A SPECIAL CASE OF A BIVARIATE COPULA: PROPERTIES, DEPENDENCE MEASURES, AND CLIMATE DATA APPLICATIONS

ROLLY N. APDO[1,*], ROLANDO N. PALUGA[2]

[1]Department of Secondary Education, Caraga State University, Philippines

[2]Department of Mathematics, Caraga State University, Philippines

[*]Corresponding author: rnapdo@carsu.edu.ph

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Abstract. Copulas are commonly used to represent the interdependencies among multiple random variables. Kim and Sungur (2004) proposed one copula, known as Kim and Sungur's copula which illustrates the relationship between two random variables. This study introduced a special case of Kim and Sungur's copula called the KSR copula. Some properties of the KSR copula were investigated which include the density, measures of dependence, and tail dependences. The measures of dependence including Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, Blomqvist's beta, and Schweizer and Wolff's Sigma were obtained. The Spearman's rho of the KSR copula ranged from -0.3333 to 0.3333 and Kendall's tau ranged from -0.2222 to 0.2222. The Maximum Likelihood Estimation through inversion of Kendall's tau in parameter estimation was applied. Additionally, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were employed to determine the best joint distribution of monthly rainfall and temperature in the Philippines. The findings indicated that the KSR copula offered a superior fit when compared to the other existing bivariate copulas.

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1. Introduction

One of the central research areas in probability and statistics is the study of the interdependence among multiple random variables. Traditionally, dependence has been examined using correlation analysis, which provides a numerical summary of the strength of association between a dependent variable and one or more independent variables. While correlation measures are useful, they offer only a limited, single–number summary and fail to capture the full structure and nature of dependence, particularly in complex or non-linear relationships [1]. To address these limitations, copula functions have emerged as a powerful framework for modeling and analyzing the dependence structure among

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random variables independently of their marginal distributions. Copulas allow for a more flexible and comprehensive description of interdependence, making them especially valuable in multivariate statistical modeling.

In the 1940s, the concept of copula began with Fréchet discovering the bounds of bivariate copula [2]. In 1959, Sklar built a theorem proposing that a joint distribution function can be expressed as a copula and marginal distributions. This means that the entire process of modeling a joint distribution can be simplified into modeling copulas. Additionally, Sklar proposed that a copula captures the dependence of the variable, which comes as a result of separating the joint distribution into a copula and marginals [3], [4]. The copula is the dependence function, and thus measures the dependence between two variables. Sklar also proposed that any measure of scale-invariant dependence can be defined in terms of the copula.

Copula is defined as a function which joins or couples a multivariate distribution function to its one-dimensional marginal distribution functions. It consists of a powerful tool to model dependence structures [5]. As Fisher (1997) notes in the Encyclopedia of Statistical Sciences, copulas are of interest to statisticians for two main reasons. First, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions [6]. Copulas have been recognized as key tools to analyze dependence structures in finance and other areas [7], [8]. The copula based measures of dependence includes Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, Blomqvist's beta, Schweizer and Wolff's Sigma, and lower and upper tail dependence that describe the concordance between extreme values of the variables. This is why, copula modeling has become an active research area.

Important applications of copulas include the financial and insurance like in control of risk clustering, pricing and hedging of credit sensitive instruments, particularly n-th to default credit derivatives and Collateralized Debt Obligations (CDOs), pricing and hedging of basket derivatives and structured products, credit portfolio management, credit and market risk measurement [9]. Also applied in deriving the best joint distribution between rainfall and temperature for monsoon seasons and months [10].

One of the parametric copulas proposed the last few years was introduced by Kim and Sungur (2004) known as the Kim and Sungur's copula which is a class of bivariate copulas [11]. This copula presents a general form that cannot be compared to other existing copulas applied to real data. Hence, in order to apply this copula in modelling, a special case of Kim and Sungur's copula has constructed and obtained the properties which are the density, Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, Blomqvist's beta, Schweizer and Wolff's Sigma, and lower and upper tail dependence. The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are the two most common criteria for copula model selection. These criteria are used in comparison to check copula fitting with the data at hand.

This study is organized as follows: Section 2 discusses the preliminaries which include bivariate copula, measures of dependence, tail dependence, and canonical maximum likelihood estimation and copula model selection; Section 3 discusses the main results which introduce the KSR copula and its properties; Section 4 discusses results in model fitting; and Section 5 presents the conclusion of the study.

2. Preliminaries

The definition and properties of copula are presented in this section. Particularly, the bivariate copula is defined. Finally, the formulas of the dependence measures and tail dependence are given.

2.1. Bivariate Copula.

Copula is used to model dependence between two or more variables. The copula that model the interdependence of only two variables is called bivariate copula. The definition is given as follows.

Definition 1. [12] *A **bivariate copula** is a function $C : [0,1]^2 \to [0,1]$ with the following properties:*

(*i.*) *For every $u, v \in [0,1] : C(u,0) = C(0,v) = 0$;*

(*ii.*) *For every $u, v \in [0,1] : C(u,1) = u$ and $C(1,v) = v$; and*

(*iii.*) *For every $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \le u_2$ and $v_1 \le v_2$:*

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0.$$

Theorem 2.1. [12] *Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then there exists a copula C such that $H(x,y) = C(F(x), G(y))$. Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with marginals F and G. Furthermore, if F and G are continuous, C is unique.*

Sklar's representation of H provides a useful way to model the joint behavior of X and Y by choosing C, F and G from appropriate parametric families. It is useful because the marginal distributions and the copula need not belong to the same family of distributions; they can be symmetric or skewed, continuous or discrete, fat-tailed or thin-tailed [12].

Definition 2. [13] *The **density** of the copula C is given by*

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}.$$

Kim and Sungur (2004) develop a new class of bivariate copulas given by $C(u,v) = C^*(u,v) + \theta f(u) g(v)$ where f and g are two real functions, C^* is a copula and $\Delta^* = C^*(u_1,v_1) - C^*(u_1,v_2) - C^*(u_2,v_1) + C^*(u_2,v_2)$ for all $u, v, u_1, u_2, v_1, v_2 \in [0,1]$ and $u_1 \le u_2$, $v_1 \le v_2$ [9].

The next theorem is a consequence of Kim and Sungur's copula if C^* is the independent copula defined by $C^*(u,v) = uv$. It can be shown that $\Delta^* = (u_2 - u_1)(v_2 - v_1)$.

Theorem 2.2. [14] *Let $f, g : [0,1] \to \mathbb{R}$ be nonzero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$. Then, the **Kim and Sungur's copula** is given by*

$$C(u,v) = uv + \theta f(u)g(v)$$

for $\dfrac{-1}{max\{\alpha\gamma, \beta\delta\}} \leq \theta \leq \dfrac{-1}{min\{\alpha\delta, \beta\gamma\}}$ *and* $min\{\alpha\delta, \beta\gamma\} \geq -1$ *where*
$\alpha = inf\{f'(u) : u \in A\} < 0$, $\beta = sup\{f'(u) : u \in A\} > 0$,
$\gamma = inf\{g'(v) : v \in B\} < 0$, $\delta = sup\{g'(v) : v \in B\} > 0$,
$A = \{0 \leq u \leq 1 : f'(u) \text{ exists}\}$ *and* $B = \{0 \leq v \leq 1 : g'(v) \text{ exists}\}$.

2.2. **Measures of Dependence of Copula.** There are six common copula based measures of dependence which are based on the notion of concordance. The definitions are given below.
Kruskal (1958) defined the Spearman's rho as the difference between probabilities of concordance and discordance between $(X_1, Y_1), (X_2, Y_2) \in (X, Y)$ with common joint distribution function H, copula C, and margins F and G [15]. That is, $\rho c = 3(P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0])$ which yields the expression in the following definition (see [16] for the derivation).

Definition 3. [13] *Let X and Y be continuous random variables whose copula is C. Then, the **Spearman's rho** for X and Y is given by*

$$\rho c = 12 \int_0^1 \int_0^1 C(u,v)dudv - 3.$$

Cherubini et. al. (2004) defined the Kendall's tau as the difference between probabilities of concordance and discordance between $(X_1, Y_1), (X_2, Y_2) \in (X, Y)$ [9]. That is, $\tau c = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$ which yields the expression in the following definition (see [16] for the derivation).

Definition 4. [13] *Let X and Y be continuous random variables whose copula is C. Then, the **Kendall's tau** for X and Y is given by*

$$\tau c = 4 \int_0^1 \int_0^1 c(u,v)C(u,v)dudv - 1.$$

Definition 5. [13] *Let X and Y be continuous random variables whose copula is C. Then, the **Spearman's footrule** for X and Y is given by*

$$\varphi c = 6 \int_0^1 C(u,u)du - 2.$$

Definition 6. [13] *Let X and Y be continuous random variables whose copula is C. Then, the **Gini's gamma** for X and Y is given by*

$$\gamma c = 4 \int_0^1 [C(u,u) + C(u,1-u)]du - 2.$$

The Spearman's footrule and Gini's gamma are considered as an alternative of the Spearman's rho and Kendall's tau.

Definition 7. [17] *Let X and Y be continuous random variables whose copula is C. Then, the **Blomqvist's beta** for X and Y is given by*

$$\beta = -1 + 4C\Big(\frac{1}{2}, \frac{1}{2}\Big).$$

Blomqvist's beta provides an easy computation compared with the other measures. It becomes more preferable to use especially in dealing to copulas with complex form.

Definition 8. [18] *Let X and Y be continuous random variables whose copula is C. Then, the **Schweizer and Wolff's Sigma** for X and Y is given by*

$$\sigma = 12 \int_0^1 \int_0^1 |C(u,v) - uv| du dv.$$

The Schweizer and Wolff's Sigma and Spearman's rho have similarities. The difference is that the Schweizer and Wolff's Sigma reports the absolute difference between the copula and the product copula while Spearman's rho reports signed distance.

2.3. **Tail Dependence.** The concept of tail dependence relates to the amount of dependence in the upper-right quadrant tail and the lower-left-quadrant tail of a bivariate distribution. It is a concept that is relevant for the study of dependence between extreme values. It turns out that tail dependence between two continuous random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strictly increasing transformations of X and Y. The tail dependence describes the concordance between extreme values of random variables X and Y. It is a measure dependence in joint tail of bivariate distribution.

Definition 9. [13] *For a bivariate copula C, if*

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$$

*exists and $\lambda_U \in (0,1]$, then C has an **upper tail dependence** and **upper tail independence** if $\lambda_U = 0$.*

Definition 10. [13] *For a bivariate copula C, if*

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}$$

*exists and $\lambda_L \in (0,1]$, then C has a **lower tail dependence** and **lower tail independence** if $\lambda_L = 0$.*

According to Nelsen (2006), the parameter of asymptotic lower tail dependence, noted by λ_L, is the conditional probability in the limit that one variable takes a very low value, given that the other also takes a very low value. Similarly, the parameter of the asymptotic upper tail dependence λ_U, is the conditional probability in the limit that one variable takes a very high value, given that the other also takes a very high value. The variables X and Y are said to be asymptotically independent if $\lambda_L = \lambda_U = 0$.

2.4. **Estimating θ: Canonical Maximum Likelihood Estimation.** The following concepts were taken from [6].

Let (X, Y) be the pair of two continuous random variables with joint distribution H, copula C, and margins F and G, respectively. Let the sample of size n with $X_1, X_2, ..., X_n$ be the random samples from a variable X and $Y_1, Y_2, ..., Y_n$ be the random samples from a variable Y. Then we have the pairs of random samples $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ for (X, Y). Further let R_i represents the rank of X_i among $X_1, X_2, ..., X_n$ and S_i is the rank of Y_i among $Y_1, Y_2, ..., Y_n$. Then we obtain pairs of ranks $(R_1, S_1), (R_2, S_2), ..., (R_n, S_n)$ associated with $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$. Since both variables X and Y are continuous, no ties exist implying that the ranks are well-defined.

