

COMPLEMENT OF MAXIMAL IDEAL GRAPH OF A COMMUTATIVE RING

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ABSTRACT. In this paper, a new kind of graph on a ring is introduced and investigated. Let R be a commutative ring with unity. The complement of the maximal ideal graph of a ring R , denoted by $C(MG(R))$, is an undirected simple graph with all non-trivial ideals of R as vertices and two distinct vertices I and J are adjacent if and only if their sum is a non-maximal ideal of R . In this article, we investigate the basic properties of these graphs to relate the combinatorial properties of $C(MG(R))$ to the algebraic properties of the ring R . We determine the diameter and the girth of $C(MG(R))$. We obtain some results for connectedness and bipartiteness, planar and weakly perfect of these graphs.

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1. INTRODUCTION

In 1988, Beck [1] introduced the concept of the zero-divisor graph. Since then, others have introduced and studied many researches in this area. In [2], the zero divisor graph, whose vertices are nonzero zero-divisors, was introduced and investigated by Anderson and Livingston. Barman and Rajkhowa [3] in 2019 defined the non-comaximal graph of ideals of a ring.

In 2020, Abdulqadr [4] introduced the maximal ideal graph of a commutative ring R denoted by $MG(R)$, is the undirected graph with all non-trivial ideals of R as vertices and two distinct vertices α and β are adjacent if and only if $\alpha + \beta$ is a maximal ideal of R .

There has been a lot of activity over the past several years in associating a graph to an algebraic system such as a ring or semiring [5–11].

In this paper, we introduce the complement of the maximal ideal graph of R denoted by $C(MG(R))$, the vertices correspond to the non-trivial ideals of R , and two distinct vertices are adjacent whenever

their sum is a non-maximal ideal of R . The graph $C(MG(R))$ is a complement graph of that graph in [4].

This is how the paper is structured: Section 2 introduces some definitions and preliminaries. In Section 3, we investigate fundamental graph-theoretic properties of the complement of the maximal ideal graph of a ring, including the degree of a vertex, girth, clique and chromatic numbers.

Finally, in Section 4, the connectivity of these graphs and determines its diameter are studied.

2. PRELIMINARIES

In this part, any undefined terminology in ring theory can be found in [12] and [13,14]. Beforehand defining the complement of the maximal ideal graph of a ring R , we recall some basic graphtheoretic notions that will be used throughout this article in [15] and [16–18] that will be used throughout this article as follows:

Suppose that \mathcal{G} is a graph with its vertex set $V(\mathcal{G})$ and its edge set $E(\mathcal{G})$.

The order of \mathcal{G} is the number of vertices in \mathcal{G} indicated by $|\mathcal{G}|$.

The degree of a vertex v in \mathcal{G} , indicated by $\deg(v)$, is the number of vertices of \mathcal{G} adjacent to v . The neighborhood $N(v)$ of a vertex v of \mathcal{G} is the set of all vertex adjacent to v .

Isolated vertex is a vertex in \mathcal{G} that has degree zero.

The complement graph of \mathcal{G} indicated by $\bar{\mathcal{G}}$ (in this paper, symbolized as $C(\mathcal{G})$) is the graph with vertex set $V(\mathcal{G})$ such that two vertices are adjacent in $\bar{\mathcal{G}}$ if and only if they are not adjacent in \mathcal{G} .

A complete graph is a graph in which every pair of distinct vertices are joined by an edge.

A graph \mathcal{G} is n -partite; $n \geq 1$, if it is possible to partition $V(\mathcal{G})$ into n subsets V_1, V_2, \dots, V_n such that every edge of \mathcal{G} joins a vertex of V_i and a vertex of V_j , where $i \neq j$.

A path in \mathcal{G} is a sequence of vertices v_1, v_2, \dots, v_n such that each consecutive pair (v_i, v_{i+1}) is connected by an edge, and all vertices in the sequence are distinct.

A vertex u is said to be connected to a vertex v in \mathcal{G} if there exists a $u - v$ path $P = (u = u_0, u_1, u_2, \dots, u_k = v)$ in \mathcal{G} . A graph \mathcal{G} is connected if every two of its vertices are connected.

An isolated vertex in \mathcal{G} is a vertex that has no edges connected to it. For a connected graph \mathcal{G} , the distance between two vertices u and v denoted by $d(u, v)$ is the minimum of the lengths of the $u - v$ paths of \mathcal{G} .

