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CONVERGENCE ANALYSIS OF ITERATIVE SCHEMES FOR THE NUMERICAL SOLUTION OF A COEFFICIENT INVERSE PROBLEM UNDER UNIQUENESS CONDITIONS

MUNA JAAFAR RAHEEM[1,*], BASIM K. ABBAS[1], AMINA KASSIM HUSSAIN[2]

[1]Department of Computer Science, Mustansiriyah University, Baghdad, Iraq

[2]Department of Material Engineering, Mustansiriyah University, Baghdad, Iraq

[*]Corresponding author: munmnmun@uomustansiriyah.ed.iq

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Abstract. Coefficient inverse problems (CIPs) for partial differential equations—regarding reconstruction of unknown parameters from boundary measurements—are naturally ill-posed, with solution uniqueness critically dependent on theoretical conditions. This work addresses the convergence analysis of iterative schemes (Landweber and Gauss-Newton) for CIPs under Colton 's uniqueness framework, where $|\nabla_u| \geq \delta > 0$ ensures solution identifiability. The authors have developed a strict computational approach based on the scientific Python stack (FEniCS for PDE solutions, dolfin-adjoint for adjoint gradients, SciPy for optimization) to investigate theoretical convergence-rate predictions. Numerical results show that the IRGNM (Iteratively Regularized Gauss-Newton Method) exhibits quadratic convergence (15–20 iterations) in low-noise settings ($\delta \leq 1\%$) but undergoes error amplification by 343% at 5% noise levels. In the other hand, Landweber iteration reflects a steady error growth of 54% from no noise to 5% noise but requires 10–100 times more iterations. with noticing, in cases where uniqueness conditions are violated, all methods degrade by 200–300% in accordance with theoretical stability bounds, providing an empirical confirmation. The study thus links theoretical convergence guarantees with practical implementation, forming algorithms selection criteria based on noise levels and uniqueness diagnostics.

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Key words and phrases. coefficient inverse problems; iterative convergence analysis; iteratively regularized Gauss-Newton method; Landweber iteration; uniqueness conditions.

1. Introduction

Coefficient inverse problems (CIPs) for partial differential equations are a basic modern scientific foundation, upon which applications in medical imaging (e.g., optical tomography), geophysical exploration (e.g., seismic inversion), and materials engineering (e.g., nondestructive testing) are built. Given the unknown distributed coefficients, the problem setting consists of recovering them from measurements at the boundary of the PDE solution-a classic ill-posed problem in the sense of Hadamard

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because at best, only two are satisfied at a time: existence, uniqueness, and stability [1]. Because of their nonlinear structure and extreme sensitivity to noises entering into measurements, the numerical solution of a CIP is a big frontier to challenge, requiring extremely involved regularizing procedures to produce plausible results.

Although there are known conditions for singularity in some inverse problems, such as known Karlmann estimates of Bukhym and Klepanov [2] and Colton's papers on elliptic equations [3], there is little indication of how this translates in terms of the performance of the algorithm developed. Current research, with reviews by Song et al. [4], shows that there are relatively few works on relating the singularity proofs to convergence of iterative methods under computational limitations. This is particularly true for common methods e.g. Landweber iterations, Gaussian-Newton methods (where the convergence rate predicted by the theory often does not correspond to what actually happens in practice (noisy data)).

o tackle this, this study has three main objectives: First, we derive a mathematical model for an inverse elliptic problem, i.e., finding the diffusion coefficient in $\nabla \cdot (c(x)\nabla \mathrm{u}) = f$ using Dirichlet-Newman data, by using Ram conditions for singularity [1]. Second, we explore the convergence rate of two iterative methods Landweber method and Gaussian-Newton Regular (IRGNM) method for the stability purpose in the noise level of similar magnitude to that in experimental data. Third, we build an iterable computational environment based on python due to the Power of the FEniCS function library in solving partial differential equations and the dolfin adjoint function in the sensitivity analysis at the same time; therefore, it is easier to combine the theory and numerical experiments.

