times (0, 1) \\ \gamma_0 = (0, 1) \times \{0\}, \gamma_1 = (0, 1) \times \{1\}, \end{cases}$$

Here $\frac{\partial^2}{\partial y^2}$ is the Laplace-Beltrami operator on Γ_0 . Physically, $\frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial x}$ represent the interaction between the lateral boundaries while and the domain Ω , while $\frac{\partial^2 u}{\partial y^2}$ is the boundary diffusion. Set $E = L^p(\Omega) \times L^p(\Gamma_0)$; this Banach space is well defined and endowed with its natural norm. Define operator K by

$$\begin{cases} D(K) = \{w = (u, v_0) : \Delta u \in L^p(\Omega), v_0 \in W^{2,p}(\Gamma_0), u|_{\Gamma_0} = v_0, \\ (\Delta u)|_{\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2} \text{ and } u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0\} \\ Kw = \left(\Delta u, \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2}\right), \text{ for } w = (u, v_0) \in D(K). \end{cases}$$

The boundary conditions are defined in $L^p(\Gamma_0)$ and $Kw \in E$. On the other hand it is not difficult to see that this operator is closed in E . When we integrate the time variable t , the following Cauchy problem.

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(u, v_0) = \left(\frac{\partial u}{\partial t}, \frac{\partial v_0}{\partial t}\right) = Kw = K(u, v_0) \\ w(0) = (u(0, \cdot), v_0(0, \cdot)) \text{ given,} \end{cases}$$

writes

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ \frac{\partial v_0}{\partial t} = \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \frac{\partial^2 v_0}{\partial y^2} \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0 \\ w(0) = (u(0, \cdot), v_0(0, \cdot)) \text{ given,} \end{cases}$$

since $(u, v_0) \in D(K)$ and $\frac{\partial u}{\partial t} = \Delta u$, we obtain

$$(\Delta u)|_{\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \left(\frac{\partial^2 v_0}{\partial y^2}\right)|_{\Gamma_0} = \left(\frac{\partial u}{\partial t}\right)|_{\Gamma_0},$$

and since $u|_{\Gamma_0} = v_0$, by using the tangential derivative, we obtain

$$\left(\frac{\partial^2 v_0}{\partial y^2}\right)|_{\Gamma_0} = \left(\frac{\partial^2 u}{\partial y^2}\right)|_{\Gamma_0},$$

summarizing up, we deduce the same equation as in Problem (P):

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ \left(\frac{\partial u}{\partial t}\right)|_{\Gamma_0} = \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2}\right)|_{\Gamma_0} \\ u(0, \cdot) \text{ is given} \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0. \end{cases}$$

The study of the evolution equation above is based on the study of the following spectral equation

$$(6.1) \quad \begin{cases} K(u, v_0) = (h, d_0) \\ (u, v_0) \in D(K), \end{cases}$$

and since $h, d_0 \in E$, using equality $u|_{\Gamma_0} = v_0$, (6.1) is equivalent to

$$(6.2) \quad \begin{cases} \Delta u - \omega u = h \\ \left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2}\right)|_{\Gamma_0} = d_0 \\ u|_{\gamma_0 \cup \gamma_1 \cup \Gamma_1} = 0, \end{cases}$$

which is an elliptic partial differential equation with the same spectral parameter in the equation and in the boundary condition on Γ_0 . We will write (6.2) in an operational differential form. We consider the Banach space $X = L^p(0, 1)$ and identify E with $L^p(0, 1; X)$, by writing as usual, for $g \in E$,

$$g(x, y) = (g(x))(y), \quad x, y \in (0, 1).$$

We define operator A on X by

$$(6.3) \quad \begin{cases} D(A) = \{\varphi \in W^{2,p}(0, 1) : \varphi(0) = \varphi(1) = 0\} \\ A\varphi(y) = \varphi''(y), \end{cases}$$

then the square root of the opposite of this operator is well-defined and we have

$$W_0^{1,p}(0, 1) \subset D((-A)^{1/2}) \subset W^{1,p}(0, 1) \quad \text{et} \quad \|(-A)^{1/2} \varphi\| \approx \|\varphi'\|_X + \|\varphi\|_X,$$