The pairs of ranks are then used in obtaining the maximum value of the log-likelihood function for the parameter θ. We define it below.

Definition 11. [17] *Suppose that the copula C associated with (X, Y) has a density c. If the margins F and G are known, then the* ***log-likelihood function*** *for θ is given by*

$$l(\theta) = \sum_{i=1}^{n} \ln \left[c\Big(F(X_i), G(Y_i)\Big) \right].$$

The margins F and G are rarely known in practice. Oakes (1994) suggested to simply replace the margins F and G with their empirical versions. Genest et al. (1995) suitably modified the idea and employed the rescaled versions of the empirical margins given by

$F_n(x) = \frac{1}{n+1} \sum_{i=1}^{n} 1(X_i \leq x)$ and $G_n(y) = \frac{1}{n+1} \sum_{i=1}^{n} 1(Y_i \leq y)$

for all $x, y \in \mathbb{R}$ where $1(A)$ is the indicator function of event A. The rescaled versions of the empirical margins can be reduced to $F_n(x) = \frac{R_i}{n+1}$ and $G_n(y) = \frac{S_i}{n+1}$ for all $i \in \{1, 2, ..., n\}$ where R_i and S_i are already defined as ranks for X_i and Y_i, respectively. The version for log-likelihood function when F and G are not known is define below.

Definition 12. [17] *If margins the F and G are not known, then the* ***log pseudo-likelihood function*** *for θ is used that is given by*

$$l^*(\theta) = \sum_{i=1}^{n} \ln \left[c\Big(\frac{R_i}{n+1}, \frac{S_i}{n+1}\Big) \right].$$

The canonical maximum likelihood estimator of θ is θ_n which is the value for bivariate data with sample size n that maximizes the log pseudo-likelihood function l^*.

There are instances that the copula has no density. Meaning, maximization of the log pseudo-likelihood l^* is computationally intensive and estimating θ is very difficult. However, there are ways to provide an appropriate starting value for log pseudo-likelihood estimation. One is the inversion of measures of dependence. The most popular are the inversions of Kendall's tau and Blomqvist's beta because their forms are often very clear.

For the inversion of Kendall's tau, the Kendall's tau of copula C is first calculated and then equate the parameter θ in terms of Kendall's tau. For example, if the Kendall's tau of copula C is $\tau c = \dfrac{\theta}{3}$, then $\theta = 3\tau c$. The estimate for parameter θ is $\theta_n = 3\tau c_n$. If the Kendall's tau of the bivariate data of sample size n is $\tau c_n = 0.1$, then $\theta_n = 3(0.1) = 0.3$ which is now the estimate for θ and is used to maximize the log pseudo-likelihood function l^*.

The asymptotic behaviour of θ_n can be derived through the properties of Kendall's tau. The propositions are presented in the next page.

Proposition 1. [17] *If $n \to \infty$, then $n^{1/2}(\tau c_n - \tau c)$ converges weakly to a centered Gaussian random variable with variance*

$$\sigma^2_{\tau c} = 16\Big\{ \int_0^1 \int_0^1 \Big[C(u,v) + \overline{C}(u,v)\Big]^2 dC(u,v) - \Big[\int_0^1 \int_0^1 \Big\{C(u,v) + \overline{C}(u,v)\Big\} dC(u,v)\Big]^2 \Big\}$$

where $\overline{C}(u,v) = 1 - u - v + C(u,v)$.

The function $\overline{C}(u,v) = 1 - u - v + C(u,v)$ is called the survival function. It is important to note that τc and τc_n represent the population value and sample estimate of Kendall's tau, respectively.

Proposition 2. [17] *If C is a copula with Kendall's tau correlation τc, then $n^{1/2}(\theta_n - \theta)$ converges in distribution, as $n \to \infty$, to a Gaussian random variable with mean zero and variance $\{g'(\tau c)\}^2 \sigma^2_{\tau c}$.*
Here, the function $g(\tau c) = \theta$.

For the inversion of Blomqvist's beta, same process in the inversion of Kendall's tau is applied. It consists of solving β_n which is the Blomqvist's beta for bivariate data with sample size n. Then, θ_n of β_n provides an estimate for the parameter θ.

The asymptotic behaviour of θ_n can be derived through the properties of Blomqvist's beta. The propositions are presented below.

Proposition 3. [17] *If $n \to \infty$, then $n^{1/2}(\beta_n - \beta)$ converges weakly to a centered Gaussian random variable with variance*

$$\sigma^2_{\beta} = 16\Big[C(\tfrac{1}{2},\tfrac{1}{2})\Big\{1 - C(\tfrac{1}{2},\tfrac{1}{2})\Big\} + \frac{1}{4}\Big\{C_1(\tfrac{1}{2},\tfrac{1}{2}) - C_2(\tfrac{1}{2},\tfrac{1}{2})\Big\}^2 + C(\tfrac{1}{2},\tfrac{1}{2})\Big\{ - C_1(\tfrac{1}{2},\tfrac{1}{2}) - C_2(\tfrac{1}{2},\tfrac{1}{2}) + 2C_1(\tfrac{1}{2},\tfrac{1}{2})C_2(\tfrac{1}{2},\tfrac{1}{2})\Big\}\Big].$$

The functions $C_1(u,v) = \dfrac{\partial C(u,v)}{\partial u}$ and $C_2(u,v) = \dfrac{\partial C(u,v)}{\partial v}$. Note that β and β_n represent the population value and sample estimate of Blomqvist's beta, respectively.

Proposition 4. [17] *If C is a copula with Blomqvist's beta β, then $n^{1/2}(\theta_n - \theta)$ converges in distribution, as $n \to \infty$, to a Gaussian random variable with mean zero and variance $\{g'(\beta)\}^2\sigma_\beta^2$. The function $g(\beta) = \theta$.*

The inversion of other measures of dependence can also be used aside from the mentioned two measures. Using Blomqvist's beta is easy to compute compared to the Kendall's tau which may have an intensive computations. However, Kendall's tau outperformed Blomqvist's beta in parameter estimation through a simulation study in [17]. In this study, the inversion of Kendall's tau is used in estimating the parameter and obtaining the maximum log pseudo-likelihood value.

2.5. **Copula Model Selection.** The following concepts were taken from [16] and [19].

There are many copulas have been introduced and suggested. Nadarajah et. al. (2016) provides a compendium of copulas. The question is, given a dataset at hand, which among the copulas is the best fit copula with the data. Among the proposed approaches for copula model selection are Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Although, the criteria do not provide any understanding of the power of the decision rule employed but they provide the comparison of the fitting of different copulas.

Various simulation studies are done to identify the best criterion in copula selection. In the study of Fang et. al. (2013), AIC is superior to select the copula that provides the best fit. However, in the study of Shumway and Stoffer (2011), BIC performs better in large samples while AIC performs better in small samples. In this study, both criteria are used in selecting the best fit copula model.

In the previous section, the process in estimating the parameter that maximizes the log pseudo-likelihood is already discussed. The value of the log pseudo-likelihood is utilize to choose the best fit bivariate copula for the given dataset at hand among the candidate copula models. We define it below.

Definition 13. [16] *Let $l(\theta)$ be the maximum value of the log-likelihood and p is the number of free parameters. The Akaike Information Criterion (AIC) is defined as*

$$AIC = -2l(\theta) + 2p.$$

Definition 14. [16] *Let $l(\theta)$ be the maximum value of the log-likelihood and p is the number of free parameters and n is the sample size. The Bayesian Information Criterion (BIC) is defined as*

$$BIC = -2l(\theta) + p\log(n).$$

The version for log-likelihood which is the log pseudo-likelihood is used for the computation of the criteria values. The criteria simply state that the copula with smallest AIC or BIC value provides the best fit.

3. Special Case of Kim and Sungur's Copula

A special case of Kim and Sungur's copula called the KSR copula is defined in this chapter. Properties of the KSR copula like the density; measures of dependence such as Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, Blomqvist's beta, Schweizer and Wolff's Sigma; and lower and upper tail dependence are obtained and interpreted.

3.1. The KSR Copula.

Recall that the new class of bivariate copulas introduced by Kim and Sungur has the form $C(u,v) = C^*(u,v) + \theta f(u)g(v)$. Theorem 2.2 presented a special case of Kim and Sungur's copula defined by $C(u,v) = uv + \theta f(u)g(v)$ with some conditions. In this study, we specify margins $f(u) = au(u-1)$ and $g(v) = b^v + (1-b)v - 1$. We will show that with these margins, we can construct a special case called KSR copula with some conditions.

Lemma 1. *Let $b > 1, 0 < a \le \frac{1}{b\ln b - b + 1}, \frac{-1}{a(b\ln b - b+1)} \le \theta \le \frac{1}{a(b\ln b - b + 1)}$ and $C^{KSR}(u,v) = uv + \theta au(u-1)[b^v + (1-b)v - 1]$. Then, $0 \le C^{KSR}(u,v) \le 1$ on the set $D = [0,1]^2$.*

Proof. Let $b > 1, 0 < a \le \frac{1}{b\ln b - b + 1}, \frac{-1}{a(b\ln b - b+1)} \le \theta \le \frac{1}{a(b\ln b - b + 1)}$ and $C^{KSR}(u,v) = uv + \theta au(u-1)[b^v + (1-b)v - 1]$. Let $f(u) = au(u-1)$. Clearly, $au(u-1) \le 0$ for $u \in [0,1]$. Let $g(v) = b^v + (1-b)v - 1$. The v-intercepts of g are 0 and 1. This implies that $g(v)$ is either negative or positive value for $v \in (0,1)$. We pick $v = \frac{1}{2}$, so that $b^{\frac{1}{2}} + (1-b)\frac{1}{2} - 1 = (\sqrt{b} + \frac{1}{2}) - (b+1) < 0$. This means that $b^v + (1-b)v - 1 \le 0$ for $v \in [0,1]$. Hence, $au(u-1)[b^v + (1-b)v - 1] \ge 0$.

Let us consider the three cases for the values of θ - zero, positive and negative.

Case 1: For $\theta = 0$, we get $C^{KSR}(u,v) = uv$. Clearly, $0 \le C^{KSR}(u,v) \le 1$.

Case 2: For $0 < \theta \le \frac{1}{a(b\ln b - b+1)}$, we choose $\theta = \frac{1}{a(b\ln b - b + 1)}$. Then

$$C^{KSR}(u,v) = uv + \frac{u(u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1}.$$

Taking the first derivatives with respect to u and v, we have

$$C_u{}^{KSR}(u,v) = v + \frac{(2u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1} \text{ and}$$

$$C_v{}^{KSR}(u,v) = u + \frac{u(u-1)(b^v \ln b - b + 1)}{b\ln b - b + 1},$$

respectively. To obtain the critical points $(u,v) \in D$, we set $C_u{}^{KSR}(u,v) = 0$ and $C_v{}^{KSR}(u,v) = 0$ and solve for u and v. Now,

$$v + \frac{(2u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1} = 0 \text{ gives } v = 0.$$

Substituting $v = 0$ to

$$u + \frac{u(u-1)(b^v \ln b - b + 1)}{b\ln b - b + 1} = 0 \text{ gives } u = 0, \frac{(1-b)\ln b}{\ln b - b + 1}.$$

But $u = \frac{(1-b)\ln b}{\ln b - b + 1} > 1$ for $b > 1$. This means that $(0,0)$ is the only critical point in D. Plugging in to the function, we have $C^{KSR}(0,0) = 0$. Next, we investigate the values of C^{KSR} on the boundaries of D.