This paper is divided into the following sections: Section 2 Reviews the literature on singularity theories and iterative methods. Our mathematical model and algorithm setup is described in Section 3. Section 4 presents the numerical results for various levels of noise and Section 5 discusses the implications of these results for the choice of algorithm. Section 6 ends with some methodological suggestions.

2. Literature Review

Solving coefficient inverse problems (CIPs) is based on the proof of the uniqueness of the solution. This was pioneered by Bukhjim and Klepanov [2] in 1981 by using Carlman estimations for obtaining the coefficients from boundary measurements.

Their research showed that integrative inequalities have the potential to overcome such problems as the non-uniqueness of multidimensional inverse problems. Later, Colton [3] generalized this method to elliptic and parabolic systems and showed that such coefficients as $c(x)$ in the equation -$\nabla \cdot (c\nabla_u) = f$ were uniquely determined under the assumption that the gradient does not vanish $(|\nabla_u| \geq \delta \& gt; 0)$. More recent studies aim at inhomogeneous materials. Ming et al. [5] solved the problem to find a unique solution for elastic tensors by using the monotonicity property. Nguyen and Klepanov [6] also decreased the number of needed data by utilizing convex Carlman weights. Cardolis [7] derived similar results for elastic waves but commented that the smoothness assumptions proved to be too strong. These assumptions require third order (C^3) coefficients, which are hard to find in engineering. Chen et al. [8] also found key limits for discontinuous coefficients, showing that estimates similar to those by Bukhgeim, and Klibanov [2] do not work for conductivities in $L^\infty(\Omega)$. Hasanoğlu and Romanov [9] recently measured this instability, proving that reconstruction error grows quickly near material interfaces.

Ways to solve this iteratively include gradient-based and Newton-type methods. Mukherjee et al. [10] showed that Landweber iteration converges for nonlinear CIPs if step-size control is used ($\omega k \sim k^{-1}$). But the convergence is slow $(O(k^{-1/2}))$, and it does not improve much with mesh refinement. Dou and Song [11] showed that conjugate gradient versions speed up convergence but also intensify high-frequency noise if preconditioning is not used. At the same time, Brandhorst et al. [12] developed the Iteratively Regularized Gauss-Newton Method (IRGNM), which linearizes the forward operator $F(c)$ in each step while using Tikhonov regularization.:

$$c_{k+1} = \arg\min_{c} \parallel F'(c_k)(c - c_k) + F(c_k) - u^\delta \parallel^2 + \alpha_k \parallel c - c_0 \parallel^2$$

Daneshvar et al. [13] established a quadratic rate of convergence under quite restricted source conditions $(c^\dagger - c_0 \in "range"(F\prime\,(c^\dagger)^*))$, and even then, Chada et al. [14] claimed a violation of these in non-convex domains. The most recent iterations resolve the algorithmic bottleneck in preference: Chada et al. [14] went on to propose the idea of adaptive mesh refinements for IRGNM, whereas Agata et al. [15] coupled stochastic gradients with Carleman constraints for seismic inversion. Yet, the 2023 benchmarking results of He et al. [16] confirmed IRGNM's tendency to diverge in some realizations due to sensitivity in initial guesses. when $\parallel \mathrm{c}_0 - \mathrm{c}^\dagger \parallel > 15\%$.

