Asia Pac. J. Math. 2026 **13**:66

ON THE KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS FOR FUZZY WEIGHTED INVEX OPTIMIZATION

PARFAIT DOUBASSI TRAORÉ, KOUNHINIR SOMÉ*

Department of Sciences and Technologies, Laboratoire de Mathématiques Informatique et Application (L@MIA), Université Norbert Zongo, Koudougou, Burkina Faso

*Corresponding author: sokous11@gmail.com

Received Feb. 28, 2026

Abstract. In this paper, we propose a novel optimization technique for solving fuzzy optimization problems. By introducing generalized weighted invexity assumptions, we establish sufficient Karush-Kuhn-Tucker (KKT) optimality conditions that ensure the theoretical soundness of the proposed approach. Furthermore, the approach reduces computational complexity through a α-cut integral transformation and guarantees the existence of solutions even in non-convex settings. To evaluate its effectiveness and practical relevance, we solve a set of test problems and provide a comparative analysis with existing methods from the literature. The results highlight the robustness, efficiency, and wide applicability of the proposed method in handling complex fuzzy optimization models.

2020 Mathematics Subject Classification. 90C70; 90C46; 26E50.

Key words and phrases. fuzzy optimization; weighted invex functions; KKT conditions.

1. Introduction

Optimization seeks to enhance existing systems by determining the best possible solutions under specific constraints. Over time, optimization theory has established a solid framework for modeling real-world processes and systems through deterministic models, thereby supporting effective decision making and improving performance. Nevertheless, many operational research problems in modern applications—such as economics, investment planning, engineering design, inventory management, mechanical systems, transportation, logistics, physics, and ecology—are often ill defined due to data uncertainty. Mathematical programming problems involving uncertainty have been widely investigated in various fields, and numerous methodological approaches have been proposed to address them. In this context, optimization based on fuzzy functions offers an appropriate framework. Based on the fuzzy sets theory, introduced by L. Zadeh [23], this approach has since been refined and extended by numerous researchers.

DOI: 10.28924/APJM/13-66

Given the central role of differentiability and convexity in fuzzy optimization, considerable research has focused on generalized derivatives and convexity properties of fuzzy-valued functions. Wu [22] and Bede [4] investigated the concept of gH-differentiability and its fundamental properties. Stefanini [20] further extended this framework by introducing fuzzy gH-differentiability, establishing local optimality conditions via the gH directional derivative, and deriving Karush–Kuhn–Tucker (KKT) conditions for constrained fuzzy optimization problems. A limitation of this approach, however, is that the resulting solutions may not always preserve the structure of fuzzy-valued functions. To overcome this issue, several authors [5, 7, 19] proposed refined definitions and alternative differentiability concepts to ensure that the outputs remain proper fuzzy-valued functions.

In this context, Mazandarani et al. [10] introduced a new notion of differentiability in fuzzy analysis, referred to as the *granular derivative* (or *gr-derivative*). This definition builds on the concepts of granular difference and horizontal membership function, as formulated by Piegat and Landowski [16]. Over the past few years, a growing body of research has been developed around gr-differentiability. For instance, Mazandarani [9] applied the gr-integral and gr-derivative framework to investigate the fuzzy Bang–Bang control problem. Mustafa [12] derived granular Euler–Lagrange equations for fuzzy variational problems and established Pontryagin-type necessary conditions for fuzzy optimal control. More recently, Zhang [25] examined a class of fuzzy optimization problems under gr-differentiability and derived KKT optimality conditions for such problems.

In addition, fuzzy models of real-world processes based on granular differentiability, along with associated solution methods, have gained increasing attention (see, e.g., Khatua et al. [8]; Najariyan and Zhao [13]; Tripathi et al. [21]; Zhang et al. [24], among others). Within this framework, Fangfang et al. [18] investigated nonconvex fuzzy optimization problems under gr-differentiability, while Antczak [2] established optimality conditions for fuzzy mathematical programs involving gr-differentiable functions.

Since the granular fuzzy derivative is not based on standard interval arithmetic and avoids several of its inherent limitations, it provides a more robust framework for analyzing fuzzy-valued functions. In light of this advantage and motivated by the results of Mishra and Giorgio [11] on Karush–Kuhn–Tucker–type conditions for unconstrained invex multiobjective optimization problems, we find it natural and advantageous to combine these two approaches.