• On $v = 0$ and $0 \leq u \leq 1$, we get $C^{KSR}(u, 0) = 0$.

• On $v = 1$ and $0 \leq u \leq 1$, we get $C^{KSR}(u, 1) = u$. The minimum is at $u = 0$ where $C^{KSR}(0, 1) = 0$ and the maximum is at $u = 1$ where $C^{KSR}(1, 1) = 1$.

• On $u = 0$ and $0 \leq v \leq 1$, we get $C^{KSR}(0, v) = 0$.

• On $u = 1$ and $0 \leq v \leq 1$, we get $C^{KSR}(1, v) = v$. The minimum is at $v = 0$ where $C^{KSR}(1, 0) = 0$ and the maximum is at $v = 1$ where $C^{KSR}(1, 1) = 1$.

In this case, we have shown that the minimum value of $C^{KSR}(u, v)$ in D is 0 at points $(0, 0)$, $(0, 1)$ and $(1, 0)$ while the maximum value is 1 at point $(1, 1)$. Hence, $0 \leq C^{KSR}(u, v) \leq 1$.

Case 3: For $\frac{-1}{a(b\ln b - b + 1)} \leq \theta < 0$, we choose $\theta = \frac{-1}{a(b\ln b - b + 1)}$. Then

$$C^{KSR}(u, v) = uv - \frac{u(u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1}.$$

The first derivatives with respect to u and v are

$$C_u{}^{KSR}(u, v) = v - \frac{(2u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1} \text{ and}$$

$$C_v{}^{KSR}(u, v) = u - \frac{u(u-1)(b^v \ln b - b + 1)}{b\ln b - b + 1},$$

respectively. Setting $C_u{}^{KSR}(u, v) = 0$ and $C_v{}^{KSR}(u, v) = 0$, we have

$$v - \frac{(2u-1)[b^v + (1-b)v - 1]}{b\ln b - b + 1} = 0 \text{ which gives } v = 0.$$

Substituting $v = 0$ to

$$u - \frac{u(u-1)(b^v \ln b - b + 1)}{b\ln b - b + 1} = 0 \text{ gives } u = 0, \frac{(b+1)\ln b - 2b + 2}{\ln b - b + 1}.$$

But $u = \frac{(b+1)\ln b - 2b + 2}{\ln b - b + 1} < 0$ for $b > 1$. This means $(0, 0)$ is the only critical point in D and $C^{KSR}(0, 0) = 0$. Next, we investigate the values of C^{KSR} on the boundaries of D. Similar results in case 2 are found.

• On $v = 0$ and $0 \leq u \leq 1$, we get $C^{KSR}(u, 0) = 0$.

• On $v = 1$ and $0 \leq u \leq 1$, we get $C^{KSR}(u, 1) = u$. The minimum is at $u = 0$ where $C^{KSR}(0, 1) = 0$ and the maximum is at $u = 1$ where $C^{KSR}(1, 1) = 1$.

• On $u = 0$ and $0 \leq v \leq 1$, we get $C^{KSR}(0, v) = 0$.

• On $u = 1$ and $0 \leq v \leq 1$, we get $C^{KSR}(1, v) = v$. The minimum is at $v = 0$ where $C^{KSR}(1, 0) = 0$ and the maximum is at $v = 1$ where $C^{KSR}(1, 1) = 1$.

Hence, $0 \leq C^{KSR}(u, v) \leq 1$.

Since $0 \leq C^{KSR}(u, v) \leq 1$ for the maximum and minimum values of θ, the inequality holds for any $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. □

Theorem 3.1. *Let* $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, *and* $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$. *Then,*

$$C^{KSR}(u,v) = uv + \theta a u(u-1)[b^v + (1-b)v - 1]$$

is a copula.

Proof. Suppose $a > 0$ and $b > 0$. Let $f, g : [0,1] \to \mathbb{R}$ where $f(u) = au(u-1)$ and $g(v) = b^v + (1-b)v - 1$. Clearly, f and g are nonzero functions, where $f(0) = f(1) = g(0) = g(1) = 0$. Consequently, f and g are differentiable functions since $f'(u) = 2au - a$ and $g'(v) = (\ln b)b^v - b + 1$ exist for every $u, v \in [0,1]$.

Observe that f' is a polynomial function and g' is a sum of exponential and polynomial functions. Consequently, f' is continuous on $[0,1]$ and g' is continuous on $[0,1]$. Also, f and g are continuously differentiable functions. Thus, f and g are absolutely continuous functions. Next, we will show that the function C^{KSR} defined by

$$C^{KSR}(u,v) = uv + \theta a u(u-1)[b^v + (1-b)v - 1]$$

is a copula by satisfying the conditions in Theorem 2.2. First, we have $\dfrac{-1}{\max\{\alpha\gamma, \beta\delta\}} \leq \theta \leq \dfrac{-1}{\min\{\alpha\delta, \beta\gamma\}}$ where

$$\begin{aligned} A &= \{0 \leq u \leq 1 : f'(u) \text{ exists }\} \\ &= \{0 \leq u \leq 1 : 2au - a \text{ exists }\} \\ &= [0,1] \text{ and} \\ B &= \{0 \leq v \leq 1 : g'(v) \text{ exists }\} \\ &= \{0 \leq v \leq 1 : (\ln b)b^v - b + 1 \text{ exists }\} \\ &= [0,1]. \end{aligned}$$

Then,

$$\alpha = \inf\{2au - a : u \in [0,1]\}, \qquad \beta = \sup\{2au - a : u \in [0,1]\},$$
$$\gamma = \inf\{(\ln b)b^v - b + 1 : v \in [0,1]\}, \delta = \sup\{(\ln b)b^v - b + 1 : v \in [0,1]\}.$$

Observe that the first derivative of f' is $f''(u) = 2a > 0$ for all $u \in [0,1]$. Also, $g''(v) = (\ln b)^2 b^v > 0$ for all $v \in [0,1]$. Now, f' and g' are increasing on $[0,1]$. It follows that

$$\alpha = f'(0) = -a < 0, \qquad \beta = f'(1) = a > 0,$$
$$\gamma = g'(0) = \ln b - b + 1 < 0, \qquad \delta = g'(1) = b\ln b - b + 1 > 0.$$

Now,

$$\frac{-1}{\max\{-a(\ln b - b + 1), a(b\ln b - b + 1)\}} \leq \theta \leq \frac{-1}{\min\{-a(b\ln b - b + 1), a(\ln b - b + 1)\}}$$

and $\min\{-a(b\ln b - b + 1), a(\ln b - b + 1)\} \geq -1$.

For $\max\{-a(\ln b - b + 1), a(b\ln b - b + 1)\}$, we let

$$h(b) = (b\ln b - b + 1) - (-\ln b + b - 1) = (\ln b)(b+1) - 2b + 2.$$

Observe that h is continuous on $b > 0$. Taking the derivative of h, we have $h'(b) = \ln b + \frac{1}{b} - 1 \geq 0$ for all $b > 0$. Also, h is increasing on $b > 0$. Observe that $h(1) = 0$. Since h is continuous and increasing on $b > 0$, we have the following inequalities

$(\ln b)(b+1) - 2b + 2 > 0$ for all $b > 1$ which is equivalent to

$a(b\ln b - b + 1) > -a(\ln b - b + 1)$ and

$(\ln b)(b+1) - 2b + 2 < 0$ for all $0 < b < 1$ which is equivalent to

$-a(\ln b - b + 1) > a(b\ln b - b + 1)$.

The same reasoning applied for $\min\{-a(b\ln b - b + 1), a(\ln b - b + 1)\} \geq -1$ which gives

$-a(b\ln b - b + 1) < a(\ln b - b + 1)$ for all $b > 1$ and

$a(\ln b - b + 1) < -a(b\ln b - b + 1)$ for all $0 < b < 1$.

Here, we choose $b > 1$. Then,

$\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{-1}{-a(b\ln b - b + 1)}$ and $-a(b\ln b - b + 1) \geq -1$

which are equivalent to

$\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$ and $a \leq \dfrac{-1}{b\ln b - b + 1}$. In Lemma 3.1.1, we have shown that $0 \leq C^{KSR}(u,v) \leq 1$ for these conditions of θ and a.

Therefore,

$$C^{KSR}(u,v) = uv + \theta a u(u-1)[b^v + (1-b)v - 1]$$

for $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, and $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$ is a copula. □

The function $C^{KSR}(u,v) = uv + \theta a u(u-1)[b^v + (1-b)v - 1]$ is called the **KSR copula**.

Another special case can be constructed for $0 < b < 1$ which can be proven similarly from above theorem. But this study focuses only for $b > 1$. The next results present the properties of the KSR copula with the interpretations.

Theorem 3.2. *Let $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, and $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$. Then, the density of the KSR copula is given by*

$$c^{KSR}(u,v) = 1 + \theta a(2u-1)[(\ln b)b^v - b + 1].$$

Proof. Let $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, and $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$. The density of the KSR copula can be obtained by solving $\dfrac{\partial^2 C^{KSR}(u,v)}{\partial u \partial v}$. Taking the first partial derivative with respect to v, we have

$$\begin{aligned}\frac{\partial C^{KSR}(u,v)}{\partial v} &= \frac{\partial\{uv + \theta a u(u-1)[b^v + (1-b)v - 1]\}}{\partial v} \\ &= \frac{\partial[uv + \theta a u^2 b^v - \theta a u b^v + \theta a(1-b)u^2 v - \theta a(1-b)uv - \theta a u^2 + \theta a u]}{\partial v} \\ &= u + \theta a u^2(\ln b)b^v - \theta a u(\ln b)b^v + \theta a(1-b)u^2 - \theta a(1-b)u.\end{aligned}$$

Then, the partial derivative with respect to u of $\dfrac{\partial C^{KSR}(u,v)}{\partial v}$ is

$$\frac{\partial^2 C^{KSR}(u,v)}{\partial u \partial v} = \frac{\partial[u + \theta a u^2(\ln b)b^v - \theta a u(\ln b)b^v + \theta a(1-b)u^2 - \theta a(1-b)u]}{\partial u}$$

$$= 1 + 2\theta a(\ln b)ub^v - \theta a(\ln b)b^v + 2\theta a(1-b)u - \theta a(1-b)$$
$$= 1 + \theta a(2u-1)[(\ln b)b^v - b + 1].$$