The eccentricity of a vertex v in a connected graph \mathcal{G} is $e(v) = \text{Max}_{u \in V(\mathcal{G})} d(u, v)$. The diameter of a connected graph \mathcal{G} is defined by $\text{diam}(\mathcal{G}) = \text{Max}_{v \in V(\mathcal{G})} e(v)$. The radius of the connected graph \mathcal{G} , indicated by $\text{rad}(\mathcal{G})$, is a minimum eccentricity among all vertices of the graph. The center of a connected graph \mathcal{G} , denoted by $\text{Cent}(\mathcal{G})$, is a set of all vertices whose eccentricity is equal to the radius of the \mathcal{G} .

The girth of \mathcal{G} , indicated by $\text{gr}(\mathcal{G})$, is the length of the shortest cycle in \mathcal{G} .

A component of a connected graph \mathcal{G} is the maximal connected subgraph of \mathcal{G} , and the number of components of \mathcal{G} is denoted by $\kappa(\mathcal{G})$. A vertex v in a connected graph \mathcal{G} is named cut-vertex if $\kappa(\mathcal{G} - v) > \kappa(\mathcal{G})$.

A graph \mathcal{G} is planar if it can be embedded in the plane. The clique number of \mathcal{G} , denoted by $\omega(\mathcal{G})$, is the order of the maximal complete subgraph of \mathcal{G} . A coloring of \mathcal{G} is an assignment of color to vertices such that no two adjacent vertices share the same color. The chromatic number of \mathcal{G} , denoted by $\chi(\mathcal{G})$, is the minimum number of colors needed to color the vertices of the graph. An independent set in \mathcal{G} is a subset of $V(\mathcal{G})$ such that no two vertices are adjacent.

A graph \mathcal{G} is a split if its vertex set can be partitioned into two disjoint sets K and I such that K is a clique and I is an independent set in \mathcal{G} .

Throughout this paper R will be a commutative ring with identity. Also, all rings are assumed to be finite commutative with identity. We mean from a non-trivial ideal of R is a nonzero proper left ideal of R . Any left ideal is ideal of R in this article, since R is commutative. A ring R is named local if it has a unique maximal ideal. For a ring R , we denote the zero ideal, the sets of all maximal and minimal ideals by $O_R, \text{Max}(R)$ and $\text{Min}(R)$, respectively. The Jacobson radical of R , indicated by $J(R)$, is the intersection of all maximal ideals of R . Recall [10] that an ideals F and H of R are comaximal if and only if $F + H = R$.

Now, we introduce the complement of the maximal ideal graph of a ring R as follows:

Definition 2.1. Let R be a finite commutative ring with identity. The complement of maximal ideal graph of R , denoted by $C(MG(R))$, is the simple undirected graph whose vertex set is the set of non-trivial ideals of R , and two vertices are adjacent if their sum is a non-maximal ideal of R .

Example 2.2. Consider \mathbb{Z}_{24} the ring of integers modulo 24.

The complement of the maximal ideal graph of \mathbb{Z}_{24} , indicated by $C(MG(\mathbb{Z}_{24}))$:

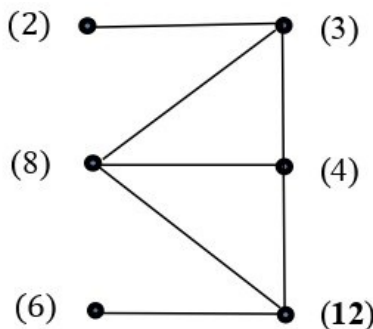


FIGURE 1. $C(MG(\mathbb{Z}_{24}))$

Note that, if F and H are comaximal ideals, at the moment they are adjacent in $C(MG(R))$.

3. SOME BASIC PROPERTIES OF $C(MG(R))$

In this section, we explore some basic properties of $C(MG(R))$. We begin by the following Lemma.

Lemma 3.1. *Let Q and T be two distinct vertices in $C(MG(R))$. Then Q and T are adjacent in each of the following cases:*

- (i) $Q \not\subseteq T$ and $T \in \text{Max}(R)$.
- (ii) $Q \subset T$ and $T \notin \text{Max}(R)$.
- (iii) $Q, T \notin \text{Max}(R)$ and $Q + T \notin \text{Max}(R)$

Theorem 3.2. *Suppose S, U and K be three distinct vertices in the graph $C(MG(R))$.*

- (a) *Let $K + S \neq R$*
 - 1. *If $S + U \neq R$ and $K \subset S$, then*
 $S \in N(U) \Rightarrow K \in N(U)$.
 - 2. *If $K + S, K + U \neq R$, then*
 $K \in N(S) \cap N(U) \Rightarrow K \in N(S \cap U)$.
 - 3. *If $K + (S + U) \neq R$, then*
 $K \in N(S + U) \Rightarrow K \in N(S) \cap N(U)$.
- (b) *Let $K \in \text{Max}(R)$ and $S \cdot U \neq O_R$. Then the following are equivalent:*
 - 1. $K \in N(S \cdot U)$.
 - 2. $K \in N(S \cap U)$.
 - 3. $K \in N(S) \cap N(U)$.
- (c) *If $S + U \neq R$ and $K \in \text{Max}(R)$, then*
 - 1. $K \in N(S) \cap N(U) \Rightarrow K \in N(S + U)$.
 - 2. $K \in N(S + U) \Rightarrow K \in N(S) \cup N(U)$.