Thus, in this paper, we propose Karush–Kuhn–Tucker (KKT) optimality conditions for constrained weighted invex fuzzy optimization problems. The proposed approach offers a significant computational advantage by reducing the number of functions to be optimized through the use of integral of α-cut transformations. Moreover, it enables the establishment of existence results for optimal solutions even in the absence of convexity. The main contributions and highlights of this work are as follows:

- Definition of a new type of generalized differentiable convex fuzzy-parameterized problem;
- Proposal of a new algorithm using the horizontal membership function to solve fuzzy optimization problems;
- Establishment of Karush-Kuhn-Tucker conditions for these types of problems;
- Resolution of numerical examples and comparison with other methods.

In the remainder of this paper, after the preliminaries in Section 2 on the basic concepts of fuzzy optimization, we establish the optimality conditions for non-dominated solutions in Section 3, illustrated by numerical applications. Section 4 presents the conclusion, together with a few remarks outlining future research directions.

2. Preliminaries

This section is dedicated to presenting some definitions and theorems that will be used in the rest of the work.

2.1. **Fuzzy Number Space.**

In this section, $\mathbb{K} \subseteq \mathbb{R}^n$ represents the n-dimensional Euclidean space, E represents the set of fuzzy numbers and E^n is the n-dimensional space of fuzzy numbers. The set of α-cuts of a fuzzy number $\tilde{u}$ is denoted by $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ where $\alpha \in [0,1]$ and $\underline{u}^\alpha$, $\overline{u}^\alpha$ are called the lower and upper bounds of $[u]^\alpha$, respectively.
The following lines focus on the horizontal membership function (HMF).

Definition 2.1. [8–10,21,24] Let $\tilde{u} : [a,b] \subseteq \mathbb{R} \to [0,1]$ be a fuzzy number. The horizontal membership function $u^{gr} : [0,1] \times [0,1] \to [a,b]$ is a representation of $\tilde{u}(x)$ with $u^{gr}(\alpha, \beta_u) = x$, where "$gr$" represents the information granule included in $x \in [a,b]$, α is the degree of membership of x to $\tilde{u}(x)$, $\beta_u \in [0,1]$ is the relative distance measurement variable, and $u^{gr}(\alpha, \beta_u) = \underline{u}^\alpha + (\overline{u}^\alpha - \underline{u}^\alpha)\beta_u$ for all $\alpha \in [0,1]$.

The horizontal membership function is also denoted by $H(\tilde{u}) = u^{gr}(\alpha, \beta_u) = \underline{u}^\alpha + (\overline{u}^\alpha - \underline{u}^\alpha)\beta_u$. For a triangular fuzzy number $\tilde{u} = (p,q,r)$ where $p,q,r \in \mathbb{R}$ with $p \leq q \leq r$, the horizontal membership function for each $\alpha, \beta_u \in [0,1]$ is given by $H(\tilde{u}) = [p + (q-p)\alpha] + [(1-\alpha)(r-p)]\beta_u$.

2.2. **Fuzzy Optimization.**

Definition 2.2. [6] A fuzzy coefficient optimization problem is called a mathematical problem of the form

$$(FO) \begin{cases} \min \tilde{F}(x) \\ s.t \\ g_j(x) \leq 0; j = \overline{1,s} \text{ and } x \in \mathbb{K} \subset \mathbb{R}^n. \end{cases} \tag{1}$$

We recall that $\mathbb{K}$ is an open set. $\tilde{F} : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ is a fuzzy function and $g_j, j = \overline{1,s}$ are real-valued functions on $\mathbb{K}$.

The set of admissible solutions for the problem (FO) is given by:

$$\mathbb{S} = \left\{ x \in \mathbb{K} \mid g_j(x) \leq 0, j = \overline{1,s} \right\}. \quad (2)$$

This set is non-empty and compact.

Definition 2.3. [10] Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are equal if and only if $H(\tilde{u}) = H(\tilde{v}) \; \forall \, \beta_u = \beta_v \; \in [0,1]$ and $\alpha \in [0,1]$.

Definition 2.4. [18] Let $\tilde{u}$ and $\tilde{v} \in E$. We have $\tilde{u} \preceq \tilde{v}$ if and only if $H(\tilde{u}) \leq H(\tilde{v}) \; \forall \, \beta_u = \beta_v \in [0,1]$ and $\alpha \in [0,1]$; $\tilde{u} \prec \tilde{v}$ if and only if $H(\tilde{u}) < H(\tilde{v})$ and $H(\tilde{u}) \neq H(\tilde{v}) \; \forall \, \beta_u = \beta_v \in [0,1]$ and $\alpha \in [0,1]$.