Therefore,

$$c^{KSR}(u,v) = 1 + \theta a(2u-1)[(\ln b)b^v - b + 1]$$

is the density of the KSR copula. □

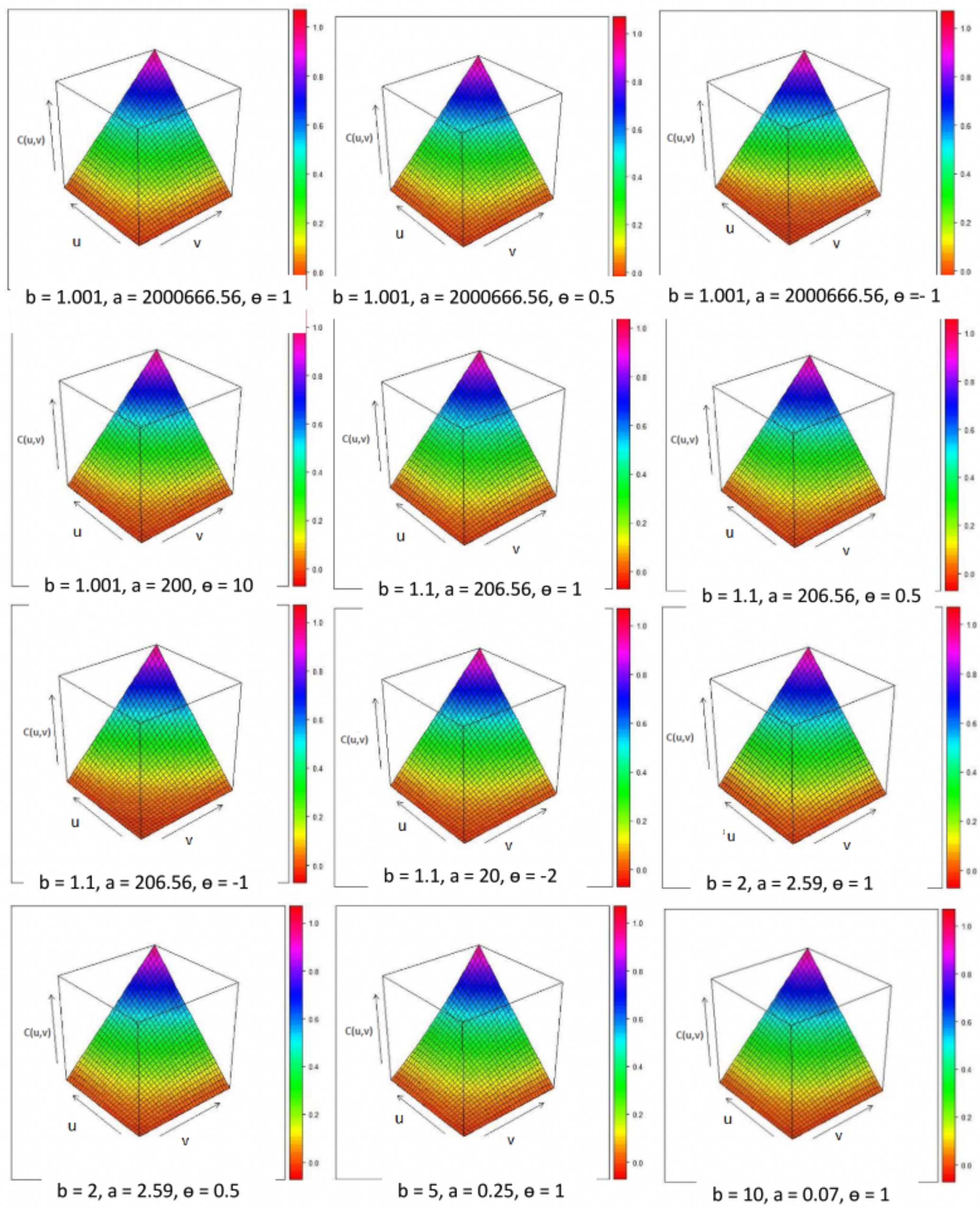


Figure 1. The KSR Copula

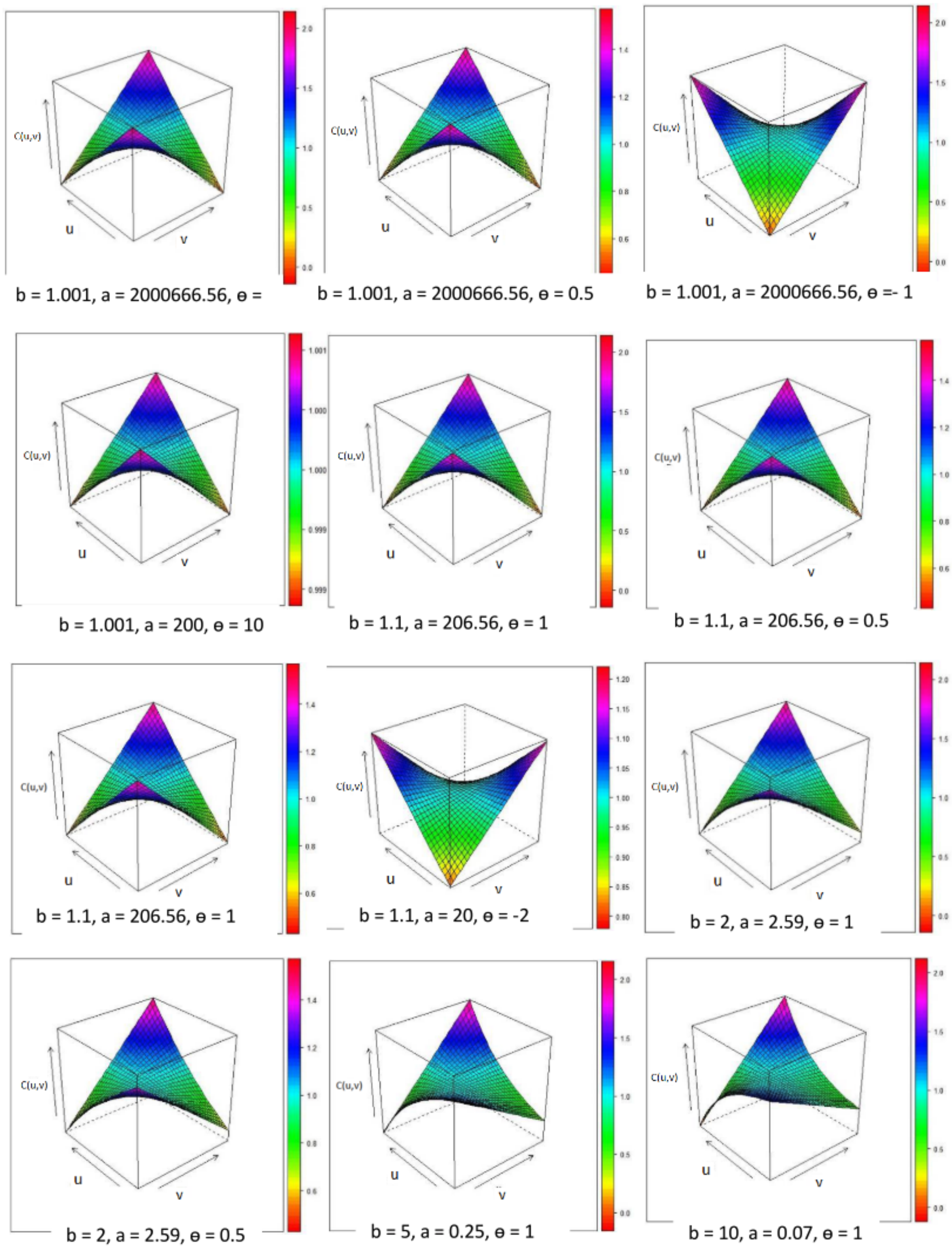


Figure 2. The Density of the KSR Copula

Figure 1 shows the graph of the KSR copula while Figure 2 shows its density. It specifies values for a, b and θ for graphical purposes. It is clear to see that the possible values of the KSR copula are within 0 and 1.

3.2. Measures of Dependence of the KSR Copula.

Theorem 3.3. *Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Then, the Spearman's rho of the KSR copula is given by*

$$\rho c^{KSR} = \theta a\Big[\frac{2(1-b)}{\ln b} + b + 1\Big].$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. The Spearman's rho can be obtained using the formula $\rho c^{KSR} = 12\int_0^1\int_0^1 C^{KSR}(u,v)dudv - 3$. First, we compute $\int_0^1 C^{KSR}(u,v)du$.

$$\begin{aligned}\int_0^1 C^{KSR}(u,v)du &= \int_0^1 [uv + \theta au^2b^v - \theta aub^v + \theta a(1-b)u^2v - \theta a(1-b)uv \\ &\quad -\theta au^2 + \theta au]du \\ &= \Big[\frac{u^2v}{2} + \frac{\theta au^3b^v}{3} - \frac{\theta au^2b^v}{2} + \frac{\theta a(1-b)u^3v}{3} - \frac{\theta a(1-b)u^2v}{2} \\ &= \frac{v}{2} - \frac{\theta ab^v}{6} - \frac{\theta a(1-b)v}{6} + \frac{\theta a}{6}.\end{aligned}$$

Second, we will take the double integral $\int_0^1\int_0^1 C^{KSR}(u,v)dudv$. Then, we have

$$\begin{aligned}\int_0^1\int_0^1 C^{KSR}(u,v)dudv &= \int_0^1 \Big[\frac{v}{2} - \frac{\theta ab^v}{6} - \frac{\theta a(1-b)v}{6} + \frac{\theta a}{6}\Big]dv \\ &= \Big[\frac{v^2}{4} - \frac{\theta a(b^v)}{6\ln b} - \frac{\theta a(1-b)v^2}{12} + \frac{\theta av}{6}\Big]\Big|_0^1 \\ &= \frac{1}{4} - \frac{\theta ab}{6\ln b} - \frac{\theta a(1-b)}{12} + \frac{\theta a}{6} + \frac{\theta a}{6\ln b}.\end{aligned}$$

Finally,

$$\begin{aligned}\rho c^{KSR} &= 12\Big[\frac{1}{4} - \frac{\theta ab}{6\ln b} - \frac{\theta a(1-b)}{12} + \frac{\theta a}{6} + \frac{\theta a}{6\ln b}\Big] - 3 \\ &= \theta a\Big[\frac{2(1-b)}{\ln b} + b + 1\Big].\end{aligned}$$

Therefore,

$$\rho c^{KSR} = \theta a\Big[\frac{2(1-b)}{\ln b} + b + 1\Big]$$

is the Spearman's rho of the KSR copula. □

Choosing b very close to 1 with any value of a and the maximum value of θ give the largest Spearman's rho of the KSR copula which is 0.3333. On the other hand, for b very close to 1 with any value of a and minimum value of θ give the smallest Spearman's rho of the KSR copula which is -0.3333. Further, choosing θ between its maximum and the minimum values gives a dependence measure between -0.3333 and 0.3333. But, the positive Spearman's rho values become smaller and the negative Spearman's rho values become bigger as b gets away from 1. This implies that KSR copula can be best fitted with data having a Spearman's rho within -0.3333 and 0.3333.

Theorem 3.4. *Let* $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$*, and* $\dfrac{-1}{a(b\ln b - b+1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b+1)}$*. Then, the Kendall's tau of the KSR copula is given by*

$$\tau c^{KSR} = \frac{2\theta a}{3}\Big[1 + b + \frac{2(1-b)}{\ln b}\Big].$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b+1}$, and $\frac{-1}{a(b\ln b - b+1)} \leq \theta \leq \frac{1}{a(b\ln b - b+1)}$. The Kendall's tau can be obtained using the formula

$\tau c^{KSR} = 4\int_0^1\int_0^1 c^{KSR}(u,v)C^{KSR}(u,v)dudv - 1$. Now,

$$\begin{aligned}
c^{KSR}(u,v)C^{KSR}(u,v) &= \Big[uv + \theta a u^2 b^v - \theta a u b^v + \theta a(1-b)u^2 v - \theta a(1-b)uv \\
&\quad -\theta a u^2 + \theta a u\Big]\Big[1 + \theta a(2u-1)[(\ln b)b^v - b + 1]\Big] \\
&= uv + \theta a u^2 b^v - \theta a u b^v + 3\theta a(1-b)u^2 v - 2\theta a(1-b)uv \\
&\quad -\theta a u^2 + \theta a u + 2\theta a(\ln b)u^2 v b^v + 2\theta^2 a^2(\ln b)u^3 b^{2v} \\
&\quad -2\theta^2 a^2(\ln b)u^2 b^{2v} + 2\theta^2 a^2(1-b)(\ln b)u^3 v b^v \\
&\quad -3\theta^2 a^2(1-b)(\ln b)u^2 v b^v - 2\theta^2 a^2(\ln b)u^3 b^v - \theta a(\ln b)uvb^v \\
&\quad -\theta^2 a^2(\ln b)u^2 b^{2v} + \theta^2 a^2(\ln b)ub^{2v} + \theta^2 a^2(1-b)(\ln b)uvb^v \\
&\quad +3\theta^2 a^2(\ln b)u^2 b^v - \theta^2 a^2(\ln b)ub^v + 2\theta^2 a^2(1-b)u^3 b^v \\
&\quad -3\theta^2 a^2(1-b)u^2 b^v + 2\theta^2 a^2(1-b)^2 u^3 v - 3\theta^2 a^2(1-b)^2 u^2 v \\
&\quad -2\theta^2 a^2(1-b)u^3 + 3\theta^2 a^2(1-b)u^2 + \theta^2 a^2(1-b)ub^v \\
&\quad +\theta^2 a^2(1-b)^2 uv - \theta^2 a^2(1-b)u.
\end{aligned}$$