Proof. (a) Suppose that $K \subset S$ and $S \in N(U)$. Then $K + U \subseteq S + U$ and $S + U \notin \text{Max}(R)$.

Since $S + U \neq R$, the ideal $K + U \notin \text{Max}(R)$. This means that $K \in N(U)$.

The proofs of the other parts follow in a similar manner.

(b) If $K \in N(S \cdot U)$, then $S \cdot U \not\subseteq K$.

Since $S \cdot U \subseteq S \cap U$, we have $S \cap U \not\subseteq K$. Thus, $K \in N(S \cap U)$.

Similarly, if $K \in N(S \cap U)$, then $S, U \not\subseteq K$ and hence, $K \in N(S) \cap N(U)$.

Now, let $K \in N(S) \cap N(U)$. Then $S, U \not\subseteq K$. Thus, there exist $a \in S, b \in U$ such that $a, b \notin K$.

Obviously, $a \cdot b \notin K$, because K is prime. Hence, $a \cdot b \in S \cdot U$ implies that $S \cdot U \not\subseteq K$. Thus, $K \in N(S \cdot U)$.

(c) The proof follows in a similar way as in part (a) and (b). □

Proposition 3.3. *If $S_1 \subset S_2 \subset \dots \subset S_n$ is a chain of non-trivial ideals of R such that S_n is a nonmaximal ideal, then the graph induced by $\{S_1, S_2, \dots, S_n\}$ is complete in $C(MG(R))$.*

Remark 3.4. If $\text{Max}(R) = \{M_1, M_2, \dots, M_n\}$ and $J(R) = O_R$, then $C(MG(R))$ contains an $|\text{Max}(R)|$ -partite subgraph induced by $V(C(MG(R)))$ with partition

$$A_i = \{S \in V(C(MG(R))) : S \subseteq M_i\}; i = 1, 2, \dots, n.$$

Proposition 3.5. Let $\text{Max}(R) = \{M_1, M_2\}$ and $S + T = M_i$, for every $S, T \subseteq M_i$, where $i \in \{1, 2\}$. Then the graph $C(MG(R))$ is a bi-partite. Furthermore, if $J(R) = O_R$, then $C(MG(R))$ is complete bi-partite.

Proof. The proof follows from Lemma 3.1 and Remark 3.4. □

Proposition 3.6. If $V(C(MG(R))) = \text{Max}(R) \cup \text{Min}(R)$ and $|C(MG(R))| \geq 3$, then no two distinct minimal ideals are adjacent in $C(MG(R))$. Moreover, the graph $C(MG(R))$ is $(|\text{Max}(R)| + 1)$ -partite graph.

Proof. Let $S, T \in \text{Min}(R)$ with $S \neq T$.

Now $S + T \neq R$. Since $V(C(MG(R))) = \text{Max}(R) \cup \text{Min}(R)$, either $S + T \in \text{Max}(R)$ or $S + T \in \text{Min}(R)$.

Since $S, T \subseteq S + T$, we must have $S + T \in \text{Max}(R)$. Thus, S and T are not adjacent in $C(MG(R))$.

Let $A = \text{Min}(R)$ and $A_M = \{M\}$ for every $M \in \text{Max}(R)$.

Since the subgraph of $C(MG(R))$ induced by $\text{Max}(R)$ is complete, and $\text{Min}(R)$ is an independent set, the graph $C(MG(R))$ is $(|\text{Max}(R)| + 1)$ -partite graph. □

Proposition 3.7. Suppose that $|\text{Max}(R)| \geq 2$ and $V(C(MG(R))) = \text{Max}(R) \cup \text{Min}(R)$. Then

- (1) The graph $C(MG(R))$ is split.
- (2) The clique and chromatic numbers of $C(MG(R))$ are $\omega(C(MG(R))) = \chi(C(MG(R))) = |\text{Max}(R)|$.
- (3) The independent number of $C(MG(R))$ is $\alpha(C(MG(R))) = |\text{Min}(R)|$.

