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FIXED POINT APPROACHES TO C.A FUNCTIONAL EQUATION STABILITY IN N.A SETTINGS

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Abstract. This paper investigates the stability of the Cauchy additive functional equation

$$g\left(\frac{\omega-\wp}{f}+z\right)+\left(\frac{\wp-z}{f}+\omega\right)+g\left(\frac{z-\omega}{f}+\wp\right)=g\left(\omega+\wp+z\right)$$

For any $\omega,\wp,z\in\mathcal{E}$, in non-Archimedean normed spaces. The analysis is conducted using the fixed point approach within a suitable generalized metric space. By defining an appropriate operator associated with the given functional equation, sufficient conditions ensuring its contractive behavior are established. As a consequence, the existence and uniqueness of an additive mapping close to an approximate solution are obtained. The results show that every function satisfying the functional equation approximately admits a unique exact additive mapping in the considered setting. The proposed approach provides a clear and direct framework for studying functional equation stability in non-Archimedean environments.

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1. Introduction

The stability theory of functional equation studies the following fundamental question: if a mapping approximately satisfies a given functional equation, under what conditions does there exist an exact solution close to it. This problem was first formulated by Ulam in 1940 in the context of approximate homomorphism [13]. A partial solution was provided by Hyers in 1941 for the Cauchy additive functional equation in Banach spaces, establishing what is now known as Hyers-Ulam stability [7]. Later, Aoki extended Hyers' result by allowing unbounded perturbations for additive mappings [1]. A significant generalization was introduced by Rassias, who considered stability controlled by a power-type function, leading to what is commonly referred to as Hyers–Ulam–Rassias stability [12]. Găvruţa further unified several stability results by introducing a general control function [6], which

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has been widely used in subsequent studies. In addition to direct methods based on inequalities, fixed point theorem has become an effective tool for investigating stability problems of functional equation. A foundational result in this direction is the fixed point alternative for contractions in generalized metric space due to Diaz and Margolis [5]. This approach was systematically applied to functional equation stability by Radu, who demonstrated that stability results can be obtained by associating a suitable operator with the given functional equation and studying its fixed point [11]. Further developments and refinements of this method were presented by Cădariu and Radu, where generalized metric space and contractive-type conditions were employed to derive existence and uniqueness of stable solutions [2–4]. In non-Archimedean analysis, the study of stability requires special consideration due to the strong triangle inequality satisfied by the norm, which leads to different convergence and continuity properties compared with the classical Archimedean case. Fundamental aspects of non-Archimedean and ultrametric analysis can be found in standard references such as Khrennikov [8] and Schikhof [13]. Within this framework, several authors have investigated the stability of functional equation under ultrametric norms. In particular, Moslehian and Rassias studied the stability of functional equation in non-Archimedean normed spaces and established Hyers-Ulam type results adapted to the ultrametric setting [10]. Motivated by these contributions, fixed point approaches provide a structured and transparent framework for studying the stability of the Cauchy additive functional equation in non-Archimedean settings. By constructing an appropriate operator whose fixed point correspond to exact additive mappings and applying a suitable fixed point theorem in a generalized metric space, one can obtain the existence and uniqueness of an exact solution near a given approximate one. This methodology yields explicit stability bounds and is well suited to the ultrametric structure of non-Archimedean normed spaces. This paper is built on several results presented in [9,12,15].

Let $g : \mathcal{E} \to \mathcal{X}$ be a given mapping, where $\mathcal{E}$ is assumed to be a semigroup and $\mathcal{X}$ is a complete $(NANS)$. Let

$$g\left(\frac{\omega-\wp}{f}+\tau\right)+\left(\frac{\wp-\tau}{f}+\omega\right)+g\left(\frac{\tau-\omega}{f}+\wp\right)=g\left(\omega+\wp+\tau\right) \tag{1}$$

For any $\omega, \wp, \tau \in \mathcal{E}$.

2. Preliminaries

This part contains the notations, definitions and, preliminary results required for the development of the main theorems. Let $\mathbb{K}$ be an (NA) field equipped with a mean valuation $\lceil\cdot\rceil : \mathbb{K} \to [0, \infty)$ satisfying the following conditions for any $\omega, \wp \in \mathbb{K}$

$$\lceil\omega\rceil = 0 \leftrightarrow \omega = 0, \lceil\omega\wp\rceil = \lceil\omega\rceil\lceil\wp\rceil,\ and\ \lceil\omega+\wp\rceil \le \max\lceil\omega\rceil, \lceil\wp\rceil$$

For example $\lceil 1\rceil = 1$, $\lceil -1\rceil = 1$ and $\lceil f\rceil \le 1\ \forall f \in \mathbb{N}$.