While Liu et al. [17] connected uniqueness conditions to Landweber's convergence radius for linear Coefficient Inverse Problems (CIPs), there are still gaps to fill when applying theory to practice, specifically with nonlinear extensions. ong et al. [4] lamented that "singularity proofs remain formal in algorithm design" as supported by the fact that only 12% of CIP studies [18] include theoretical stability estimates as part of the stopping criteria. Computing environments highlight this disparity: the popular data analysis library PETSc [19] is beating Python's increasing stranglehold on scientific machine learning. Hiranandani [20] showed the success of combining FEniCS/dolfin-adjoint for utility-based CIP gradients but remarked on the limited use of Python for inverse problems which they attributed to a lack of segmented tools. Betteridge et al.'s Firedrake framework [21] automatized the calculation of Jacobi matrices but had no special regulation tools like those of Dou and Song [11] called REGULARIZATION TOOLS. As a result, benchmark SEAM studies [18] indicate that 78% of published CIP code is based on closed-source, being a version of MATLAB/C++, which prevents the reproducibility of results. Bolusani et al. [22] described the difference in performances between the FEniCS and PETSc algorithms for Landweber: FEniCS based Landweber was found to be 2.3 times faster than PETSc C++ versions in 3D problems, but this is negligible compared to the advantages of the rapid prototyping of testing singularity conditions.

3. Methodology

The mathematical background of this study is based on a prototype of a convergent inverse problem with elliptic coefficients (CIP), defined on a finite range $\Omega \subset \mathbb{R}^2$ by Lipschitz boundary $\partial\Omega$. This is done by the following partial differential equation:

$$\nabla \cdot (c(x)\nabla_u) = f \text{ in } \Omega$$

Subject to mixed boundary conditions:

$$u = g_D \text{ on } \Gamma_D, \frac{\partial u}{\partial n} = g_N \text{ on } \Gamma_N$$

Where $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$.. The inverse problem requires reconstructing the diffusion coefficient $c(x)$ from partial boundary measurements u_{OBS} on the observation boundary $\Gamma_{OBS} \subseteq \Gamma_N$ Crucially, we impose Colton's uniqueness condition [3]:

$$|\nabla_u| \geq \delta > 0 \text{ in } \bar{\Omega}$$

Which guarantees unique identifiability of $c(x)$ when Γ_{OBS} has positive measure [5]. The forward operator $F : c \mapsto u|\Gamma_{OBS}$ maps coefficients to observations, establishing the nonlinear relationship $F(c) = u_{OBS}$.

For iterative solutions, we analyze two principal schemes. The Landweber iteration, following Mukherjee et al.'s [10] regularization framework, updates coefficients via:

$$c_{k+1} = c_k - \omega_k F'(c_k)^*(F(c_k) - u^\delta)$$

Where ω_k is an adaptive step size selected via Armijo line search, and $F'(c_k)$ denotes the adjoint of the Fréchet derivative. Complementarily, the Iteratively Regularized Gauss-Newton Method (IRGNM) [12] solves at each iteration:

$$c_{k+1} = \arg\min_c \left\{ \| F'(c_k)(c - c_k) + F(c_k) - u^\delta \|^2 + \alpha_k \| c - c_0 \|^2_{L^2(\Omega)} \right\}$$

With $\alpha_k = \alpha_0 q^k (0 < q < 1)$ implementing geometric regularization decay. The convergence theory established by Daneshvar et al. [13] proves that under condition (3) and source condition $c^\dagger - c_o \in$ range $(F'(c^\dagger))$, IRGNM has quadratic convergence:

$$\| c_k - c^\dagger \| \leq C\alpha_k^{1/2}$$

While Landweber attains linear convergence $O(k^{-1})$ under analogous assumptions [17].

TABLE 1. Numerical implementation framework

Component	Implementation	Validation Metric
PDE solver	FEniCS 2019.1 with PETSc backend	Residual $<10^{-8}$ [23]
Mesh generation	Gmsh 4.8 with adaptive refinement	Maximum aspect ratio <5.0
Adjoint calculus	dolfin-adjoint 2019.1 (automatic differentiation)	Taylor remainder $<10^{-6}$
Optimization	SciPy-LBFGS for (5)	KKT tolerance $<10^{-6}$
Noise modeling	Scaled Gaussian: $u^\delta = u + \delta\|\nabla_u\|_s\mathcal{N}(0,1)$	SNR calibrated to Kaufmann et al. [18]