Next is we take $\int_0^1 c^{KSR}(u,v)C^{KSR}(u,v)du$ and we get

$$\begin{aligned}
\int_0^1 c^{KSR}(u,v)C^{KSR}(u,v)du &= \Big[\frac{u^2 v}{2} + \frac{\theta a u^3 b^v}{3} - \frac{\theta a u^2 b^v}{2} + \theta a(1-b)u^3 v - \theta a(1-b)u^2 v \\
&\quad -\frac{\theta a u^3}{3} + \frac{\theta a u^2}{2} + \frac{2\theta a(\ln b)u^3 v b^v}{3} + \frac{\theta^2 a^2(\ln b)u^4 b^{2v}}{2} \\
&\quad -\frac{2\theta^2 a^2(\ln b)u^3 b^{2v}}{3} + \frac{\theta^2 a^2(1-b)(\ln b)u^4 v b^v}{2} \\
&\quad -\theta^2 a^2(1-b)(\ln b)u^3 v b^v - \frac{\theta^2 a^2(\ln b)u^4 v b^v}{2} \\
&\quad -\frac{\theta a(\ln b)u^2 v b^v}{2} - \frac{\theta^2 a^2(\ln b)u^3 b^{2v}}{3} + \frac{\theta^2 a^2(\ln b)u^2 b^{2v}}{2} \\
&\quad +\frac{\theta^2 a^2(1-b)(\ln b)u^2 v b^v}{2} + \theta^2 a^2(\ln b)u^3 b^v - \frac{\theta^2 a^2(\ln b)u^2 b^v}{2} \\
&\quad +\frac{\theta^2 a^2(1-b)u^4 b^v}{2} - \theta^2 a^2(1-b)u^3 b^v + \frac{\theta^2 a^2(1-b)^2 u^4 v}{2} \\
&\quad -\theta^2 a^2(1-b)^2 u^3 v - \frac{\theta^2 a^2(1-b)u^4}{2} + \theta^2 a^2(1-b)u^3 \\
&\quad +\frac{\theta^2 a^2(1-b)u^2 b^v}{2} + \frac{\theta^2 a^2(1-b)^2 u^2 v}{2} - \frac{\theta^2 a^2(1-b)u^2}{2}\Big]\Big|_0^1 \\
&= \frac{\theta a}{6} + \frac{v}{2} - \frac{\theta a b^v}{6} + \frac{\theta a(\ln b)vb^v}{6}.
\end{aligned}$$

Then, calculating the double integral $\int_0^1\int_0^1 c^{KSR}(u,v)C^{KSR}(u,v)dudv$ we have

$$\int_0^1\int_0^1 c^{KSR}(u,v)C^{KSR}(u,v)dudv = \int_0^1\Big[\frac{\theta a}{6} + \frac{v}{2} - \frac{\theta a b^v}{6} + \frac{\theta a(\ln b)vb^v}{6}\Big]dv$$

$$= \Big[\frac{\theta a v}{6} + \frac{v^2}{4} - \frac{\theta a b^v}{6\ln b} + \frac{\theta a(\ln b) v b^v}{6\ln b} - \frac{\theta a(\ln b) b^v}{6(\ln b)^2}\Big]\Big|_0^1$$
$$= \frac{1}{4} + \frac{\theta a}{6} + \frac{\theta a b}{6} - \frac{\theta a b}{3\ln b} + \frac{\theta a}{3\ln b}.$$

Finally,

$$\tau c^{KSR} = 4\Big(\frac{1}{4} + \frac{\theta a}{6} + \frac{\theta a b}{6} - \frac{\theta a b}{3\ln b} + \frac{\theta a}{3\ln b}\Big) - 1$$
$$= \frac{2\theta a}{3} + \frac{2\theta a b}{3} - \frac{4\theta a b}{3\ln b} + \frac{4\theta a}{3\ln b}$$
$$= \frac{2\theta a}{3}\Big[1 + b + \frac{2(1-b)}{\ln b}\Big].$$

Therefore,

$$\tau c^{KSR} = \frac{2\theta a}{3}\Big[1 + b + \frac{2(1-b)}{\ln b}\Big]$$

is the Kendall's tau of the KSR copula. □

The maximum and minimum Kendall's tau are 0.2222 and -0.2222, respectively. For the same trend as of Spearman's rho, the maximum Kendall's tau is obtained as b gets very close to 1 with any value of a and the maximum value of θ. The minimum Kendall's tau is obtained as b gets very close to 1 with any value of a and the minimum value of θ. For any values of the parameters, Kendall's tau values range from -0.2222 to 0.2222 implying that the KSR copula can be best fitted to bivariate data with Kendall's tau within -0.2222 and 0.2222.

The other measures of dependence follow the same trend of selection of parameter values in the Spearman's rho and Kendall's tau.

Theorem 3.5. *Let* $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, *and* $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. *Then, the Spearman's footrule of the KSR copula is given by*

$$\varphi c^{KSR} = \theta a\Big[\frac{b+1}{2} - \frac{6(b+1)}{(\ln b)^2} + \frac{12(b-1)}{(\ln b)^3}\Big].$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. The Spearman's footrule of the KSR copula can be obtained using the formula $\varphi c^{KSR} = 6\int_0^1 C^{KSR}(u,u)du - 2$. Observe that, $C^{KSR}(u,u) = u^2 + \theta a u^2 b^u + \theta a u b^u + \theta a(1-b)u^3 - \theta a(1-b)u^2 - \theta a u^2 + \theta a u$. Taking the first integral $\int_0^1 C^{KSR}(u,u)du$, we get

$$\int_0^1 C^{KSR}(u,u)du = \int_0^1 \Big[u^2 - \theta a u^2 b^u + \theta a u b^u + \theta a(1-b)u^3 - \theta a(1-b)u^2$$
$$-\theta a u^2 + \theta a u\Big] du$$
$$= \Big[\frac{u^3}{3} + \theta a\Big(\frac{u^2 b^u}{\ln b} - \frac{2ub^u}{(\ln b)^2} + \frac{2b^u}{(\ln b)^3}\Big) - \theta a\Big(\frac{ub^u}{\ln b} - \frac{b^u}{(\ln b)^2}\Big)$$
$$+\frac{\theta a(1-b)u^4}{4} - \frac{\theta a(1-b)u^3}{3} - \frac{\theta a u^3}{3} + \frac{\theta a u^2}{2}\Big]\Big|_0^1$$
$$= \frac{1}{3} - \frac{\theta a(b+1)}{(\ln b)^2} + \frac{2\theta a(b-1)}{(\ln b)^3} + \frac{\theta a(b+1)}{12}.$$

Then, we have

$$\begin{aligned}\varphi c^{KSR} &= 6\Big[\frac{1}{3} - \frac{\theta a(b+1)}{(\ln b)^2} + \frac{2\theta a(b-1)}{(\ln b)^3} + \frac{\theta a(b+1)}{12}\Big] - 2\\ &= \frac{-6\theta a(b+1)}{(\ln b)^2} + \frac{12\theta a(b-1)}{(\ln b)^3} + \frac{2\theta a(b+1)}{2}\\ &= \theta a\Big[\frac{b+1}{2} - \frac{6(b+1)}{(\ln b)^2} + \frac{12(b-1)}{(\ln b)^3}\Big].\end{aligned}$$

Therefore,

$$\varphi c^{KSR} = \theta a\Big[\frac{b+1}{2} - \frac{6(b+1)}{(\ln b)^2} + \frac{12(b-1)}{(\ln b)^3}\Big]$$

is the Spearman's footrule of the KSR copula. □

The Spearman's footrule of the KSR copula ranges from -0.1934 to 0.1934. This implies that using Spearman's footrule, KSR copula can be best fitted to bivariate data with Spearman's footrule within -0.1934 and 0.1934.

Theorem 3.6. *Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Then, the Gini's gamma of the KSR copula is given by*

$$\gamma c^{KSR} = 2\theta a\Big[\frac{b+1}{3} - \frac{4(b+1)}{(\ln b)^2} + \frac{8(b-1)}{(\ln b)^3}\Big].$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. The Gini's gamma can be obtained by using the formula,

$$\gamma c^{KSR} = 4\int_0^1 \Big[C^{KSR}(u,u) + C^{KSR}(u,1-u)\Big]du - 2.$$

Observe that

$$\begin{aligned}C^{KSR}(u,u) + C^{KSR}(u,1-u) &= \Big[u^2 + \theta a u^2 b^u - \theta a u b^u + \theta a(1-b)u^3 - \theta a(1-b)u^2\\ &\quad -\theta a u^2 + \theta a u\Big] + \Big[u(1-u) + \theta a u^2 b^{1-u} - \theta a u b^{1-u}\\ &\quad +\theta a(1-b)u^2(1-u) - \theta a(1-b)u(1-u) - \theta a u^2 + \theta a u\Big]\\ &= -\theta a(1+b)u^2 + \theta a u^2 b^u - \theta a u b^u + (1+\theta a+\theta a b)u\\ &\quad +\theta a u^2 b^{1-u} - \theta a u b^{1-u}.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_0^1 \Big[C^{KSR}(u,u) + C^{KSR}(u,1-u)\Big]du &= \int_0^1 \Big[-\theta a(1+b)u^2 + \theta a u^2 b^u - \theta a u b^u\\ &\quad +(1+\theta a+\theta a b)u + \theta a u^2 b^{1-u} - \theta a u b^{1-u}\Big]du\\ &= \Big[-\frac{\theta a(b+1)u^3}{3} + \theta a\Big(\frac{u^2 b^u}{\ln b} - \frac{2ub^u}{(\ln b)^2} + \frac{2b^u}{(\ln b)^3}\Big)\\ &\quad -\theta a\Big(\frac{ub^u}{\ln b} - \frac{b^u}{(\ln b)^2}\Big) + \frac{(1+\theta a+\theta a)u^2}{2}\\ &\quad -\theta a\Big(\frac{u^2 b^{1-u}}{\ln b} + \frac{2ub^{1-u}}{(\ln b)^2} + \frac{2b^{1-u}}{(\ln b)^3}\Big)\\ &\quad +\theta a\Big(\frac{ub^{1-u}}{\ln b} + \frac{b^{1-u}}{(\ln b)^2}\Big)\Big]\Big|_0^1\end{aligned}$$

$$= \frac{1}{2} + \frac{\theta a(b+1)}{6} - \frac{2\theta a(b+1)}{(\ln b)^2} + \frac{4\theta a(b-1)}{(\ln b)^3}.$$

It follows that,

$$\begin{aligned}\gamma c^{KSR} &= 4\Big[\frac{1}{2} + \frac{\theta a(b+1)}{6} - \frac{2\theta a(b+1)}{(\ln b)^2} + \frac{4\theta a(b-1)}{(\ln b)^3}\Big] - 2 \\ &= \frac{2\theta a(b+1)}{3} - \frac{8\theta a(b+1)}{(\ln b)^2} + \frac{16\theta a(b-1)}{(\ln b)^3} \\ &= 2\theta a\Big[\frac{b+1}{3} - \frac{4(b+1)}{(\ln b)^2} + \frac{8(b-1)}{(\ln b)^3}\Big].\end{aligned}$$

Therefore,

$$\gamma c^{KSR} = 2\theta a\Big[\frac{b+1}{3} - \frac{4(b+1)}{(\ln b)^2} + \frac{8(b-1)}{(\ln b)^3}\Big]$$

is the Gini's gamma of the KSR copula. □

The Gini's gamma of the KSR copula ranges from -0.1333 to 0.1333. This implies that using Gini's gamma, KSR copula can be best fitted to bivariate data with Gini's gamma within -0.1333 and 0.1333.