Proof. (1) From Lemma 3.1, the induced subgraph of $C(MG(R))$ by $\text{Max}(R)$ is a complete. On the other hand, by Proposition 3.6, $\text{Min}(R)$ is an independent set. Hence, $C(MG(R))$ is split.

(2) By First part, the clique number of $C(MG(R))$ is $\omega(C(MG(R))) = |\text{Max}(R)|$. To find the chromatic number of $C(MG(R))$, assume that A and B are induced subgraphs of $C(MG(R))$ by $\text{Max}(R)$ and $\text{Min}(R)$, respectively. Then A is complete and B is independent.

If $N \in \text{Min}(R)$, then $N \subset M$, for some $M \in \text{Max}(R)$. It follows that M and N are not adjacent in $C(MG(R))$. Then the vertices N and M gives the same color. Hence, the chromatic number of $C(MG(R))$ is $\chi(C(MG(R))) = |\text{Max}(R)|$.

(3) The independent number of $C(MG(R))$ follows from Proposition 3.6. □

Proposition 3.8. The girth $\text{gr}(C(MG(R))) = 3$, in each of the following cases:

- (1) $|\text{Max}(R)| \neq 2$ and $C(MG(R))$ contains a cycle.
- (2) $V(C(MG(R))) \setminus \text{Max}(R)$ is a totally ordered set congaing at least three elements.
- (3) $|\text{Max}(R)| = 2$, $|C(MG(R))| \geq 5$ and $V(C(MG(R))) \setminus \text{Max}(R) \subset J(R)$.

Proof. Let $C: S_1, S_2, \dots, S_n, S_1$ be a cycle in $C(MG(R))$ with $|\text{Max}(R)| \neq 2$.

If $|\text{Max}(R)| = 2$, then M is nilpotent. Thus, $C: M^2, M^3, M^4, M^2$ is a cycle.

If $|\text{Max}(R)| \geq 3$, then $C(MG(R))$ contains a complete subgraph induced by $\text{Max}(R)$.

In both cases, $\text{gr}(C(MG(R))) = 3$.

The proof of parts (2) and (3) follow from Lemma 3.1. □

Theorem 3.9. *Let $R \cong F_1 \times F_2 \times \dots \times F_t$, where $F_i (i = 1, 2, \dots, t)$ is a field. Then $C(MG(R))$ is planar only if and only if $t \leq 3$.*

Proof. Let $R = F_1 \times F_2 \times \dots \times F_t$, where F_1, F_2, \dots, F_t are distinct fields.

If $t > 3$, then $C(MG(R))$ contains a $K_{3,3}$ with partite sets

$$V_1 = \left\{ F_1 \times O_{F_2} \times O_{F_3} \times O_{F_4}, O_{F_1} \times F_2 \times O_{F_3} \times O_{F_4}, O_{F_1} \times O_{F_2} \times F_3 \times F_4 \right\}$$

$$V_2 = \left\{ O_{F_1} \times O_{F_2} \times F_3 \times O_{F_4}, O_{F_1} \times O_{F_2} \times O_{F_3} \times F_4, F_1 \times F_2 \times O_{F_3} \times O_{F_4} \right\}.$$

Since $K_{3,3}$ is non-planar, the graph $C(MG(R))$ is a non-planar graph.

Now, suppose that $t = 3$. Then the graph $C(MG(R))$ is:

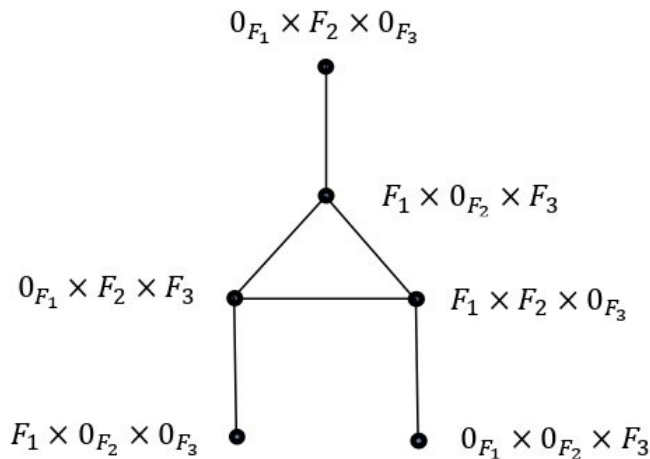


FIGURE 2. $C(MG(F_1 \times F_2 \times F_3))$

From Figure 2, one has $C(MG(R))$ is a planar graph.