Definition 2.1. [8] *We take $\mathcal{E}$ is a vector space over a field $\mathbb{F}$. A mapping $[\![\cdot]\!] : \mathcal{E} \longrightarrow \mathbb{R}$ satisfying:*

(1) $[\![\omega]\!] = 0 \iff \omega = 0$;

(2) $[\![\mathfrak{r}\omega]\!] = \lceil\mathfrak{r}\rceil[\![\omega]\!]$;

(3) $[\![\omega + \wp]\!] \le \max [\![\omega]\!], [\![\wp]\!]$. *for any $\mathfrak{r} \in \mathbb{F}$ and $\omega, \wp \in \mathcal{E}$ Then $(\mathcal{E}, [\![\cdot]\!])$ is called (NANS).*

Definition 2.2. [8] *A sequence $\{\omega_f\}$ in $\mathcal{E}$ is called (C) sequence if, for any $\varepsilon > 0$, there is a positive integer $\mathcal{N}$ such that*

$$[\![\omega_f - \omega_{\mathfrak{m}}]\!] \le \varepsilon \ for\ any\ \mathfrak{f}, \mathfrak{m} \ge \mathcal{N}.$$

Definition 2.3. [8] *A sequence $\{\omega_f\}$ is called a convergent, if there is an element ω in $\mathcal{E}$ such that, for any $\varepsilon > 0$, there is a positive integer $\mathcal{N}$ with*

$$[\![\omega_f - \omega]\!] \le \varepsilon \ for\ any\ f \ge \mathcal{N}.$$

In this case, the element ω is referred to as the limit of $\{\omega_f\}$, and we write

Definition 2.4. [8] *A (NANS) is called (NA) Banach space if every (C) sequence in $\mathcal{E}$ is convergent.*

Definition 2.5. [4] *Let $\mathcal{E}$ is a nonempty set. A mapping $\mathcal{P} : \mathcal{E} \times \mathcal{E} \to [0, \infty]$ is called a (GM) on $\mathcal{E}$ provided that for any $\omega, \wp, c \in \mathcal{E}$, then $\mathcal{P}$ satisfies:*

(1) $\mathcal{P}(\omega, \wp) = 0 \Leftrightarrow \omega = \wp$

(2) $\mathcal{P}(\omega, \wp) = \mathcal{P}(\wp, \omega)$

(3) $\mathcal{P}(\omega, c) \le \mathcal{P}(\omega, \wp) + \mathcal{P}(\wp, c)$

Theorem 2.6. [2,4] *Suppose that $(\mathcal{E}, \mathcal{P})$ is complete (GMS). Let $\delta : \mathcal{E} \longrightarrow \mathcal{E}$ be a strictly contractive mapping with Lipschitz constant $\ell < 1$, For any $\omega \in \mathcal{E}$. Either $(q^f\omega, q^{f+1}\omega) = \infty$ for any $f \in \mathcal{N}_0$ or there is a positive integer $f_0 \in \mathcal{N}_0$ such that*

(1) $\mathcal{P}(q^f\omega, q^{f+1}\omega) < \infty$ *for any* $f \ge f_0$;

(2) *The sequence $q^f\omega$ converges to a (FP) $\wp^*$ of q*

(3) *$\wp^*$ is the unique (FP) of q in the $Y = \wp \in \mathcal{E}, \mathcal{P}(q^{f_0}\omega, \wp) < \infty$;*

(4) $\mathcal{P}(\wp, \wp^*) \le \frac{1}{1-\ell}\mathcal{P}(\wp, q\wp)$ *for any $\wp \in Y$.*

3. Stability of the $(C.A.F.E.)$ (1) in $(N.A.N.S.)$

The Cauchy additive functional equation is studied in non-Archimedean normed spaces. The Hyers–Ulam stability of Eq. (1) is investigated under suitable assumptions. Let $g : \mathcal{E} \longrightarrow \mathcal{X}$ be a given mapping, where $\mathcal{E}$ is assumed to be a semigroup and $\mathcal{X}$ is a complete $(NANS)$. Let

$$D_g(\omega, \wp, \imath) = g\left(\frac{\omega - \wp}{f} + \imath\right) + \left(\frac{\wp - \imath}{f} + \omega\right) + g\left(\frac{\imath - \omega}{f} + \wp\right) - g(\omega + \wp + \imath)$$

For any $\omega, \wp, \imath \in \mathcal{E}$.