Definition 2.5. [6] Let x^* be an efficient solution of the problem (FO), that is, $x^* \in \mathbb{R}$. We say that x^* is a non-dominated solution of the problem (FO) if there does not exist $x \in \mathbb{S}\{x^*\} \mid \tilde{F}(x) \preceq \tilde{F}(x^*)$.

Definition 2.6. [6] Let x^* be an efficient solution of the problem (FO), that is, $x^* \in \mathbb{R}$. We say that x^* is a weakly non-dominated solution of the problem (FO) if there does not exist $x \in \mathbb{S}\{x^*\} \mid \tilde{F}(x) \prec \tilde{F}(x^*)$.

Proposition 2.7. [6] *If $x^* \in \mathbb{S}$ is a non-dominated solution of the problem (FO), then x^* is a weakly non-dominated solution of the problem (FO).*

Proof. If x^* is a non-dominated solution of (FO), we have $\forall x \in \mathbb{S} \backslash \{x^*\}$,
$\tilde{F}(x^*) \preceq \tilde{F}(x) \Rightarrow [F(x^*)]^\alpha \leq [F(x)]^\alpha$ with $F(x^*) \neq F(x)$, $\underline{F}_\alpha(x^*) \leq \underline{F}_\alpha(x)$
and $\overline{F}_\alpha(x^*) \leq \overline{F}_\alpha(x) \;\; \forall \alpha \in [0,1]$.
There exists $\alpha^* \in [0,1]$ such that $\underline{F}_{\alpha^*}(x^*) < \underline{F}_{\alpha^*}(x)$ or $\overline{F}_{\alpha^*}(x^*) < \overline{F}_{\alpha^*}(x)$.
Hence, $\tilde{F}(x^*) \prec \tilde{F}(x) \; \forall x \in \mathbb{S} \backslash \{x^*\}$. Consequently, if x^* is a non-dominated solution of (FO), it is also a weakly non-dominated solution of (FO). □

Remark 2.8. [6] The converse of this proposition is not true.

2.3. **Karush-Kuhn-Tucker Optimality Conditions for the Convex Case.**

Theorem 2.9. [3] *Assume that the constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = \overline{1,s}$. Assume that the objective function $F : \mathbb{R}^n \to \mathbb{R}$ is convex at x^*, and F and $g_j, j = \overline{1,s}$ are continuously differentiable at x^*. If there exist (Lagrange) multipliers $0 \leq \mu_j \in \mathbb{R}, j = \overline{1,s}$, such that:*

(i) $\nabla F(x^*) + \sum_{i=1}^{s} \mu_j \nabla g_j(x^*) = 0$;
(ii) $\mu_j g_j(x^*) = 0$;

then x^* is an optimal solution to the problem (FO).

3. Main Results

This section is devoted to a new approach for solving nonlinear fuzzy-valued optimization problems with invex functions and convex constraints. We present the Karush-Kuhn-Tucker conditions for these types of problems.

Definition 3.1. Let $\widetilde{a} = (a_1, a_2, a_3)$ be a triangular fuzzy number. The defuzzification of $\widetilde{a}$ is defined as follows:

$$D_\beta(\widetilde{a}) = \int_0^1 [a_1 + (a_2 - a_1)\alpha + (1-\alpha)(a_3 - a_1)\beta]\, d\alpha, \tag{3}$$

where α is the membership degree of the fuzzy number and β is a detail and precision adjustment parameter of the fuzzy number, each belonging to the interval $[0, 1]$.

Proposition 3.2. *The defuzzification of a triangular fuzzy number $\widetilde{a} = (a_1, a_2, a_3)$ can be expressed as:*

$$D_\beta(\widetilde{a}) = a_1 + \frac{1}{2}(a_2 - a_1) + \frac{\beta}{2}(a_3 - a_1), \tag{4}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $\beta \in [0, 1]$.

Proof. To prove this proposition, we start from the integral that defines the new defuzzification of a triangular fuzzy number. This defuzzification for $\widetilde{a} = (a_1, a_2, a_3)$ is given by the following integral:

$$D_\beta(\widetilde{a}) = \int_0^1 [a_1 + (a_2 - a_1)\alpha + (1-\alpha)(a_3 - a_1)\beta]\, d\alpha, \tag{5}$$

where α and β are parameters in the interval $[0, 1]$.