Similarly, $C(MG(R))$ is a planar graph, when $t = 1, 2$. □

Remark 3.10. *If $R \cong N_1 \oplus N_2 \oplus \dots \oplus N_m$, where $N_1, N_2, \dots, N_m (m \geq 2)$ are minimal ideals of R , then*

- (1) *Each maximal ideal is adjacent to a unique minimal ideal in $C(MG(R))$.*
- (2) *Each minimal ideal is adjacent to a unique maximal ideal in $C(MG(R))$.*

Theorem 3.11. *Let $R \cong N_1 \oplus N_2 \oplus \dots \oplus N_m$, where $N_1, N_2, \dots, N_m (m \geq 2)$ are minimal ideals of R . Then*

(1) Every two minimal ideals are adjacent if and only if $m \neq 3$.

$$(2) \deg(I) = \begin{cases} 2^{m-1} - 1 & \text{if } I \in \text{Max}(R) \\ 1 & \text{if } m \in \{2, 3\}, I \in \text{Min}(R) \\ 2^m - 2m - 1 & \text{if } m \geq 4, I \in \text{Min}(R) \\ 2^m - 2^k(m - k) - 3 & I = \sum_{i=1}^k S_i \end{cases}$$

for every distinct minimal ideal S_1, S_2, \dots, S_k with $1 < k < m - 1$.

Proof. (1) If $m = 2$, then $N_1, N_2 \in \text{Max}(R)$.

Thus, $N_1 + N_2 \notin \text{Max}(R)$. This implies that N_1 and N_2 are adjacent in $C(MG(R))$.

If $m = 3$, then $N_i + N_j \in \text{Max}(R)$, for every $i, j = 1, 2, 3$ with $i \neq j$.

In this case, every two distinct minimal ideals are not adjacent in $C(MG(R))$.

If $m \geq 4$, then every maximal ideal is the sum of at least three minimal ideals. This implies that $N_i + N_j \notin \text{Max}(R)$, for every $i \neq j$. Thus, N_i and N_j are adjacent in $C(MG(R))$.

(2) Let $I \in \text{Max}(R)$. Then

$$I = (N_1 + N_2 + \dots + N_{j-1}) + (N_{j+1} + N_{j+2} + \dots + N_m).$$

For some $j \in \{1, 2, \dots, m\}$.

First, I is adjacent to all other maximal ideals in $C(MG(R))$. Also, by Remark 3.10, I is adjacent to the only minimal ideal N_j . Now, we show the adjacency of I with other ideals.

For any $S \subseteq \{1, \dots, m\}$ with $2 \leq |S| \leq m - 2$. Define

$$(1) \quad N_S = \oplus_{i=1}^m I_i, \quad I_i = \begin{cases} O_R, & i \in S \\ N_i, & i \notin S \end{cases}$$

which represents all non-trivial ideals except a maximal and minimal ideal of R .

The number of ideals that defined by (1) adjacent to I is

$$\begin{aligned} \sum_{k=2}^{m-2} \binom{m-1}{k} &= 2^{m-1} - \left(\binom{m-1}{0} + \binom{m-1}{1} + \binom{m-1}{m-1} \right) \\ &= 2^{m-1} - m - 1 \end{aligned}$$

Thus, the total number of ideals adjacent to I is

$$\deg(I) = (m - 1) + (1) + (2^{m-1} - m - 1) = 2^{m-1} - 1.$$

Let $I \in \text{Min}(R)$. Then $I = N_j$, for some $j \in \{1, 2, \dots, m\}$.

If $m = 2$, then $\deg(I) = 1$.

If $m = 3$, then by Remark 3.10 and the first part of the theorem, $\deg(I) = 1$.

Assume that $m \geq 4$. Then I is adjacent to a unique maximal ideal and $m - 1$ minimal ideals in

$C(MG(R))$. From (1) we have:

$$N_j + N_S = \begin{cases} N_S & \text{if } j \in S \\ \bigoplus_{i \notin S \cup \{j\}} N_i \oplus N_j & \text{if } j \notin S \end{cases}$$

By the same way of previews case, there are $\sum_{k=2}^{m-2} \binom{m}{k} - (m-1)$ vertices in the form N_S adjacent to N_j . Thus,

$$\begin{aligned} \deg(N_j) &= 1 + (m-1) + \left(\sum_{k=2}^{m-2} \binom{m}{k} - (m-1) \right) \\ &= 2^m - 2m - 2 \end{aligned}$$

Similarly, the degree of every vertex not in $\text{Max}(R) \cup \text{Min}(R)$ can be computed. \square

4. THE CONNECTEDNESS OF $C(MG(R))$

In this section, we determine some conditions under which $C(MG(R))$ is disconnected or connected.