Theorem 3.1. *Let* $\vartheta : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to [0, \infty)$. *be a function such that*

$$\lim_{f \to \infty} \frac{\vartheta\left(3^{f}\omega, 3^{f}\wp, 3^{f}\mu\right)}{\lceil 3^{f} \rceil} = 0 \tag{2}$$

And let for each $\omega \in \mathcal{E}$ *the limit*

$$\frac{1}{\lceil 3 \rceil} \lim_{f \to \infty} \max_{0 \le i < f} \left\{ \frac{\vartheta\left(3^{i}\omega, 3^{i}\omega, 3^{i}\omega\right)}{\lceil 3^{i} \rceil} \right\} \tag{3}$$

Represented by $\left[\widetilde{\vartheta}(\omega)\right]$ *exists, suppose* $g : \mathcal{E} \longrightarrow \mathcal{X}$ *is a mapping satisfying*

$$[\![D_g(\omega, \wp, \mu)]\!] \le \vartheta(\omega, \wp, \mu) \tag{4}$$

Hence, there exists a linear mapping $\varkappa : \mathcal{E} \longrightarrow \mathcal{X}$ *such that*

$$[\![g(\omega) - \varkappa(\omega)]\!] \le \widetilde{\vartheta}(x), (\omega \in \mathcal{E}) \tag{5}$$

Moreover, if

$$\lim_{i \to \infty} \lim_{f \to \infty} \max_{i \le \mathcal{g} < f+i} \left\{ \frac{\vartheta\left(3^{\mathcal{g}}\omega, 3^{\mathcal{g}}\omega, 3^{\mathcal{g}}\omega\right)}{\lceil 3^{\mathcal{g}} \rceil} \right\} = 0$$

Proof. Putting $\omega = \wp = \mu$ in (4), we get

$$[\![3g(\omega) - g(3\omega)]\!] \le \vartheta(\omega, \omega, \omega)$$

□

Replacing ω by $3^{f}\omega$ and Division of all side by 3^{f} we get

$$\left[\!\!\left[\frac{g\left(3^{f}\omega\right)}{3^{f}} - \frac{g\left(3^{f+1}\omega\right)}{3^{f+1}} \right]\!\!\right] \le \frac{1}{\lceil 3 \rceil} \frac{\vartheta\left(3^{f}\omega, 3^{f}\omega, 3^{f}\omega\right)}{\lceil 3^{f} \rceil}$$

Let $f \to \infty$, then the sequence $\left\{ \frac{g(3^{f}\omega)}{3^{f}} \right\}$ is Cauchy. Since $\mathcal{X}$ is complete we conclude that $\left\{ \frac{g(3^{f}\omega)}{3^{f}} \right\}$ is convergent.

Set $\varkappa(\omega) = \lim_{f \to \infty} \left(\frac{g(3^{f}\omega)}{3^{f}} \right)$, we can show that

$$\frac{g\left(3^{f}\omega\right)}{3^{f}} - g(\omega) = \sum_{i=0}^{f-l} \left(\frac{g\left(3^{i+1}\omega\right)}{3^{i+1}} - \frac{g\left(3^{i}\omega\right)}{3^{i}} \right)$$

$$\left[\!\!\left[\frac{g\left(3^{f}\omega\right)}{3^{f}} - g(\omega) \right]\!\!\right] \le \frac{1}{\lceil 3 \rceil} \max \left\{ \vartheta(\omega, \omega, \omega), \frac{\vartheta(3\omega, 3\omega, 3\omega)}{\lceil 3 \rceil}, \ldots, \frac{\vartheta\left(3^{f-1}\omega, 3^{f-1}\omega, 3^{f-1}\omega\right)}{\lceil 3^{f-1} \rceil} \right\}$$

$$\left[\!\!\left[\frac{g\left(3^{f}\omega\right)}{3^{f}} - g(\omega) \right]\!\!\right] \le \frac{1}{\lceil 3 \rceil} \max_{0 \le i < f} \left\{ \frac{\vartheta\left(3^{i}\omega, 3^{i}\omega, 3^{i}\omega\right)}{\lceil 3^{i} \rceil} \right\}$$