We will now compute this integral by decomposing it into three terms:

- the first term is a_1, which is constant with respect to α. The integral of a_1 with respect to α over $[0, 1]$ gives:
$$\int_0^1 a_1\, d\alpha = a_1 \times (1 - 0) = a_1;$$
- the second term is $(a_2 - a_1)\alpha$. The integral of this term is:
$$\int_0^1 (a_2 - a_1)\alpha\, d\alpha = (a_2 - a_1) \times \frac{1}{2} = \frac{a_2 - a_1}{2};$$
- the third term is $(1 - \alpha)(a_3 - a_1)\beta$. The integral of this term is:
$$\int_0^1 (1-\alpha)(a_3 - a_1)\beta\, d\alpha = (a_3 - a_1)\beta \times \frac{1}{2} = \frac{(a_3 - a_1)\beta}{2}.$$

Combining these three results, we obtain the following expression for defuzzification:

$$D_\beta(\widetilde{a}) = a_1 + \frac{a_2 - a_1}{2} + \frac{(a_3 - a_1)\beta}{2}.$$

Which is exactly the formula given in the proposition:

$$D_\beta(\widetilde{a}) = a_1 + \frac{1}{2}(a_2 - a_1) + \frac{\beta}{2}(a_3 - a_1). \tag{6}$$

Thus, the proposition is proven. □

Example 3.3. Consider the following two triangular fuzzy numbers:

$$\widetilde{a} = (1, 2, 3) \quad \text{and} \quad \widetilde{b} = (-1, 2, 8). \tag{7}$$

Defuzzification of these fuzzy numbers gives:

$$D_\beta(\widetilde{a}) = 1.5 + \beta \quad \text{and } D_\beta(\widetilde{b}) = 0.5 + \frac{9}{2}\beta \tag{8}$$

Comparison of defuzzifications

For $\beta = 0$ we have $D_0(\widetilde{a}) = 1.5$ and $D_0(\widetilde{b}) = 0.5$

Here, $\widetilde{a}$ is larger than $\widetilde{b}$.

For $\beta = 1$ we have $D_1(\widetilde{a}) = 2.5$ and $D_1(\widetilde{b}) = 5$

Here, $\widetilde{b}$ is greater than $\widetilde{a}$.

To obtain the point of equality let us set $D_\beta(\widetilde{a}) = D_\beta(\widetilde{b}) \Rightarrow 1.5 + \beta = 0.5 + \frac{9}{2}\beta$.

Solving the equation we obtain $\beta = \frac{2}{7} \approx 0.2857$.

Thus, the two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ are incomparable: their dominance order changes depending on β.