In the next result, we demonstrate two cases of disconnectedness for the graph that possesses an isolated vertex

Theorem 4.1. (1) *If R is local with maximal ideal Q , then Q is isolated vertex in $C(MG(R))$. Besides, if R is a principle ideal ring, then $C(MG(R)) \setminus Q$ is complete.*
 (2) *Let $\{\text{Min}(R), \text{Max}(R)\}$ be a partition of $V(C(MG(R)))$. If $J(R)$ is a vertex in $C(MG(R))$, then it is isolated and $C(MG(R)) \setminus J(R)$ is connected.*

Proof. (1) Presume that $\text{Max}(R) = \{Q\}$ and every ideal L contains in Q . This yields that $L + Q = Q$ and so Q is not adjacent to L in $C(MG(R))$.

If R is a principal ideal ring, then the set of all ideals consists a chain. By Proposition 3.3, the graph $C(MG(R)) \setminus Q$ is complete.

(2) Presume that $\{\text{Min}(R), \text{Max}(R)\}$ is a partition of $V(C(MG(R)))$. If R is local, then by the first part of the theorem, $J(R)$ is isolated vertex in $C(MG(R))$. Now, suppose that R is non-local. Then $J(R) \in \text{Min}(R)$. If $L \in \text{Min}(R)$, then by Proposition 3.6, L and $J(R)$ are not adjacent in $C(MG(R))$. Otherwise, we have $J(R) + L = L \in \text{Max}(R)$. Thus, $J(R)$ is not adjacent to I in $C(MG(R))$. It remains to show that $C(MG(R)) \setminus J(R)$ is connected. First, the subgraph of $C(MG(R))$ induced by $\text{Max}(R)$ is complete. Then, $L \not\subseteq J(R)$. As a result, $L \not\subseteq Q$ for some $Q \in \text{Max}(R)$. Then L and Q are adjacent in $C(MG(R)) \setminus J(R)$. So $C(MG(R)) \setminus J(R)$ is connected. \square

Proposition 4.2. *Let $\text{Max}(R) = \{M_1, M_2\}$ with $M_1 \neq M_2$. If $V(C(MG(R))) \setminus \text{Max}(R) \subset M_1 \cap M_2$, then $C(MG(R))$ consists of an edge $\{M_1, M_2\}$ and a complete component induced by $V(C(MG(R))) \setminus \text{Max}(R)$.*

Proof. First, M_1 and M_2 are comaximal ideals. So, they are adjacent in $C(MG(R))$.

Now, let $K, L \in V(C(MG(R))) \setminus \text{Max}(R)$ with $K \neq L$.

Since $K, K + L \subset M_1 \cap M_2$, none of M_1 and M_2 are adjacent to K , and K and L are adjacent in $C(MG(R))$.

Henceforth, $C(MG(R))$ consists of an edge $\{M_1, M_2\}$ and a complete component induced by the vertex set $V(C(MG(R))) \setminus \text{Max}(R)$. \square

From now on, we current all cases in which the graph is connected.

Theorem 4.3. *If $J(R) = O_R$, then $C(MG(R))$ is connected with $\text{diam}(C(MG(R))) \leq 3$.*

Proof. Let $I, J \in V(C(MG(R)))$ with $I \neq J$.

If $I + J \notin \text{Max}(R)$, then they are adjacent in $C(MG(R))$.

Now, suppose that $I + J \in \text{Max}(R)$.

We have two cases for I and J :

Case1. $I \in \text{Max}(R)$ and $J \notin \text{Max}(R)$

First, we have $J \subset I$ and $J \not\subset J(R)$. Thus, there exists $M \in \text{Max}(R)$ such that $J \not\subset M$.

Now, $I, J \neq M$. From Lemma 3.1, The vertex M is adjacent to both I and J in $C(MG(R))$ by Lemma 3.1, since $M + J = M + I = R$. Thus (I, M, J) is a path in $C(MG(R))$.

Similarly, a path of length 2 can be obtained in $C(MG(R))$, when $I \notin \text{Max}(R)$ and $J \in \text{Max}(R)$.

Case2. $I, J \notin \text{Max}(R)$

Since $I, J \not\subset J(R)$, there exist $M_1, M_2 \in \text{Max}(R)$ such that $I \not\subset M_1$ and $J \not\subset M_2$. If $M_1 \neq M_2$, then by Lemma 3.1, (I, M_1, M_2, J) is a path in $C(MG(R))$. Otherwise, (I, M_1, J) is a path in $C(MG(R))$.