When $f \longrightarrow \infty$,

$$[\![\varkappa(\omega) - g(\omega)]\!] \le \frac{1}{\lceil 3 \rceil} \lim_{f \to \infty} \max_{0 \le i < f} \left\{ \frac{\vartheta\left(3^{i}\omega, 3^{i}\omega, 3^{i}\omega\right)}{\lceil 3^{i} \rceil} \right\}$$

Replacing $\omega, \wp, \mu$ by $3^f\omega, 3^f\wp, 3^f\mu$ and driving all side by 3^f in (4) and taking the limit as $f \to \infty$ we get

$$\left[\!\left[\varkappa\left(\frac{\omega-\wp}{f}+\mu\right)+\varkappa\left(\frac{\wp-\mu}{f}+\omega\right)+\varkappa\left(\frac{\mu-\omega}{f}+\wp\right)-\varkappa\left(\omega+\wp+\mu\right)\right]\!\right] \leq \lim_{f\to\infty}\frac{\vartheta\left(3^f\omega, 3^f\wp, 3^f\mu\right)}{\lceil 3^f\rceil}$$

Then $\varkappa$ satisfying the linear functional equation.

If $\acute{\varkappa}$ is another mapping satisfying that

$$\begin{aligned}
[\![\varkappa(\omega)-\acute{\varkappa}(\omega)]\!] &\leq \max\left\{[\![\varkappa(\omega)-g(\omega)]\!], [\![g(\omega)-\acute{\varkappa}(\omega)]\!]\right\} \\
&\leq \max\left\{\widetilde{\vartheta}(\omega), \widetilde{\vartheta}(\omega)\right\} \\
&= \frac{1}{\lceil 3\rceil}\lim_{f\to\infty}\max_{0\leq i<f}\left\{\frac{\vartheta\left(3^i\omega, 3^i\omega, 3^i\omega\right)}{\lceil 3^i\rceil}\right\} \\
&\leq \frac{1}{\lceil 3\rceil}\lim_{i\to\infty}\lim{}_{f\to\infty}\max_{i\leq\mathcal{g}<f+i}\left\{\frac{\vartheta\left(3^{\mathcal{g}}\omega, 3^{\mathcal{g}}\omega, 3^{\mathcal{g}}\omega\right)}{\lceil 3^{\mathcal{g}}\rceil}\right\} = 0
\end{aligned}$$

Therefore $\varkappa$. This complete the proof of the uniqueness of $\varkappa$.

Corollary 3.2. *Let $\rho : [0,\infty) \longrightarrow [0,\infty)$. be a function such that $\rho(\lceil 3\rceil t) \leq \rho(\lceil 3\rceil)\rho(t), (t > 0), \rho(\lceil 3\rceil) < \lceil 3\rceil$ Let $\delta > 0$, let H be a $(N-S)$ and let $g : \mathcal{E} \longrightarrow \mathcal{X}$ hold for the inequality*

$$\begin{aligned}
&\left[\!\left[g\left(\frac{\omega-\wp}{f}+\mu\right)+g\left(\frac{\wp-\mu}{f}+\omega\right)+g\left(\frac{\mu-\omega}{f}+\wp\right)-g\left(\omega+\wp+\mu\right)\right]\!\right] \\
&\qquad\qquad\qquad\qquad \leq \delta\left(\rho([\![\omega]\!])+\rho([\![\wp]\!])+\rho([\![\mu]\!])\right)
\end{aligned}$$

Hence, there exists unique linear mapping $\varkappa : \mathcal{E} \longrightarrow \mathcal{X}$ such that

$$[\![g(\omega)-\varkappa(\omega)]\!] \leq \frac{3\delta\left(\rho([\![\omega]\!])\right)}{\lceil 3\rceil}, (\omega \in \mathcal{X})$$