Theorem 3.7. *Let $b > 1$, $0 < a \leq \frac{1}{b\ln b - b + 1}$, and*

$$\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}.$$

Then, the Schweizer and Wolff's Sigma of the KSR copula is given by

$$\sigma^{\text{KSR}} = \begin{cases} \theta a(1+b) + \dfrac{2\theta a(1-b)}{\ln b}, & \text{if } 0 \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}, \\ -\theta a(1+b) - \dfrac{2\theta a(1-b)}{\ln b}, & \text{if } \dfrac{-1}{a(b\ln b - b + 1)} \leq \theta < 0. \end{cases}$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b+1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Observe that

$$\begin{aligned}C^{KSR}\Big(\frac{1}{2}, \frac{1}{2}\Big) &= \frac{1}{2}\Big(\frac{1}{2}\Big) + \theta a\Big(\frac{1}{2}\Big)\Big(\frac{1}{2} - 1\Big)\Big[b^{\frac{1}{2}} + (1-b)(\tfrac{1}{2}) - 1\Big] \\ &= \frac{1}{4} + \frac{\theta a}{8}(b - 2\sqrt{b} + 1).\end{aligned}$$

Plugging in $C^{KSR}\Big(\frac{1}{2}, \frac{1}{2}\Big)$ to the Blomqvist's beta formula, we have

$$\begin{aligned}\beta^{KSR} &= -1 + 4\Big[\frac{1}{4} + \frac{\theta a}{8}\Big(b - 2\sqrt{b} + 1\Big)\Big] \\ &= \frac{\theta a}{2}(b - 2\sqrt{b} + 1).\end{aligned}$$

Therefore,

$$\beta^{KSR} = \frac{\theta a}{2}(b - 2\sqrt{b} + 1)$$

is the Blomqvist's beta of the KSR copula. □

The Blomqvist's beta of the KSR copula ranges from -0.25 to 0.25 implying that using Blomqvist's beta, KSR copula can be best fitted to bivariate data with Blomqvist's beta within -0.25 and 0.25.

Theorem 3.8. *Let* $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, *and* $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$. *Then, the Schweizer and Wolff's Sigma of the KSR copula is given by*

$$\sigma^{KSR} = \begin{cases} \theta a(1+b) + \dfrac{2\theta a(1-b)}{\ln b} & \text{if } 0 \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)} \text{ and} \\ \\ -\theta a(1+b) - \dfrac{2\theta a(1-b)}{\ln b} & \text{if } \dfrac{-1}{a(b\ln b - b + 1)} \leq \theta < 0. \end{cases}$$

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Plugging in the KSR copula to the Schweizer and Wolff's Sigma formula, we have

$$\sigma^{KSR} = 12\int_0^1\int_0^1 |C^{KSR}(u,v) - uv|dudv$$
$$= 12\int_0^1\int_0^1 |\theta au(u-1)[b^v + (1-b)v - 1]|dudv$$

$$= \begin{cases} 12\int_0^1\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])dudv \\ \qquad \text{if } \theta au(u-1)[b^v + (1-b)v - 1] \geq 0 \text{ and} \\ \\ -12\int_0^1\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])dudv \\ \qquad \text{if } \theta au(u-1)[b^v + (1-b)v - 1] < 0. \end{cases}$$

Note that $0 \leq u, v \leq 1$ and $b > 1$. Clearly, $b^v + (1-b)v - 1 \leq 0$. Observe that $\theta au(u-1) \leq 0$ if $\theta \geq 0$ and $\theta au(u-1) > 0$ if $\theta < 0$. Then,

$$\sigma^{KSR} = \begin{cases} 12\int_0^1\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])dudv \\ \qquad \text{if } 0 \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)} \text{ and} \\ \\ -12\int_0^1\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])dudv \\ \qquad \text{if } \dfrac{-1}{a(b\ln b - b + 1)} \leq \theta < 0. \end{cases}$$

Taking the first definite integral, we get

$$\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])du = [b^v + (1-b)v - 1]\Big(\frac{\theta au^3}{3} - \frac{\theta au^2}{2}\Big)\Big|_0^1$$
$$= \frac{-\theta a}{6}[b^v + (1-b)v - 1].$$

Then,

$$\int_0^1\int_0^1 (\theta au(u-1)[b^v + (1-b)v - 1])dudv = \int_0^1 \frac{-\theta a}{6}[b^v + (1-b)v - 1]dv$$
$$= \frac{-\theta a}{6}\Big(\frac{b^v}{\ln b} - \frac{(1-b)v^2}{2} - v\Big)\Big|_0^1$$
$$= \frac{-\theta a}{6}\Big(\frac{b}{\ln b} - \frac{b}{2} - \frac{1}{2} - \frac{1}{\ln b}\Big).$$

Finally,

$$\sigma^{KSR} = \begin{cases} 12\Big[\frac{-\theta a}{6}\Big(\frac{b}{\ln b} - \frac{b}{2} - \frac{1}{2} - \frac{1}{\ln b}\Big)\Big] \\ \text{if } 0 \leq \theta \leq \frac{1}{a(b\ln b - b + 1)} \text{ and} \\ \\ -12\Big[\frac{-\theta a}{6}\Big(\frac{b}{\ln b} - \frac{b}{2} - \frac{1}{2} - \frac{1}{\ln b}\Big)\Big] \\ \text{if } \frac{-1}{a(b\ln b - b + 1)} \leq \theta < 0 \end{cases}$$

$$= \begin{cases} \theta a(1+b) + \frac{2\theta a(1-b)}{\ln b} & \text{if } 0 \leq \theta \leq \frac{1}{a(b\ln b - b + 1)} \text{ and} \\ \\ -\theta a(1+b) - \frac{2\theta a(1-b)}{\ln b} & \text{if } \frac{-1}{a(b\ln b - b + 1)} \leq \theta < 0. \end{cases}$$

Therefore,

$$\sigma^{KSR} = \begin{cases} \theta a(1+b) + \frac{2\theta a(1-b)}{\ln b} & \text{if } 0 \leq \theta \leq \frac{1}{a(b\ln b - b + 1)} \text{ and} \\ \\ -\theta a(1+b) - \frac{2\theta a(1-b)}{\ln b} & \text{if } \frac{-1}{a(b\ln b - b + 1)} \leq \theta < 0. \end{cases}$$

is the Schweizer and Wolff's Sigma of the KSR copula. □

The Schweizer and Wolff's Sigma of the KSR copula also ranges from -0.3333 to 0.3333 implying that using this measure of dependence, KSR copula can be best fitted to bivariate data with Schweizer and Wolff's Sigma within -0.3333 to 0.3333.

It is important to observe that KSR copula becomes independent copula if $\theta = 0$, that is, $C^{KSR}(u,v) = uv$. This means that the measure of dependence for KSR copula at $\theta = 0$ becomes independence.

3.3. **Tail Dependence of Copula of the KSR Copula.**

Theorem 3.9. *Let* $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, *and* $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. *Then, the KSR copula has an upper tail independence.*

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Then, we have

$$\begin{aligned} \lambda_U{}^{KSR} &= \lim_{u\to 1^-} \frac{1 - 2u + C^{KSR}(u,u)}{1-u} \\ &= \lim_{u\to 1^-} \Big[\frac{u^2 + \theta a u^2 b^u - \theta a u b^u + \theta a(1-b)u^3}{1-u} \\ &\qquad - \frac{\theta a(1-b)u^2 - \theta a u^2 + \theta a u - 2u + 1}{1-u}\Big] \\ &= \lim_{u\to 1^-} \Big[\frac{1 - u - \theta a u b^u - \theta a u(1-b)u^2 + \theta a u}{u-1}\Big](u-1) \\ &= \lim_{u\to 1^-} \Big[1 - u - \theta a u b^u - \theta a(1-b)u^2 + \theta a u\Big] \end{aligned}$$

$$= 0$$

Thus, C^{KSR} has an upper tail independence. □

Theorem 3.10. *Let $b > 1, 0 < a \leq \dfrac{1}{b\ln b - b + 1}$, and $\dfrac{-1}{a(b\ln b - b + 1)} \leq \theta \leq \dfrac{1}{a(b\ln b - b + 1)}$. Then, the KSR copula has a lower tail independence.*

Proof. Let $b > 1, 0 < a \leq \frac{1}{b\ln b - b + 1}$, and $\frac{-1}{a(b\ln b - b+1)} \leq \theta \leq \frac{1}{a(b\ln b - b + 1)}$. Then, we have

$$\begin{aligned}
{\lambda_L}^{KSR} &= \lim_{u\to 0^+} \frac{C^{KSR}(u,u)}{u} \\
&= \lim_{u\to 0^+} \Big[\frac{u^2 + \theta a u^2 b^u - \theta a u b^u + \theta a(1-b)u^3}{u} \\
&\quad - \frac{\theta a(1-b)u^2 - \theta a u^2 + \theta a u}{u}\Big] \\
&= \lim_{u\to 0^+} \Big[\frac{u + \theta a u b^u - \theta a b^u + \theta a(1-b)u^2}{u} \\
&\quad - \frac{\theta a(1-b)u - \theta a u + \theta a}{u}\Big](u) \\
&= \lim_{u\to 0^+} \Big[u + \theta a u b^u - \theta a b^u + \theta a(1-b)u^2 - \theta a(1-b)u \\
&\quad -\theta a u + \theta a\Big] \\
&= 0
\end{aligned}$$

Thus, C^{KSR} has a lower tail independence.

□

Since ${\lambda_L}^{KSR} = {\lambda_U}^{KSR} = 0$, the variables are said to be asymptotically independent. This means that there is no probability in the limit that one variable takes a very low value, given that the other also takes a very low value. Similarly, there is no probability in the limit that one variable takes a very high value, given that the other also takes a very high value.