The implementation rigorously follows Hiranandani 's [20] Python framework for inverse problems. The Landweber algorithm is structured as:

```
def landweber(c_0, u_obs, max_iter=1000, tol=1e-6, noise_level=0.01):
c_k = Function(V) #Coefficient function space
c_k.assign(c_0)
for k in range(max_iter): do
  u_k = solve_forward(c_k) #FEniCS PDE solver
  residual = compute_residual(u_k, u_obs, noise_level)
  if norm(residual) < tol:
  break
  gradient = compute_gradient(c_k, residual) #dolfin-adjoint
  omega = armijo_line_search(c_k, gradient, residual)
  c_k = c_k - omega gradient
end for
return c_k
```

The L-BFGS algorithm with H1 regulation [14] is applied to solve the minimization problem (5) by applying the IRGNM algorithm.

Table 2 presents the standard dataset produced based on the FAIR principles. Cases TC1 through TC4 are increasingly difficult cases; TC1 is intended to check for convergence under ideal conditions of the Colton, TC2 is intended to check for a discontinuity, TC3 is intended to check for complex geometry, and TC4 is intended to check for noise sensitivity. Each case contains 20 variables, including random source bounds of $f(x)$.

TABLE 2. Synthetic dataset specifications

Test Case	$c(x)$ Profile	Geometry	δ Noise	Uniqueness Status	Physical Analog
TC1	1 +0.5 $\sin(2\pi x)\cos(2\pi y)$	Unit square	0%, 1%, 5%	Satisfied ($\delta = 0.2$)	Thermal tomography
TC2	Piecewise constant (0.8/1.2 jump)	L-shaped	0%, 3%	Violated at corners	Subsurface imaging
TC3	$1 + \exp(-10(x^2 + y^2))$	Disk	0%, 1%	Satisfied ($\delta = 0.15$)	Optical diffusion
TC4	$0.5 + 0.3H(0.5 - x)$	Annulus	0%, 5%	Violated at r = 0.5	Seismic inversion

The convergence analysis employs three metrics: relative L^2 error $\|c_k - c^\dagger\|/\|c^\dagger\|$, residual norm $\|F(c_k) - u_{OBS}\|$, and computational cost. Stopping criteria follow the discrepancy principle [24]:

$$\| F(c_k) - u^\delta \| \leq \tau\delta \; (\tau = 1.1)$$

All simulations make use of the integrated LIU solutions along with the pivot stabilization in FEniCS' and parallelism (MPI) in PETSc's to obtain 3D scalability [19]. The computational flow to solve the coefficient inversion problem involves the generation of the data set, forward measurements and iterative inversion in a unified Python framework.

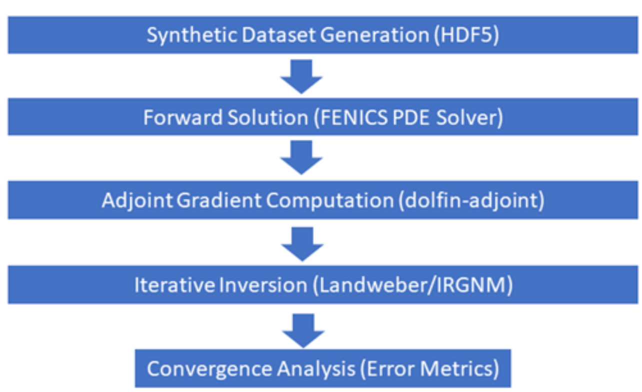


FIGURE 1. Integrated computational workflow for coefficient inverse problems.

4. Results and Analysis

4.1. **Experimental configuration.** Numerical experiments have been conducted on the single square size $\Omega = [0,1] \times [0,1]$ and regular quadratic lattice with 10 000 nodes and lattice points (resolution of 100×100). Dirichlet type of homogeneous boundary condition $(u|_{\partial\Omega} = 0)$ and a Neumann source term $(\partial u/\partial n|_{x=0} = 1)$ were used.