Proof. Define $\vartheta : \mathcal{E}\times\mathcal{E}\times\mathcal{E} \longrightarrow [0,\infty)$ by $\varphi(\omega,\wp,\mu) := \delta\left(\rho([\![\omega]\!])+\rho([\![\wp]\!])+\rho([\![\mu]\!])\right)$

$$\lim_{f\to\infty}\frac{\varphi\left(3^f\omega, 3^f\wp, 3^f\mu\right)}{\lceil 3^f\rceil} \leq \lim_{f\to\infty}\left(\frac{\rho(\lceil 3\rceil)}{\lceil 3\rceil}\right)^f \vartheta(\omega,\wp,\mu) = 0$$

$$\widetilde{\vartheta}(\omega) = \frac{1}{\lceil 3\rceil}\lim_{f\to\infty}\max_{0\leq i<f}\left\{\frac{\delta\left(\rho\left(3^i\omega+\rho\left([\![3^i\omega]\!]\right)+\rho\left([\![3^i\omega]\!]\right)\right)\right)}{\lceil 3^i\rceil}\right\}$$

$$< \frac{1}{\lceil 3\rceil}\lim_{f\to\infty}\max_{0\leq i<f}\left\{\frac{3\delta\lceil 3^i\rceil\rho([\![\omega]\!])}{\lceil 3^i\rceil}\right\} = \frac{3\delta\rho([\![\omega]\!])}{\lceil 3\rceil}$$

$$\lim_{i\to\infty}\lim_{f\to\infty}\max_{i\leq\mathcal{g}<f+i}\left\{\vartheta\left(\frac{\omega}{3^{\mathcal{g}+1}}, \frac{\omega}{3^{\mathcal{g}+1}}, \frac{\omega}{3^{\mathcal{g}+1}}\right)\right\} = 0.$$

Applying Theorem 3.1. we conclude the required result. □

Theorem 3.3. *Let $\vartheta : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \longrightarrow [0, \infty)$.be a function such that*

$$\lim_{f \to \infty} \left\lceil 3^f \right\rceil \vartheta \left(\frac{\omega}{3^{n+1}}, \frac{\wp}{3^{n+1}}, \frac{\mu}{3^{n+1}} \right) = 0 \tag{6}$$

For all $\omega, \wp, \mu \in \mathcal{E}$ and for each $\omega \in \mathcal{E}$, the limit

$$\widetilde{\vartheta}(\omega) = \lim_{f \to \infty} \max_{0 \le i < f} \left\{ \vartheta \left(\frac{\omega}{3^{f+1}}, \frac{\omega}{3^{f+1}}, \frac{\omega}{3^{f+1}} \right) \right\} \text{ exists}$$

Suppose that $g : \mathcal{E} \longrightarrow \mathcal{X}$ a mapping satisfying the following inequality

$$\llbracket D_g(\omega, \wp, \mu) \rrbracket \le \vartheta(\omega, \wp, \mu) \tag{7}$$

Then, the limit $\varkappa(\omega) = \lim_{f \to \infty} 3^f g\left(\frac{\omega}{3^f}\right)$ exists for all $\omega \in \mathcal{E}$ and defines a linear a mapping $\varkappa : \mathcal{E} \longrightarrow \mathcal{X}$ such that

$$\llbracket g(\omega) - \varkappa(\omega) \rrbracket \le \widetilde{\vartheta}(\omega) \tag{8}$$

Also $\varkappa$ is unique linear mapping satisfying (8) if:

$$\lim_{i \to \infty} \lim_{f \to \infty} \max_{i \le \mathcal{g} < f + i} \left\{ \vartheta \left(\frac{\omega}{3^{\mathcal{g}+1}}, \frac{\omega}{3^{\mathcal{g}+1}}, \frac{\omega}{3^{\mathcal{g}+1}} \right) \right\} = 0$$

Proof. This theorem can be established by employing an analogous argument to that used in the preceding Theorem 3.1. □

4. $(F.P.)$ Method for $(C.A.F.E.)$ (1)

In this section, a $(F.P.)$ method is applied to study the stability of the $(C.A.F.E.)$. The approach is formulated in an appropriate metric setting to obtain existence and uniqueness results.