3.4. **Estimating θ: The Inversion of Kendall's tau and Blomqvist's Beta of the KSR Copula.**

Theorem 3.11. *The asymptotic variance of the Kendall's tau of KSR copula is given by*

$$\begin{aligned}
\{g'(\tau c^{KSR})\}^2\sigma^2_{\tau c^{KSR}} &= \frac{9}{4a^2\Big[1 + b + \dfrac{2(1-b)}{\ln b}\Big]^2} \\
&\Big[\frac{8\theta^2a^2b^2}{15} + \frac{16\theta^2a^2b}{15} + \frac{8\theta^2a^2}{15} + \frac{8\theta ab}{3} + \frac{8\theta a}{3} - 16\Big(\frac{\theta ab}{6} + \frac{\theta a}{6} \\
&- \frac{1}{6\ln b}(-\theta a\ln b - 4\theta a) + \frac{b}{6\ln b}(\theta a \ln b - 4\theta a) + \frac{1}{2}\Big)^2 \\
&-16\Big(\frac{1}{13500\ln^4 b}(5400\theta^2a^2b\ln^2 b - 450\theta^2a^2\ln^4 b \\
&-4050\theta^2a^2\ln^3 b - 5400\theta^2a^2\ln^2 b - 2250\theta a)\Big)\Big].
\end{aligned}$$

Proof. For Kendall's tau , we compute

$$\sigma^2_{\tau c^{KSR}} = 16\Big\{ \int_0^1 \int_0^1 \Big[C^{KSR}(u,v) + \overline{C}^{KSR}(u,v)\Big]^2 dC^{KSR}(u,v)$$

$$-\Big[\int_0^1\int_0^1\{C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\}dC^{KSR}(u,v)\Big]^2\Big\}$$

where $C^{KSR}(u,v) = uv + \theta au(u-1)[b^v + (1-b)v - 1]$ and the survival function is $\overline{C}^{KSR}(u,v) = 1-u-v+uv+\theta au(u-1)[b^v+(1-b)v-1]$. Observe that

$$\Big\{C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\Big\}^2 = \Big\{2\theta au(u-1)[b^v+(1-b)v-1] \\ +2uv-u-v+1\Big\}^2.$$

Note that the density dC^{KSR} is given by

$$dC^{KSR}(u,v) = 1+\theta a(2u-1)[(\ln b)b^v - b + 1].$$

For the first double integral, we evaluate first

$$\int_0^1\Big[C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\Big]^2 dC^{KSR}(u,v)$$
$$=\int_0^1\Big\{2\theta au(u-1)[b^v+(1-b)v-1]+2uv-u-v+1\Big\}^2 \\ \Big\{1+\theta a(2u-1)[(\ln b)b^v-b+1]\Big\}dudv$$
$$=\frac{\theta^2a^2b^2v}{15}+\frac{2\theta^2a^2(\ln b)v^2b^{v+1}}{15}-\frac{\theta^2a^2(\ln b)vb^{v+1}}{15}-\frac{2\theta^2a^2vb^{v+1}}{15}-\frac{\theta^2a^2b^{v+1}}{15} \\ +\frac{\theta^2a^2b}{15}-\frac{2\theta^2a^2(\ln b)vb^{2v}}{15}+\frac{\theta^2a^2(\ln b)b^{2v}}{15}+\frac{2\theta^2a^2b^{2v}}{15}-\frac{2\theta^2a^2(\ln b)v^2b^v}{15} \\ +\frac{\theta^2a^2(\ln b)vb^v}{5}+\frac{2\theta^2a^2vb^v}{15}-\frac{\theta^2a^2(\ln b)b^v}{15}-\frac{\theta^2a^2b^v}{5}-\frac{\theta^2a^2v}{15}+\frac{\theta^2a^2}{15} \\ +\frac{2\theta ab}{6}+\frac{\theta a(\ln b)vb^v}{3}-\frac{\theta a(\ln b)b^v}{6}-\frac{\theta ab^v}{3}+\frac{2\theta a}{6}+\frac{v^2-v+1}{3}.$$

Then,

$$\int_0^1\int_0^1\Big[C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\Big]^2 dC^{KSR}(u,v)$$
$$=\frac{\theta^2a^2b^2}{30}+\frac{\theta^2a^2b}{15}+\frac{\theta^2a^2}{30}+\frac{\theta a}{6}-\frac{1}{3500\ln b}\Big(5400\theta^2a^2b\ln^2 b-450\theta^2a^2\ln^4 b \\ -4050\theta^2a^2\ln^3 b-5400\theta^2a^2\ln^2 b-2250\theta a\ln^4 b-9000\theta a\ln^3 b\Big) \\ +\frac{1}{13500\ln b}\Big(b^2(-450\theta^2a^2\ln^4 b+1350\theta^2a^2\ln^3 b)+b(900\theta^2a^2b\ln^4 b)\Big).$$

For the second double integral, we first evaluate

$$\int_0^1\{C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\}dC^{KSR}(u,v)$$
$$=\int_0^1\Big\{2\theta au(u-1)[b^v+(1-b)v-1]+2uv-u-v+1\Big\} \\ \Big\{1+\theta a(2u-1)[(\ln b)b^v-b+1]\Big\}dudv$$
$$=\frac{\theta ab}{6}+\frac{\theta a(\ln b)vb^v}{3}-\frac{\theta a(\ln b)b^v}{6}-\frac{\theta ab^v}{3}+\frac{\theta ab}{6}+\frac{1}{2}.$$

Then,

$$\int_0^1\int_0^1\{C^{KSR}(u,v)+\overline{C}^{KSR}(u,v)\}dC^{KSR}(u,v)$$
$$=\frac{\theta ab}{6}+\frac{\theta a}{6}-\frac{1}{6\ln b}(-\theta a\ln b-4\theta a)+\frac{b}{6\ln b}(\theta a\ln b-4\theta a)+\frac{1}{2}.$$

Substituting the obtained expressions to the formula, we get

$$\sigma^2_{\tau c^{KSR}} = \frac{8\theta^2a^2b^2}{15}+\frac{16\theta^2a^2b}{15}+\frac{8\theta^2a^2}{15}+\frac{8\theta ab}{3}+\frac{8\theta a}{3}-16\Big(\frac{\theta ab}{6}+\frac{\theta a}{6} \\ -\frac{1}{6\ln b}(-\theta a\ln b-4\theta a)+\frac{b}{6(\ln b)}(\theta a(\ln b)-4\theta a)+\frac{1}{2}\Big)^2 \\ -16\Big(\frac{1}{13500\ln^4 b}(5400\theta^2a^2b\ln^2 b-450\theta^2a^2\ln^4 b$$

$$-4050\theta^2a^2\ln^3 b - 5400\theta^2a^2\ln^2 b - 2250\theta a)\Big).$$

Note that $\tau c^{KSR} = \dfrac{2\theta a}{3}\Big[1+b+\dfrac{2(1-b)}{\ln b}\Big]$. Then, the resulting estimator for θ obtained by the inversion of Kendall's tau is defined as

$$\theta_n = \frac{3\tau c_n^{KSR}}{2a\Big[1+b+\dfrac{2(1-b)}{\ln b}\Big]}.$$

Given $g(\tau c^{KSR}) = \dfrac{3\tau c^{KSR}}{2a\Big[1+b+\dfrac{2(1-b)}{\ln b}\Big]}$, the asymptotic variance of θ_n is

$$\{g'(\tau c^{KSR})\}^2\sigma^2_{\tau c^{KSR}} = \frac{9}{4a^2\Big[1+b+\dfrac{2(1-b)}{\ln b}\Big]^2}$$
$$\Big[\frac{8\theta^2a^2b^2}{15}+\frac{16\theta^2a^2b}{15}+\frac{8\theta^2a^2}{15}+\frac{8\theta ab}{3}+\frac{8\theta a}{3}-16\Big(\frac{\theta ab}{6}+\frac{\theta a}{6}$$
$$-\frac{1}{6\ln b}(-\theta a\ln b-4\theta a)+\frac{b}{6\ln b}(\theta a\ln b-4\theta a)+\frac{1}{2}\Big)^2$$
$$-16\Big(\frac{1}{13500\ln^4 b}(5400\theta^2a^2b\ln^2 b-450\theta^2a^2\ln^4 b$$
$$-4050\theta^2a^2\ln^3 b-5400\theta^2a^2\ln^2 b-2250\theta a)\Big)\Big].$$

□

Theorem 3.12. *The asymptotic variance of the Blomqvist's beta of KSR copula is given by*

$$\{g'(\beta^{KSR})\}^2\sigma^2_{\beta^{KSR}} = \frac{4}{a^2(b-2\sqrt{b}+1)^2}\Big[\frac{\theta^2a^2(\sqrt{b}\ln b-b+1)^2}{4}$$
$$+2\theta a(\sqrt{b}-\frac{1}{2}b-\frac{1}{2})+\Big(-\sqrt{b}+\frac{1}{2}b+\frac{3}{2}\Big)\Big(\sqrt{b}-\frac{1}{2}b+\frac{5}{2}\Big)+2\Big].$$

Proof. For Blomqvist's beta, we compute

$$\sigma^2_{\beta^{KSR}} = 16\Big[C^{KSR}(\tfrac{1}{2},\tfrac{1}{2})\Big\{1-C^{KSR}(\tfrac{1}{2},\tfrac{1}{2})\Big\}+\frac{1}{4}\Big\{{C_1}^{KSR}(\tfrac{1}{2},\tfrac{1}{2})-{C_2}^{KSR}(\tfrac{1}{2},\tfrac{1}{2})\Big\}^2$$
$$+C^{KSR}(\tfrac{1}{2},\tfrac{1}{2})\Big\{-{C_1}^{KSR}(\tfrac{1}{2},\tfrac{1}{2})-{C_2}^{KSR}(\tfrac{1}{2},\tfrac{1}{2})+2{C_1}^{KSR}(\tfrac{1}{2},\tfrac{1}{2}){C_2}^{KSR}(\tfrac{1}{2},\tfrac{1}{2})\Big\}\Big]$$

where $C^{KSR}(u,v) = uv+\theta au(u-1)[b^v+(1-b)v-1]$,

$${C_1}^{KSR}(u,v) = v+2\theta aub^v-\theta ab^v+2\theta a(1-b)uv-\theta a(1-b)v-2\theta au+\theta a$$

and

$${C_2}^{KSR}(u,v) = u+\theta au^2((\ln b)b^v-\theta au(\ln b)b^v+\theta au(1-b)u^2-\theta a(1-b)u.$$

Then,

$$C^{KSR}(\tfrac{1}{2},\tfrac{1}{2}) = -\frac{1}{4}\Big(\sqrt{b}-\frac{b}{2}-\frac{1}{2}\Big),\ {C_1}^{KSR}(\tfrac{1}{2},\tfrac{1}{2}) = -\frac{1}{2}\ \text{and}$$
$${C_2}^{KSR}(\tfrac{1}{2},\tfrac{1}{2}) = -\frac{\theta a}{4}\Big(\sqrt{b}\ln b-b+1\Big)+\frac{1}{2}.$$

After substituting to the formula, we get

$$\sigma^2_{\beta^{KSR}} = \frac{\theta^2a^2(\sqrt{b}\ln b-b+1)}{4}+2\theta a\Big(\sqrt{b}-\frac{b}{2}-\frac{1}{2}\Big)$$
$$+\Big(-\sqrt{b}+\frac{b}{2}+\frac{3}{2}\Big)\Big(\sqrt{b}-\frac{b}{2}+\frac{5}{2})\Big)+2.$$

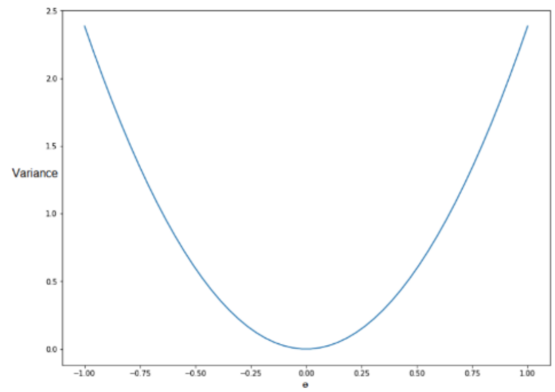
Note that $\beta^{KSR} = \dfrac{\theta a}{2}\Big(b-2\sqrt{b}+1\Big)$. Then, the resulting estimator for θ obtained by the inversion of Blomqvist's beta is defined as

$$\theta_n = \frac{2\beta_n^{KSR}}{a\Big(b - 2\sqrt{b} + 1\Big)}.$$

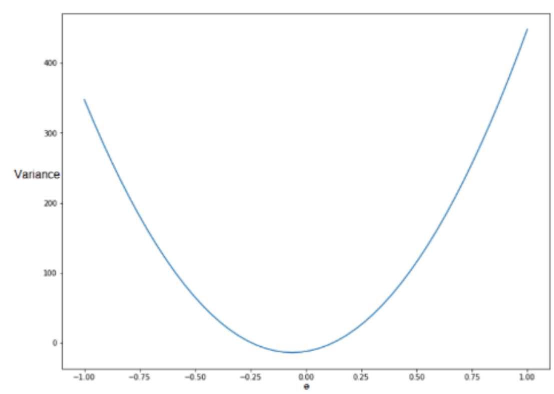
Given $g(\beta^{KSR}) = \dfrac{2\beta^{KSR}}{a\Big(b - 2\sqrt{b} + 1\Big)}$, the asymptotic variance of θ_n is

$$\{g'(\beta^{KSR})\}^2\sigma^2_{\beta^{KSR}} = \frac{4}{a^2(b - 2\sqrt{b} + 1)^2}\Big[\frac{\theta^2 a^2(\sqrt{b}\ln b - b + 1)^2}{4}$$
$$+ 2\theta a(\sqrt{b} - \frac{1}{2}b - \frac{1}{2}) + \Big(- \sqrt{b} + \frac{1}{2}b + \frac{3}{2}\Big)\Big(\sqrt{b} - \frac{1}{2}b + \frac{5}{2}\Big) + 2\Big].$$