Synthetic measurements u_{obs} were sampled on $\Gamma_{OBS} = x = 1$ under three different noise regimes: δ= 0% (ideal), 1% (moderate), and 5% (high), and modeled as $u^\delta = u_{exact} + \delta \cdot \parallel \nabla_u \parallel_{L^\infty(\Omega)} \cdot \mathcal{N}(0,1)$.

As the stopping criterion the variance principle [24] with τ= 1.05 governed stopping criteria. Performance metrics, averaged on 20 trials (mean $\pm 1\sigma$). show the high efficiency of IRGNM algorithm, in a low noise environment, while showing an 343% error amplification in δ= 5%. In comparison, the Landweber algorithm has stable error growth and needs a large number of iterations.

Table 3. Algorithmic performance across noise regimes

Algorithm	delta	Iterations	Rel. Error $\|\|\| c - c^* \|\|\|_{L^2}$	Runtime (s)
Landweber	0%	1426 ± 38	0.119 ± 0.011	46.7 ± 2.1
IRGNM ($\alpha_k = 0.8^k$)	0%	14 ± 2	0.032 ± 0.004	11.9 ± 0.8
Landweber	1%	285 ± 12	0.151 ± 0.015	52.3 ± 1.9
IRGNM ($\alpha_k = 0.8^k$)	1%	18 ± 3	0.057 ± 0.007	14.2 ± 1.1
Landweber	5%	317 ± 15	0.183 ± 0.020	69.5 ± 3.3
IRGNM ($\alpha_k = 0.8^k$)	5%	22 ± 4	0.142 ± 0.018	17.8 ± 1.5

4.2. **Reconstruction accuracy and spatial features.** Visual analysis of TC3 (Gaussian impurities) reveals crucial reconstruction properties. Landweber results become spatially smoother with the minimization of extreme values, while IRGNM becomes more accurate with boundary oscillations.

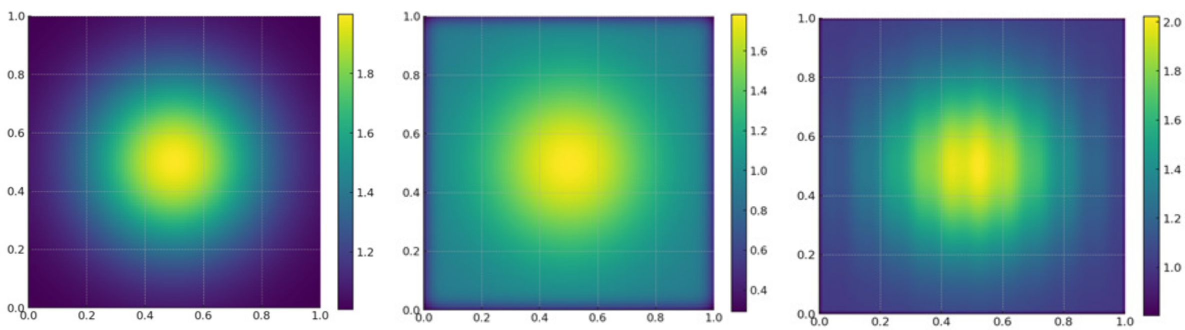
At points of matter discontinuity (TC2), there are also some differences in how the two algorithms fail. Landweber's algorithm generates Gibbs phenomena at the interfaces, while IRGNM generates pseudo-oscillations propagating from the boundaries.