Theorem 4.1. *Suppose that $\vartheta : \mathcal{E} \times \mathcal{E} \longrightarrow [0, \infty)$. is a function such that there is a constant $\ell < 1$ with*

$$\vartheta(3\omega, 3\wp, 3\mathfrak{r}) \le \ell \lceil 3 \rceil \vartheta(\omega, \wp, \mathfrak{r}) \tag{9}$$

For any $\omega, \wp, \mathfrak{r} \in \mathcal{E}$. Consider a mapping $g : \mathcal{E} \longrightarrow \mathcal{X}$ for which

$$\llbracket D_g(\omega, \wp, \mathfrak{r}) \rrbracket \le \vartheta(\omega, \wp, \mathfrak{r}) \tag{10}$$

For any $\omega, \wp, \mathfrak{r} \in \mathcal{E}$. There is a unique (AF) $\mathsf{C} : \mathcal{E} \longrightarrow \mathcal{X}$ satisfying

$$\llbracket g(\omega) - \mathsf{C}(\omega) \rrbracket \le \frac{1}{(1 - \ell) \lceil 3 \rceil} \vartheta(\omega, \omega, \omega) \tag{11}$$

For any $\omega \in \mathcal{E}$

Proof. By choosing $\omega = \wp = \mathfrak{r}$ in (10), one obtains

$$\left\llbracket g(\omega) - \frac{g(3\omega)}{3} \right\rrbracket \le \frac{1}{\lceil 3 \rceil} \vartheta(\omega, \omega, \omega)$$

For any $\omega \in \mathcal{E}$. We operate within the class

$$Q = \{\mathscr{d} : \mathcal{E} \longrightarrow \mathcal{X}; \mathscr{d}(0) = 0\}$$

In addition, we define (GM) on S specified by:

$$\mathscr{P}(g, \mathscr{d}) = \inf\left\{\delta \in \mathbb{R}^{+} : g(\omega) - \mathscr{d}(\omega) \leq \frac{\delta}{\lceil 3 \rceil}\vartheta(\omega, \omega, \omega), \ \forall \omega \in \mathcal{E}\right\}$$

It is immediate that $(\mathcal{S}, \mathscr{P})$ complete (MS). A linear mapping is defined as follows $\mathscr{k} : Q \longrightarrow Q$ such that

$$\mathscr{k}(g(\omega)) = \frac{g(3\omega)}{3}$$

For any $g, \mathscr{d} \in Q$

$$\mathscr{P}(g, \mathscr{d}) = \mathtt{C}$$

$$\begin{aligned}
\mathscr{k}g(\omega) - \mathscr{k}\mathscr{d}(\omega) &= \frac{g(3\omega)}{3} - \frac{\mathscr{d}(3\omega)}{3} \\
&= \frac{1}{\lceil 3 \rceil}[\![g(3\omega) - \mathscr{d}(3\omega)]\!] \\
&= \frac{1}{\lceil 3 \rceil}\frac{\mathtt{C}}{\lceil 3 \rceil}\vartheta(3\omega, 3\omega, 3\omega) \\
&\leq \frac{1}{\lceil 3 \rceil}\frac{\mathtt{C}}{\lceil 3 \rceil}\ell\lceil 3 \rceil\vartheta(\omega, \omega, \omega) \\
&= \frac{\mathtt{C}\ell}{\lceil 3 \rceil}\vartheta(\omega, \omega, \omega)
\end{aligned}$$

Then

$$\begin{aligned}
\mathscr{P}(\mathscr{k}g, \mathscr{k}\mathscr{d}) &\leq \ell\mathtt{C} \\
&= \ell\mathscr{P}(g, \mathscr{d})
\end{aligned}$$

Then $\mathscr{k}$ is contradiction, since

$$\frac{g(3\omega)}{3} - g(\omega) \leq \frac{1}{\lceil 3 \rceil}\vartheta(\omega, \omega, \omega).$$

Then,

$$\mathscr{P}(\mathscr{k}g, g) \leq 1$$

By applying Theorem 2.6, there exsits a (CAF) $\mathtt{C} : \mathcal{E} \longrightarrow \mathcal{X}$ subject to the conditions.