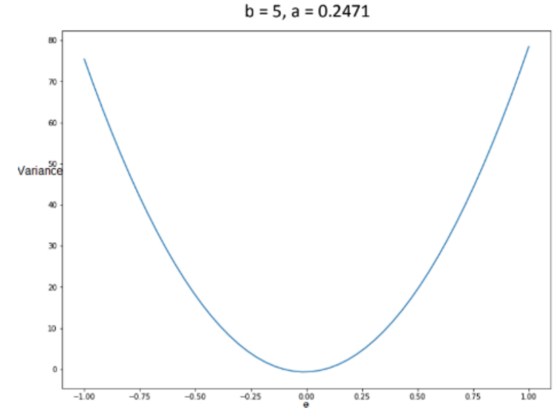
□



b = 1.001, a = 2000666.56



b = 5, a = 0.2471



b = 10, a = 0.0713

FIGURE 3. Asymptotic Variance of θ_n of τc_n for the KSR Copula

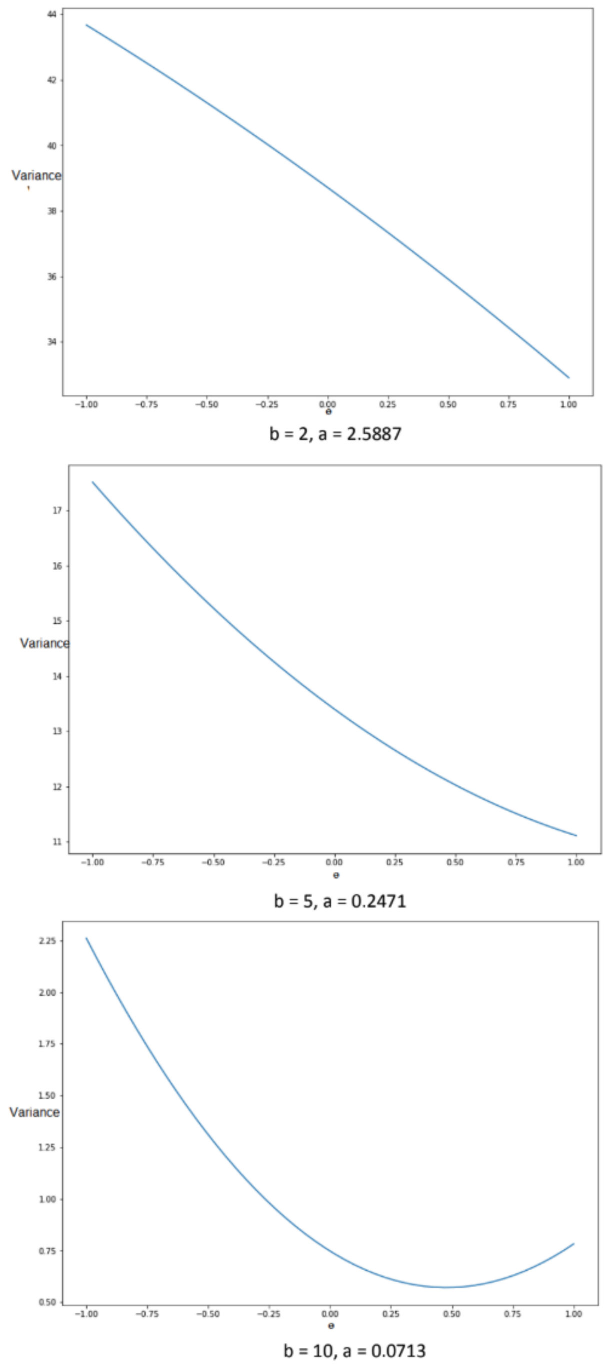


FIGURE 4. Asymptotic Variance of θ_n of β_n for the KSR Copula

4. Model Fitting

In this section, the KSR copula is compared to the existing bivariate copulas. The procedures for analysis are given in which AIC and BIC were used to determine the best fit model.

4.1. **The Data.**

The monthly rainfall (mm) and temperature (^oC) in the Philippines from January 1931 to December 2015 were used for the copula model fitting. It consists of 1020 months $(n = 1020)$.

Table 1 shows that Philippines experienced the lowest and largest monthly temperatures within 1931 and 2015 at 22.81^oC and 27.96^oC, respectively. On the other hand, the minimum monthly rainfall and maximum monthly rainfall are 14.36 mm and 487.63 mm, respectively.

Figure $5a$ presents the scatter plot between the actual monthly rainfall (Y) and temperature (X). Figure $5b$ presents the scatter plot employing the rescaled versions of empirical margins given as $u = F_n(x) = \dfrac{R_i}{n+1}$ and $v = G_n(y) = \dfrac{S_i}{n+1}$. The Kendall's tau of the data is 0.092. Hence, we can fit the KSR copula since its Kendall's tau ranges from -0.2222 to 0.2222.

Table 1. Descriptive Statistics of Rainfall and Temperature

	Minimum	Maximum	Mean	Standard deviation
Rainfall	14.36	487.63	204.30	98.13
Temperature	22.81	27.96	25.58	0.84

4.2. **Copula Model Selection.**

The procedures for analysis and results for model fitting with the data will be given. The inversion of the Kendall's tau for estimating θ will be used to obtain the AIC and BIC values. The steps are the following:

Step 1. Organize the bivariate data.

Step 2. Compute the density and the Kendall's tau of the copula.

Step 3. Use the inversion of the Kendall's tau to compute θ by substituting the Kendall's tau of the data at hand.

Step 4. Plug in the bivariate data to the density with θ obtained in Step 3.

Step 5. Compute the log pseudo-likelihood value.

Step 6. Finally, compute the AIC and BIC values. The copula with the smallest AIC or BIC value is the best fit model.

The bivariate copulas used for the analysis are Gumbel of Type A, Gumbel-Hougaard (GH), Farlie-Gumbel-Morgenstern (FGM), and KSR.

Table 2 shows the criteria values of 4 copulas. Observe that fixing b allows us to create a set of possible values for a and θ. But even if we choose any value of a with the Kendall's tau of the data at hand and θ, we still get the same AIC or BIC value.

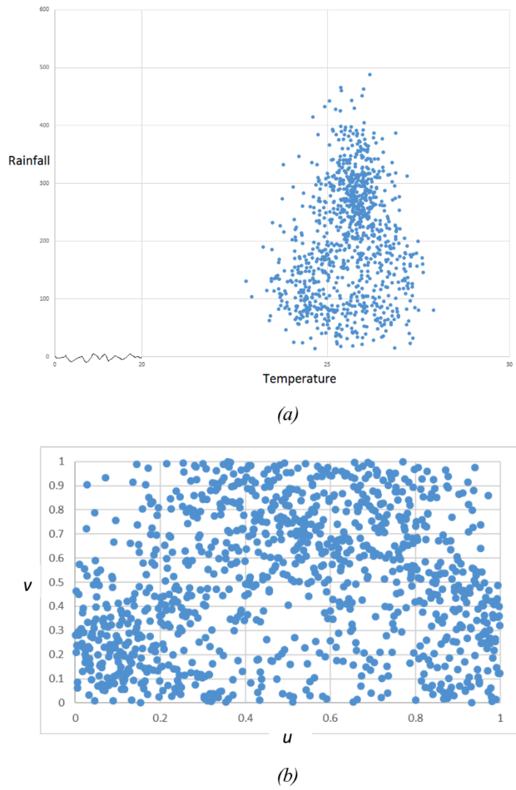


FIGURE 5. Scatterplot of the Rainfall vs Temperature

TABLE 2. Criteria Values of the Copulas

Copula	Theta	Log pseudo-likelihood Value	AIC	BIC
KSR				
$b = 1.001, a = 2000666.56$	0.4140	12.30	-22.61	-17.68
$b = 1.1, a = 206.5606$	0.4206	12.26	-22.52	-17.60
$b = 2, a = 2.5887$	0.4651	11.89	-21.78	-16.85
$b = 5, a = 0.2471$	0.5426	10.99	-19.98	-15.05
$b = 10, a = 0.0713$	0.6081	10.11	-18.22	-13.29
$b = 50, a = 0.0068$	0.7821	7.69	-13.38	-8.46
FGM	0.4140	12.30	-22.61	-17.68
Gumbel Type A	0.2620	-156.86	315.73	320.65
GH	1.1013	-8.69	19.37	24.30

For example, for $b = 1.001$, we choose $a = 2000666.56$ with the Kendall's tau of the bivariate data which is 0.092, then we get $\theta = 0.4140$, which gives us $AIC = -22.62$ and $BIC = -17.68$. The AIC and BIC values are still the same even if we choose any value of a.

Observe that KSR copula outperformed Gumbel of Type A and GH copulas for any values of b. With $b = 1.001$, KSR and FGM copulas have similar performance. Obviously, choosing b closest to 1 gets the smallest AIC or BIC values. This implies that for this kind of data, KSR with b closest to 1 is the best representative of the KSR copula in model fitting.

5. Conclusion

This study first introduced the KSR copula which is a apecial case of Kim and Sungur's copula and verified its properties, measures of dependence, and practical application on climate data in the Philippines. Dependence measures such as Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, Blomqvist's beta, and Schweizer–Wolff's sigma were explicitly constructed, demonstrating that the KSR copula captures moderate positive and negative dependence. KSR copula was computationally simpler than other copulas due to the parameter estimation via the inversion of Kendall's tau. Model selection using the AIC and BIC confirmed that the KSR copula gave the best fit among all bivariate copulas in modelling rainfall–temperature dependence; thus, demonstrating the copula's value in representing complex non-linear and tail dependence in climate data modeling.

This study builds on the research of Kim and Sungur (2004), who were the first to lay the groundwork on flexible copula structures, and complements other recent research, such as Wei et al. (2017), on generalized FGM copulas within a regression framework [20]. In the context of hydroclimatic studies, the copula approach has been utilized in precipitation–temperature modeling and the analysis of extreme events [21], and the KSR copula enriches this field by providing a parametric copula of minimal structure and adequate complexity. Given its performance relative to rotated and meta-elliptical copulas, the KSR copula is a suitable candidate for use in climate extremes research. Additionally, the studies comparing copula model selection reinforce the use of AIC and BIC, which illustrates the model selection process described here is less ad hoc and more robust [22], [23].

Apart from climate applications, copulas have been used in fields such as finance, insurance, and genetics, where complex dependence structures need to be captured [9], [24]. The KSR copula offers new prospects for such applications, especially for moderate tail dependence in risk management, portfolio modeling, and the study of gene interactions due to its manageable density and bounded measures of dependence. This means that the present research is not only contributing to the theoretical evolution of copulas, but also exemplifying the growing practical value of copulas and reinforcing the view that copulas are, and will continue to be, important in the analysis of multivariate dependence in various fields.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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