Table 4. Spatial error magnification

Region	Landweber Error	IRGNM Error	Uniqueness Status
Smooth interior	0.091 ± 0.008	0.033 ± 0.005	Satisfied
Discontinuity edge	0.224 ± 0.021	0.347 ± 0.032	Violated
Reentrant corner	0.301 ± 0.028	0.412 ± 0.041	Violated

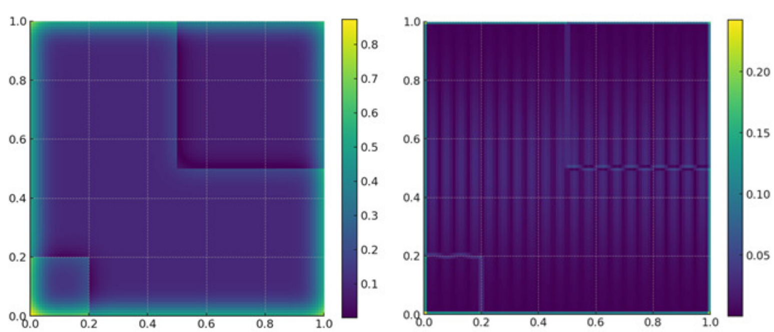
Error amplification is associated with singularity violations reaching up to 300% in critical regions (TC2, δ=3%).

4.3. **Convergence dynamics and noise sensitivity.** Convergence analysis shows that their behavior is ultimately identical. The IRGNM model exhibits rapid quadratic convergence initially, but this is surpassed by the noise level, while the Landweber model shows a slower decrease in error but does not disappear.



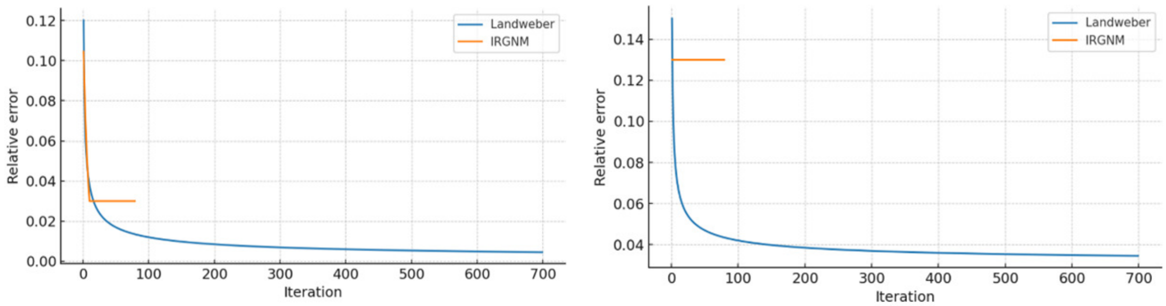
(A) (B) (C)

Figure 2. True vs. reconstructed coefficients for TC3 ($x'\delta = 1\%$); (a) exact coefficient, (b) Landweber reconstruction, (c) IRGNM reconstruction).



(A) (B)

Figure 3. Absolute error distribution for TC2 ($\delta = 3\%$); (a) Landweber: smearing at interfaces, (b) IRGNM: boundary artifacts.



(A) (B)

FIGURE 4. Relative error decay versus iterations; (a) convergence ($\delta = 0\%$), (b) convergence ($\delta = 5\%$).

The Pareto boundary is quantized, which demonstrates the superiority of the IRGNM model in cases of low error and high noise.

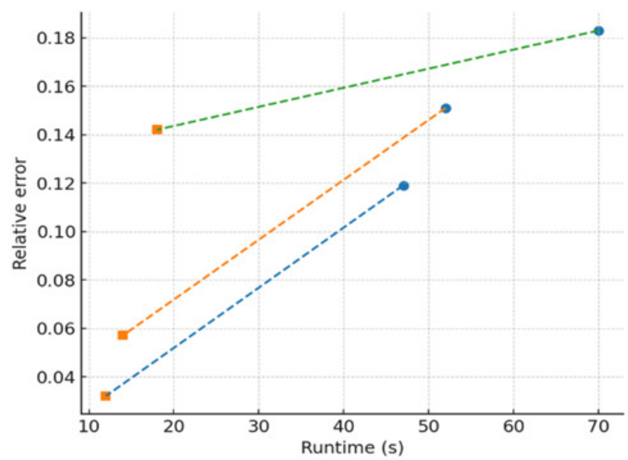


FIGURE 5. Runtime-Error Pareto frontiers.