After both cases, one has proved that $C(MG(R))$ is a connected graph by $\text{diam}(C(MG(R))) \leq 3$. \square

Theorem 4.4. *If $\text{Max}(R) = \{M_1, M_2\}$ with $M_1 \neq M_2$, then either $C(MG(R))$ is connected or it consists of two components.*

Proof. If $M_1 \cap M_2 = O_R$, then by Theorem 4.3, $C(MG(R))$ is connected with $\text{diam}(C(MG(R))) \leq 3$.

Now, suppose that $M_1 \cap M_2 \neq O_R$.

Define

$$A_1 = \{I \in V(C(MG(R))) : I \subseteq M_1, I \not\subseteq M_2\},$$

$$A_2 = \{J \in V(C(MG(R))) : J \subseteq M_2, J \not\subseteq M_1\} \text{ and}$$

$$B = \{K \in V(C(MG(R))) : K \subseteq M_1 \cap M_2\}.$$

By the previous case, every vertex in A_1 is adjacent to every vertex in A_2 .

For any distinct $K, L \in B$, we have $K + L \subseteq M_1 \cap M_2$ which is a non-maximal ideal. Hence the

subgraph induced by B is complete. Now, define $B^* = \{K \in B : K + I, K + J \in \text{Max}(R) \text{ for all } I \in A_1 \text{ and } J \in A_2\}$.

First, suppose that $B^* \subset B$. If $B^* = \emptyset$, then $C(MG(R))$ is connected. Otherwise, there exists $L \in B \setminus B^*$ such that $L + I, L + J \notin \text{Max}(R)$ for some $I \in A_1$ and $J \in A_2$. In this case, $C(MG(R))$ is connected. Finally, if $B^* = B$, then edge between B and $A_1 \cup A_2$. Thus, $C(MG(R))$ has exactly two components $A_1 \cup A_2$ and B . \square

Theorem 4.5. *Let $J(R) \neq O_R$, with $V(C(MG(R))) = \text{Min}(R) \cup \text{Max}(R)$, and $\text{Min}(R) \cap \text{Max}(R) \neq \emptyset$. At that time $C(MG(R))$ is a connected graph and $\text{diam}(C(MG(R))) \leq 2$.*

Proof. Let $I, J \in V(C(MG(R)))$ with $I \neq J$.

If $I + J \notin \text{Max}(R)$, then they are adjacent in $C(MG(R))$.

Now, assume that $I + J \in \text{Max}(R)$.

Case1. $I \in \text{Max}(R)$ and $J \in \text{Min}(R)$.

In this case, $J \subset I, I \notin \text{Min}(R)$ and $J \notin \text{Max}(R)$.

Since $\text{Min}(R) \cap \text{Max}(R) \neq \emptyset$, we have $S \in \text{Max}(R) \cap \text{Min}(R)$. The minimality of S gives that $J \not\subset S$. It follows from the maximality of S and Lemma 3.1 that J and S are adjacent in $C(MG(R))$. Also, I and S are adjacent vertices in $C(MG(R))$. Thus, (I, S, J) is a path in $C(MG(R))$. Similarly, $J \in \text{Max}(R)$.

Similarly, a path of length 2 can be obtained in $C(MG(R))$, when $I \in \text{Min}(R)$ and $J \in \text{Max}(R)$.

Case2. $I, J \in \text{Min}(R)$. In the same way of Case1, we get a path (I, S, J) in $C(MG(R))$.

From both cases, we have shown that $C(MG(R))$ is connected and $\text{diam}(C(MG(R))) \leq 2$. \square

From Theorem 4.4, the graph $C(MG(R))$ may be connected. The next result determines the upper bound of $\text{diam}(C(MG(R)))$ whenever it is connected.

Theorem 4.6. *Let $\text{Max}(R) = \{P_1, P_2\}$, with $P_1 \neq P_2$. If $C(MG(R))$ is connected graph, then $\text{diam}(C(MG(R))) \leq 4$.*

Proof. Let $C(MG(R))$ be a connected graph.

If $V(C(MG(R))) = \text{Max}(R)$, then $\text{diam}(C(MG(R))) = 1$.

Assume that $I \notin \text{Max}(R)$, is a vertex in $C(MG(R))$.

If $P_1 \cap P_2 = O_R$, then by Theorem 4.3, $\text{diam}(C(MG(R))) \leq 3$.

Now, assume that $P_1 \cap P_2 \neq O_R$.

Since $C(MG(R))$ is connected, by Theorem 4.1, we have $V(C(MG(R))) \neq \text{Max}(R) \cup \text{Min}(R)$.

First, $P_1 \cap P_2$ is not adjacent to any of P_1 and P_2 in $C(MG(R))$. Then there exists a vertex K adjacent to $P_1 \cap P_2$.