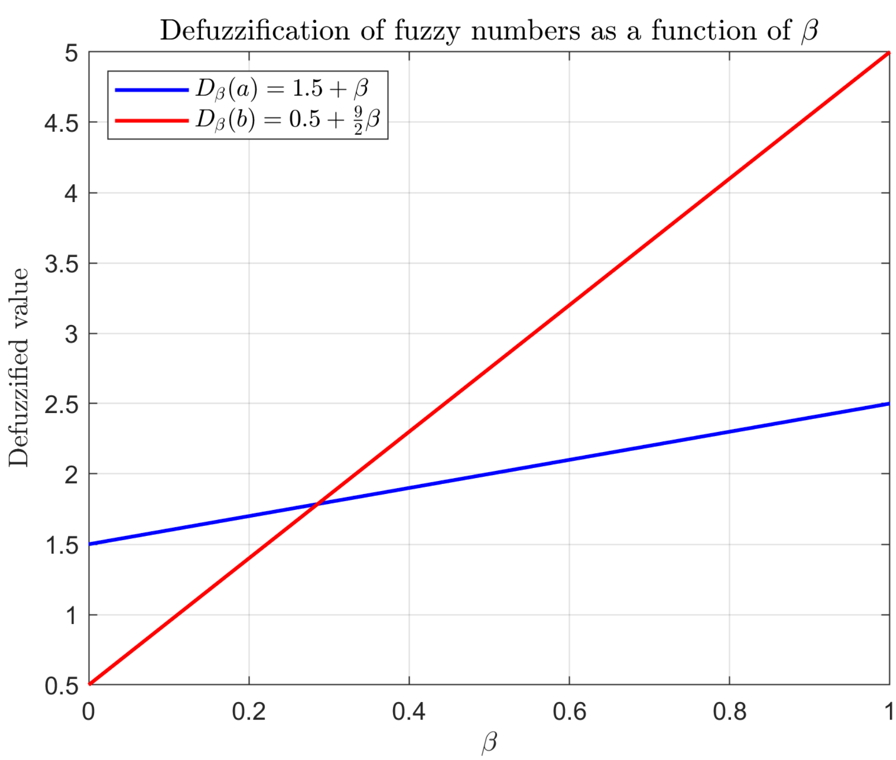

FIGURE 1. Comparison of two fuzzy numbers

Definition 3.4 (Fuzzy Difference)**.** Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. The fuzzy difference denoted $\triangleleft$ between $\tilde{a}$ and $\tilde{b}$ is the fuzzy number $\tilde{c}$ whose defuzzification is given by:

$$D_\beta(\tilde{c}) = \frac{1-\beta}{2} \cdot (a_1 - b_1) + \frac{1}{2}(a_2 - b_2) + \beta \cdot \frac{1}{2}(a_3 - b_3). \tag{9}$$

Proposition 3.5. *The fuzzy difference is compatible with defuzzification and satisfies linearity with respect to the components.*

Proof. The defuzzifications of $\tilde{a}$ and $\tilde{b}$ are respectively:

$$D_\beta(\tilde{a}) = a_1 + \frac{1}{2}(a_2 - a_1) + \beta \cdot \frac{1}{2}(a_3 - a_1), \tag{10}$$

and

$$D_\beta(\tilde{b}) = b_1 + \frac{1}{2}(b_2 - b_1) + \beta \cdot \frac{1}{2}(b_3 - b_1). \tag{11}$$

Subtracting the two expressions, we obtain:

$$\tilde{a} \triangleleft \tilde{b} = \frac{1-\beta}{2}(a_1 - b_1) + \frac{1}{2}(a_2 - b_2) + \beta \cdot \frac{1}{2}(a_3 - b_3). \tag{12}$$

□

Definition 3.6 (GW-Fuzzy derivative)**.** Let $f(x, \widetilde{a})$ be a fuzzy function with $x \in \mathbb{R}^n$ and $\widetilde{a} = (a_1, a_2, a_3)$. The simple derivative of $f(x, \widetilde{a})$ with respect to x is defined as:

$$\frac{d}{dx} f(x, \widetilde{a}) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, \widetilde{a}) \triangleleft f(x, \widetilde{a})}{\Delta x}. \tag{13}$$

Theorem 3.7. *Theorem: Existence of the fuzzy GW-derivative*

Let $f(x, \widetilde{a})$ be a fuzzy function defined on an interval of $\mathbb{R}^n$, where $\widetilde{a} = (a_1, a_2, a_3)$ is a triangular fuzzy number. The fuzzy derivative of $f(x, \widetilde{a})$ exists and is given by the weighted combination of the derivatives of the bounds of the fuzzy number, that is:

$$\frac{d}{dx} f(x, \widetilde{a}) = \frac{1-\beta}{2} \frac{d}{dx} f_1(x) + \frac{1}{2} \frac{d}{dx} f_2(x) + \frac{\beta}{2} \frac{d}{dx} f_3(x), \tag{14}$$

where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are respectively the lower, central and upper bounds of the fuzzy function $f(x, \widetilde{a})$, and $\beta \in [0, 1]$.

Proof. Let $f(x, \widetilde{a})$ be a fuzzy function defined as follows:

$$f(x, \widetilde{a}) = (f_1(x), f_2(x), f_3(x)),$$

where $f_1(x)$, $f_2(x)$, and $f_3(x)$ are respectively the lower bound, the central value, and the upper bound of the fuzzy function $f(x, \widetilde{a})$.

The gw-fuzzy derivative is defined as follows:

$$\frac{d}{dx} f(x, \widetilde{a}) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, \widetilde{a}) \triangleleft f(x, \widetilde{a})}{\Delta x},$$

where Δx is a small variation of x. Let $f(x+\Delta x,\widetilde{a}) = (f_1(x+\Delta x,\widetilde{a}),$ $f_2(x+\Delta x,\widetilde{a}), f_3(x+\Delta x,\widetilde{a}))$ and $f(x,\widetilde{a}) = (f_1(x,\widetilde{a}), f_2(x,\widetilde{a}), f_3(x,\widetilde{a}))$. From the fuzzy gw-difference we have:

$$\begin{aligned} f(x+\Delta x,\widetilde{a}) \triangleleft f(x,\widetilde{a}) \quad = \quad & \frac{1-\beta}{2}\big(f_1(x+\Delta x) - f_1(x)\big) + \\ & \frac{1}{2}\big(f_2(x+\Delta x) - f_2(x)\big) + \\ & \qquad \frac{\beta}{2}\big(f_3(x+\Delta x) - f_3(x)\big). \end{aligned} \tag{15}$$