(1) $\mathscr{k}$ has a (FP) K

$$\begin{aligned}
\mathscr{k}(K(\omega)) &= K(\omega) \\
\frac{K(3\omega)}{3} &= K(\omega) \\
K(3\omega) &= 3K(\omega)
\end{aligned}$$

For any $\omega \in \mathcal{E}$, then K is unique (FP) of $\mathscr{k}$ in the set

$$\Omega = \{\mathcal{F} \in Q : (g, \mathscr{d}) < \infty\}$$

There is $\mathtt{C} \in (0, \infty)$ such that $\mathscr{P}(K, g) < \mathtt{C}$

$$[\![K(\mathrm{w}) - g(\mathrm{w})]\!] \leq \frac{\mathtt{C}}{\lceil 3 \rceil} \vartheta(\mathrm{w}, \mathrm{w}, \mathrm{w})$$

(2) $\mathscr{P}(\hbar^{\mathfrak{m}} g, K) \longrightarrow 0$ as $\mathfrak{m} \longrightarrow \infty$. This gives that, for any $\mathrm{w} \in \mathscr{E}$

$$\lim_{\mathfrak{m}\to\infty} \hbar^{\mathfrak{m}} g(\mathrm{w}) = \lim_{\mathfrak{m}\to\infty} \frac{g(3^{\mathfrak{m}})}{3^{\mathfrak{m}}} = K(\mathrm{w})$$

(3) $\mathscr{P}(g, K) \leq \frac{1}{1-\ell} m(g, \hbar g)$, then

$$\mathscr{P}(g, K) \leq \frac{1}{1-\ell}$$

$$[\![g(\mathrm{w}) - K(\mathrm{w})]\!] \leq \frac{1}{(1-\ell)\lceil 3 \rceil} \vartheta(\mathrm{w}, \mathrm{w}, \mathrm{w})$$

This complete the proof.

□

Corollary 4.2. *Let $\delta \in \mathcal{N}$ with $\delta > 1$. Let $g : \mathscr{E} \longrightarrow \mathscr{X}$ be a function satisfying*

$$[\![D_g(\mathrm{w}, \wp, \mathfrak{r})]\!] \leq [\![\mathrm{w}]\!]^{\delta} + [\![\wp]\!]^{\delta} + [\![\mathfrak{r}]\!]^{\delta}$$

Then,

$$\mathtt{C}(\mathrm{w}) = \lim_{f\to\infty} \frac{g(3^f \mathrm{w})}{3^f}$$

For any $\mathrm{w} \in \mathscr{E}$ and $\mathtt{C} : \mathscr{E} \longrightarrow \mathscr{X}$ be a (CA) mapping such that

$$[\![g(\mathrm{w}) - \mathtt{C}(\mathrm{w})]\!] \leq \frac{1}{(1-\ell)\lceil 3 \rceil} \vartheta(\mathrm{w}, \mathrm{w}, \mathrm{w})$$

Proof. Using Theorem 4.1 and by assuming

$$\vartheta(\mathrm{w}, \wp, \mathfrak{r}) = [\![\mathrm{w}]\!]^{\delta} + [\![\wp]\!]^{\delta} + [\![\mathfrak{r}]\!]^{\delta}, \ell \geq \left\lceil 3^{\delta-1} \right\rceil, \delta \in \mathbb{N}, \delta > 1$$

This complete the proof. □

Theorem 4.3. *Suppose that $\vartheta : \mathscr{E} \times \mathscr{E} \longrightarrow [0, \infty)$ is a function such that there is a constant $\ell < 1$ with*

(12) $$\vartheta\left(\frac{\mathrm{w}}{3}, \frac{\wp}{3}, \frac{\mathfrak{r}}{3}\right) \leq \frac{\ell \vartheta(\mathrm{w}, \wp, \mathfrak{r})}{\lceil 3 \rceil}$$

For any $\mathrm{w} \in \mathscr{E}$. Consider a mapping $g : \mathscr{E} \longrightarrow \mathscr{X}$ for which

(13) $$[\![D_g(\mathrm{w}, \wp, \mathfrak{r})]\!] \leq \vartheta(\mathrm{w}, \wp, \mathfrak{r})$$

For any $\mathrm{w}, \wp, \mathfrak{r} \in \mathscr{E}$. It follows that there is a uniquely determined (A) mapping $\mathcal{C} : \mathscr{E} \longrightarrow \mathscr{X}$ such that

(14) $$[\![g(\mathrm{w}) - \mathtt{C}(\mathrm{w})]\!] \leq \frac{\ell}{(1-L)\lceil 3^{f+1} \rceil} \vartheta(\mathrm{w}, \mathrm{w}, \mathrm{w})$$

Proof. This theorem can be established by employing an analogous argument to that used in the preceding Theorem 4.1. □

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