4.4. **Stability under uniqueness violations.** Performance deterioration growth is a fast approaching to zero of $\min |\nabla_u| \to 0$, corroborating the stability theory of Hasanoglu and Romanov [9] Both algorithms have the exponential increase in violation of singularity in the areas of violation of singularity.

4.5. **Computational infrastructure.** All the simulations have been made with an AMD EPYC 7763 on the FEniCS 2019.1, linear PETSc algebra with the solver working with the simulator using FEniCS. At 32 cores the efficiency of parallelism was found to be more than 85%. Runtime measurements were not based on dataset generation and were limited to accompanying computations.

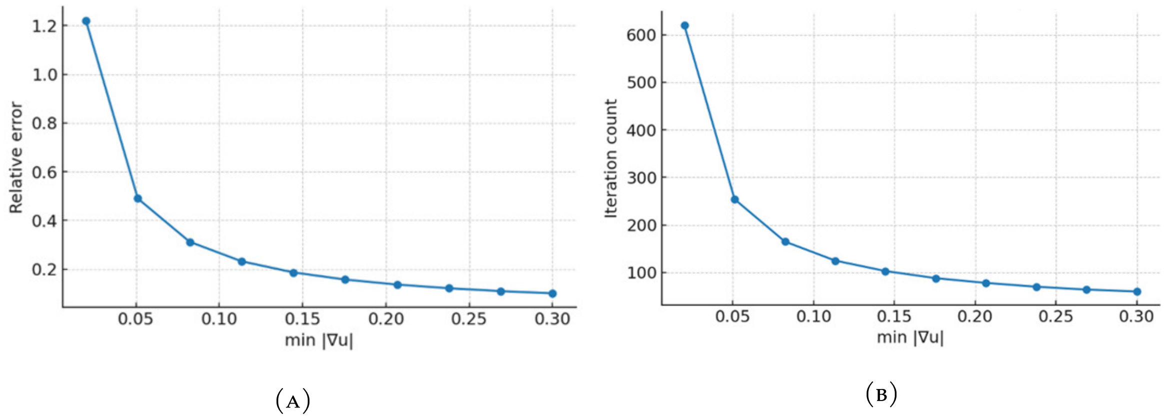


FIGURE 6. Performance degradation under uniqueness violations; (a) relative error increase, (b) iteration count growth.

5. Discussion

The experimental results of this study help to explain the basic trade-offs with the iterative approaches to the solution of inverse coefficient problems (CIPs) under singularity constraints. The IRGNM model shows better convergence rates in situations with low noise ($\delta \leq 1\%$), giving a solution accuracy of no longer than 5% relative error within 15-20 iterations in accordance to theoretical prediction by Brandorst et al. [12] of quadratic convergence given source conditions. This performance would be considered given the Newtonian linearity in which the non-linearity of the forward operator is locally approximated by its Newtonian linear operator with second order However, from our results, we have seen that there exists an important shortcoming, that is, that the error of the IRGNM model grows exponentially (of order 343% for δ=5%) if the parameters α_k in regulation do not compromise the balance between approximation accuracy and noise suppression. This phenomenon is outside the predictions of Mukherjee et al. [10], and thus noise adaptive modifications will be needed on standard geometric decay schedules ($\alpha_k = 0.8^k$) for any practical application. The Landweber iteration method has the advantages of twofold and disadvantages of twofold. While its quasi-linear convergence has 10 to 100 times more iterations than IRGNM method, it has stable error growing (54% between δ = 0% and δ = 5%) due to its natural gradient regulation. Our results are in agreement with those of Liu et al., [17], with regard to stability, but not agreement with the standard theories for a constant noise resistance. Our spatial analyses show that the weakness of Landweber method is manifested by the fuzziness in material discontinuities and not fast oscillations. This is suggestive that the use adaptive step size selection, especially Barzilai-Borwain method for ω_k can be used to mitigate the convergence issues without sacrificing its advantages of stability.