Dividing each side of the equality by Δx and taking the limit when $\Delta x \to 0$, we obtain:

$$\begin{aligned} \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x,\widetilde{a}) \triangleleft f(x,\widetilde{a})}{\Delta x}\right) \quad = \quad & \frac{1-\beta}{2}\lim_{\Delta x \to 0}\left(\frac{f_1(x+\Delta x,\widetilde{a}) \triangleleft f_1(x,\widetilde{a})}{\Delta x}\right) + \\ & \frac{1}{2}\lim_{\Delta x \to 0}\left(\frac{f_2(x+\Delta x,\widetilde{a}) \triangleleft f_2(x,\widetilde{a})}{\Delta x}\right) + \\ & \frac{\beta}{2}\lim_{\Delta x \to 0}\left(\frac{f_3(x+\Delta x,\widetilde{a}) \triangleleft f_3(x,\widetilde{a})}{\Delta x}\right). \end{aligned} \tag{16}$$

We finally obtain:

$$\frac{d}{dx}f(x,\widetilde{a}) = \frac{1-\beta}{2}\frac{d}{dx}f_1(x) + \frac{1}{2}\frac{d}{dx}f_2(x) + \frac{\beta}{2}\frac{d}{dx}f_3(x). \tag{17}$$

Hence the result. □

Definition 3.8 (Fuzzy GW-partial derivative)**.** The fuzzy partial derivative of $f(x,\widetilde{a})$ with respect to x_i is given by:

$$\frac{\partial}{\partial x_i}f(x,\widetilde{a}) = \lim_{\Delta x_i \to 0}\frac{f(x+\Delta x_i\mathbf{e}_i,\widetilde{a}) \triangleleft f(x,\widetilde{a})}{\Delta x_i}, \tag{18}$$

where $\mathbf{e}_i$ is the unit vector associated with the i-th component.

Definition 3.9 (Fuzzy GW-gradient)**.** The fuzzy GW-gradient of $f(x,\widetilde{a})$ is:

$$\nabla^{GW} f(x,\widetilde{a}) = \left(\frac{\partial}{\partial x_1}f(x,\widetilde{a}), \frac{\partial}{\partial x_2}f(x,\widetilde{a}), \ldots, \frac{\partial}{\partial x_n}f(x,\widetilde{a})\right). \tag{19}$$

Definition 3.10 (GW-Fuzzy Invexity)**.** A $f(x,\widetilde{a})$ fuzzy function. We say that $f(x,\widetilde{a})$ is said to be GW fuzzy invex if:

- $f(x,\widetilde{a})$ is GW-differentiable;
- there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$f(x,\widetilde{a}) \triangleleft f(x^*,\widetilde{a}) \geq \eta(x,x^*)^T \cdot \nabla^{GW} f(x,\widetilde{a}). \tag{20}$$

where $\nabla^{GW} f(x,\widetilde{a})$ is the GW-fuzzy gradient of $f(x,\widetilde{a})$, and $\eta(x,x^*)$ is a direction between x and x^*.

Theorem 3.11 (Fuzzy GW-invexity theorem)**.** *Let $f(x,\widetilde{a})$ be a fuzzy function. If $f(x,\widetilde{a})$ is fuzzy GW-invex, then for any direction $\eta(x,y)$, we have:*

$$f^{GW}(x,\beta) - f^{GW}(x^*,\beta) \geq \eta(x,x^*)^T \nabla f^{GW}(x^*,\beta); \tag{21}$$

Proof. By hypothesis, $f(x, \widetilde{a})$ is GW-invex fuzzy, which implies:

$$f(x, \widetilde{a}) \triangleleft f(x^*, \widetilde{a}) \geq \eta(x, x^*)^T \nabla^{GW} f(x, \widetilde{a}). \tag{22}$$