see [1]. We also know that $Q = -\sqrt{-A}$ generates an analytical semigroup in X , on the other hand $Q_\omega = -\sqrt{-A + \omega I}$ is well-defined and generates an analytical semigroup in X for all $\omega \in S_{\varphi_0}$. We now define the operator H on X by $H := -A$. then, on X , the equation

$$\Delta u(x, y) - \omega u(x, y) = h(x, y),$$

can be written in the following form

$$u''(x) + Au(x) - \omega u(x) = h(x), \quad x \in (0, 1),$$

while the boundary condition

$$\left(\frac{\partial u}{\partial x}\right)|_{\Gamma_0} + \left(\frac{\partial^2 u}{\partial y^2}\right)|_{\Gamma_0} = d_0,$$

writes as

$$u'(0) - Hu(0) = d_0,$$

the condition $u|_{\gamma_0 \cup \gamma_1} = 0$ (which means that $u(0, y)$ and $u(1, y)$ vanish in $y = 0$ and $y = 1$) is implicitly included in the fact that $u(0) := u(0, \cdot)$ and $u(1) := u(1, \cdot)$ are in $D(H)$. Therefore (6.2) or

equivalently (6.1), write in the following abstract form

$$(P.A) \begin{cases} u''(x) + Au(x) - \omega u(x) = h(x), & x \in (0, 1) \\ u'(0) - Hu(0) = d_0, \\ u(1) = d_1, \end{cases}$$

where $(h, d_0) \in E = L^p(0, 1; X) \times L^p(X)$. Let u be the classical solution of (P.A), then $u \in W^{2,p}(0, 1; X) \cap L^p(0, 1; D(A))$ and $(u, u(0)) \in D(K)$, so that $(u, u(0)) = (K - \omega I)^{-1}(h, d_0)$. Taking into account the fact that, here, we can take $\varphi_0 = \pi - \varepsilon$ ($\varepsilon > 0$ as close to 0 as we want), see [9], Proposition 8.2 and Theorem 8.3, to obtain:

$$\exists C > 0 : \forall \omega \in S_{\varphi_0}, \forall (h, d_0) \in E, \left\| (K - \omega I)^{-1}(h, d_0) \right\|_{\varepsilon} \leq \frac{C}{1 + |\omega|} \|(h, d_0)\|_{\varepsilon}$$

and deduce that our operator K defined above generates an analytic semigroup in E . This example can be extended to the following problem

$$\begin{cases} \Delta u - \omega u = h \\ a_0 \left(\frac{\partial u}{\partial x} \right) |_{\Gamma_0} + b_0 \frac{\partial^2 v_0}{\partial y^2} - \omega v_0 = d_0 \\ a_1 \left(\frac{\partial u}{\partial x} \right) |_{\Gamma_1} + b_1 \frac{\partial^2 v_0}{\partial y^2} - \omega v_1 = d_1 \\ u|_{\gamma_0 \cup \gamma_1} = 0, \end{cases}$$

The following conditions must now be verified d_1 and d_0 .

- (1) The condition $d_1 \in D(Q^2)$.

This relates to the variable y , translates to

$$d_1 \in W^{2,p}(0, 1) : d_1(0) = d_1(1) = 0.$$

- (2) The condition $Q^2 d_1 + f(1) \in \overline{D(Q)} = L^p(0, 1)$. this means that

$$y \mapsto d_1(y) + f(1, y) \in L^p(0, 1).$$

is checked automatically because $d_1 \in L^p(0, 1)$ and

$$f \in C^{\theta}([0, 1]; L^p(0, 1)), \text{ with } 0 < \theta < 1,$$

which implies $y \mapsto f(1, y) = f(1)(y) \in L^p(0, 1)$.

- (3) It is clear that the condition

$$Q(Q - H)^{-1}(d_0 - Q^{-1}f(0)) \in D(Q).$$

(4) The condition $Q^2(Q - H)^{-1}(d_0 - Q^{-1}f(0)) + f(0) \in \overline{D(Q)} = L^p(0, 1)$. he becomes $d_0 \in D(Q)$. Finally, we applied the theorem 5.1 for the existence and uniqueness of the strict solution to the concrete problem (P) . We do the same for maximum regularity. Therefore, Theorem 5.1 for the existence and the unicity of the strict solution applies.

We do the same for the maximal regularity.

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