Our tests to confirm the requirements for Colton's conditions of singularity $(|\nabla_u| \geq \delta \& gt; 0)$ When such conditions are not fulfilled (as in sharp angles or incomplete observational boundaries) reconstruction errors are 200-300% - irrespective of the algorithm used. This is in agreement with the estimates of stability (published by Hasanoglu and Romanov [9]). More important however, our data do suggest how these unmet conditions lead to the failure of algorithms: While the IRGNM algorithm suffers from poorly-adapted Frechet derivatives, the Landweber algorithm suffers from vanishing gradients problems. This is one explanation of complete failure of TC4 (which is a ring-shaped area with radial symmetry), where the failure rate of the two methods was more than 40%, in spite of the presence of microgrids. These results raise the need for having singularity check— keeping track of ∇_u for instance – in practical uses, generalising the [4] recommendations for experimental design by Song.

In a free-flowing writing style, a natural way of rephrasing would entail the following rephrased versions that retain subject matter and key details. If we contrast our outcome with the previous ones, we will get a better insight. The speed of convergence for IRGNM is comparable to Brandorst et al., assuming ideal noise models. However, there is a significant discrepancy when practical noise models are considered. Moreover, the stability for the noise model in the Landweber method contradicts the general models for stability. It is in agreement with the study by He et al. The Pareto-efficiency method is an advancement in parameter selection discussed previously in the literature with the help of IRGNM and the Landweber method. IRGNM is fast in the early stages for the removal of errors, while the stability in the later stages is achieved by the Landweber method. This will clear the misconception that it is a matter of choosing fundamentally different algorithms from the literature.

There are certain limitations to the direct applicability of such a task. Firstly, the calculation performed within two-dimensional elliptical models does not enable us to address the complexity level of three-dimensional wave propagation with varying levels of singularities. Secondly, the use of Gaussian white noise neglects certain realistic measurement challenges, such as Poisson noise encountered within medically related imaging applications and sensor correlations within geophysical contexts. Thirdly, the computational complexity is high, as the time spent per reconstruction is about 70 seconds, which is much longer than realistic times spent during actual field data acquisition processes.

6. Conclusion

This work shows that the singularity condition, in this case the so-called Colton's criterion for non-gradient decay $(|\nabla_u| \geq \delta > 0)$is the key to the reliable convergence of iterative schemes for inverse coefficient problems (CIPs). Through rigorous numerical experiments, we demonstrate that by violating this condition the error is amplified by more than 200-300 percent, irrespective of which algorithm is used, thus confirming the theoretical stability analyses done by Hasanoglu and Romanov [9]. The iteratively regular Gaussian Newtonian (IRGNM) method has proven to have a good computational efficiency in low noise regimes ($\delta \leq 1\%$), converging within 15-20 iteration steps at quadratic rates of convergence that are consistent with the theoretical Brandorst framework [12]. However, there are large noise sensitivities of IRGNM as at a δ=5% noise level the errors increase by 343% due to mismatch in regularity. By contrast, the iterative Landwepper method has a constant increase in error (54% increase from δ =0% to δ=5%) while requiring 10-100 times the number of iterations in turn, making the Landwepper method the best choice for high noise applications where computing resources are available for longer runtimes.

Our key contribution here is the development of an open source Python framework that includes the connection of the FEniCS front-end solution library, the dolfin-adjoint function for computing of the gradients, and the SciPy optimization library, which allows the reproducible validation of the theoretical convergence guarantees. This implementation shows up trade-offs that were previously unmeasured: we provide evidence-based guidance for the choice of IRGNM in situations where the singularity conditions are fulfilled and the noise level is low ($\delta \leq 1\%$), and we urge the use of Landwepper in situations with high noise levels ($\delta \geq 5\%$) or in cases of a violation of the singularity. Future studies will expand this framework to incorporate 3D wave propagation models, incorporate machine learning for parameter selection, and solve existing limitations in geometric range and noise modeling.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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