If $K \not\subset P_1 \cap P_2$, then $K \not\subset P_i$ for some $i = 1, 2$. Consequently, $K \subset P_j, j \neq i$.

Thus, $(P_1 \cap P_2, K, P_i)$ and $(P_1 \cap P_2, K, P_j)$ are paths in $C(MG(R))$.

Hence, $d(P_1 \cap P_2, P_i) = 2$ and $d(P_1 \cap P_2, P_j) = 3$.

Now, assume that $K \subseteq P_1 \cap P_2$. Since $C(MG(R))$ is connected, by Theorem 4.5, there exists a vertex $L \notin \text{Max}(R)$ such that $L \not\subseteq P_i$ for some $i = 1, 2$.

As in the previous case, $d(P_1 \cap P_2, P_i) = 2$ and $d(P_1 \cap P_2, P_j) = 3$, when $L + P_1 \cap P_2 \notin \text{Max}(R)$.

Now, let $L + P_1 \cap P_2 \in \text{Max}(R)$.

If $K + L \notin \text{Max}(R)$, then $(P_1 \cap P_2, K, L, P_i, P_j)$ is a path in $C(MG(R))$; $i \neq j$. So, $d(P_1 \cap P_2, P_j) = 4$ for $j = 1, 2$.

Similarly, if $K + L \notin \text{Max}(R)$, we can prove that $\text{diam}(C(MG(R))) \leq 4$.

In the similar way, if $K + L \in \text{Max}(R)$, then $\text{diam}(C(MG(R))) \leq 4$ is it proven. \square

The final result, demonstrates that $C(MG(R))$ is a connected by using certain conditions.

Theorem 4.7. *If $V(C(MG(R))) \neq \text{Min}(R) \cup \text{Max}(R)$ and $|\text{Max}(R)| \geq 3$, then $C(MG(R))$ is a connected graph with $\text{diam}(C(MG(R))) \leq 3$.*

Proof. Let $I, J \in V(C(MG(R)))$ with $I \neq J$.

If $I + J \notin \text{Max}(R)$, then they are adjacent in $C(MG(R))$. Assume that $I + J \in \text{Max}(R)$. If $J(R) = O_R$, then by Theorem 4.1 (2), the graph $C(MG(R))$ is connected.

Now, suppose that $J(R) \neq O_R$. We have two cases for I and J :

Case1. $I \in \text{Max}(R)$ and $J \notin \text{Max}(R)$

In this case, we have $J \subset I$. Now, we give two distinct maximal ideals $M_1, M_2 \neq I$.

Subcase1. $J \not\subseteq J(R)$. In this case, $J \not\subseteq M_1$, for some $M_1 \in \text{Max}(R)$. Then by Lemma 3.1, (I, M_1, J) is a path in $C(MG(R))$.

Subcase2. $J \subseteq J(R)$. First, we know that $J \neq J(R)$ and $J(R) \subset M_1 \cap M_2$. From Lemma 3.1, J is adjacent to $M_1 \cap M_2$ in $C(MG(R))$.

From the properties of maximal ideals, we have $M_1 \cap M_2 \not\subseteq I$. Thus, $M_1 \cap M_2$ is adjacent to I in $C(MG(R))$. In this case, $(J, M_1 \cap M_2, I)$ is a path in $C(MG(R))$.

Case2. $I, J \notin \text{Max}(R)$

In this case, none of I and J contained in $J(R)$. Suppose that $I + J = M$, where $M \in \text{Max}(R)$.

If $I, J \not\subseteq J(R)$, then $I \not\subseteq N$ and $J \not\subseteq P$ for some $N, P \in \text{Max}(R) \setminus \{M\}$. If $N = P$, then (I, N, J) is a path in $C(MG(R))$. Otherwise, (I, N, P, J) is a path in $C(MG(R))$.

Now, Suppose that $J \subseteq J(R)$ but $I \not\subseteq J(R)$. Then $I \not\subseteq Q$ for some $Q \in \text{Max}(R)$.

In the same sense of subcase2, we can determine two distinct maximal ideals $S, T \neq Q$ such that $(J, Q \cap T, S, I)$ is a path in $C(MG(R))$.

From all cases, we have proved that $C(MG(R))$ is joined such that $\text{diam}(C(MG(R))) \leq 3$. \square

5. CONCLUSIONS

In this article, we defined and studied a new type of graphs associated with commutative rings, called the complement of the maximal ideal graph of R , indicated by $C(MG(R))$. We investigated numerous important faces of this graph, including vertex degrees, chromatic, clique, and independence numbers. Besides, we studied its connectivity, planarity, Eulerian, and determined an upper bound for its diameter.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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