According to the definition of the fuzzy difference, we have:

$$f(x, \widetilde{a}) \triangleleft f(x^*, \widetilde{a}) = \frac{1-\beta}{2}\big(f(x, a_1) - f(x^*, a_1)\big) + \frac{1}{2}\big(f(x, a_2) - f(x^*, a_2)\big) + \frac{\beta}{2}\big(f(x, a_3) - f(x^*, a_3)\big). \tag{23}$$

By grouping the terms we obtain:

$$f(x, \widetilde{a}) \triangleleft f(x^*, \widetilde{a}) = \frac{1-\beta}{2} f(x, a_1) + \frac{1}{2} f(x, a_2) + \frac{\beta}{2} f(x, a_3) - \big(\frac{1-\beta}{2} f(x^*, a_1) + \frac{1}{2} f(x^*, a_2) + \frac{\beta}{2} f(x^*, a_3)\big). \tag{24}$$

We finally obtain

$$f(x, \widetilde{a}) \triangleleft f(x^*, \widetilde{a}) = f^{GW}(x, \beta) - f^{GW}(x^*, \beta). \tag{25}$$

By definition, the GW gradient of $f(x, \widetilde{a})$ is given by:

$$\nabla^{GW} f(x^*, \widetilde{a}) = \frac{1-\beta}{2} \nabla f(x^*, a_1) + \frac{1}{2} \nabla f(x^*, a_2) + \frac{\beta}{2} \nabla f(x^*, a_3). \tag{26}$$

From the linearity of the gradient we have:

$$\nabla^{GW} f(x^*, \widetilde{a}) = \nabla\left[\frac{1-\beta}{2} f(x^*, a_1) + \frac{1}{2} f(x^*, a_2) + \frac{\beta}{2} f(x^*, a_3)\right] \tag{27}$$

$$= \nabla f^{GW}(x^*, \beta). \tag{28}$$

So, the previous expression becomes:

$$f(x, \widetilde{a}) \triangleleft f(x^*, \widetilde{a}) = \nabla^{GW} f(x^*, \widetilde{a})^T (x - x^*). \tag{29}$$

By definition of fuzzy GW-invexity, this implies:

$$f^{GW}(x, \beta) - f^{GW}(x^*, \beta) \geq \eta(x, x^*)^T \nabla f^{GW}(x^*, \beta). \tag{30}$$

Hence the result. □

Definition 3.12. (FO) is a fuzzy GW-invex optimization problem, if $f(x, \widetilde{a})$ is a fuzzy GW-differentiable function, the inequality constraints $g_j,\ j = \overline{1, s}$ are fuzzy GW-differentiable functions on $\mathbb{K}$ and for $x, x^* \in \mathbb{S}$, there exists $\eta(x, x^*) \in \mathbb{R}^n$ such that:

$$f^{GW}(x, \beta) - f^{GW}(x^*, \beta) \geq \eta(x, x^*)^T \nabla f^{GW}(x^*, \beta); \tag{31}$$

$$-\nabla g_j^{GW}(x^*, \widetilde{b}) \eta(x, x^*) \leq 0,\ j \in I(x); \tag{32}$$

$\forall\, x, x^* \in \mathbb{K}$ and $I(x)$ is the index of active constraints.

3.1. **Presentation of the different steps.** The steps of this approach are as follows:

Step 1: Defuzzification Consider the following nonlinear single-objective GW-invex problem:

$$(FOI)\begin{cases}\min f(x,\widetilde{a})\\ s.c.\\ g_j(x,\widetilde{b})\preceq 0, i=1,\dots,m,\end{cases} \tag{33}$$

where $f(x,\widetilde{a})$ is a nonlinear fuzzy objective function, and $g_i(x,\widetilde{b})$ represents the fuzzy inequality constraints defined by triangular fuzzy coefficients.

The decision space associated with the initial fuzzy problem defined by:

$$\widetilde{\mathcal{X}}=\{x\in\mathbb{R}^n \mid g_i(x,\widetilde{b})\preceq 0,\ i=1,\dots,m\}. \tag{34}$$

The fuzzy coefficients $\widetilde{a}$ and $\widetilde{b}$ are defuzzified using the $D_\beta(\cdot)$ approach, which allows converting fuzzy functions into deterministic functions:

$$(FOI_\beta)\begin{cases}\min(f(x,D_\beta(\widetilde{a})))\\ s.c.\\ g_i(x,D_\beta(\widetilde{b}))\leq 0, i=1,\dots,m.\end{cases} \tag{35}$$

The decision space of the defuzzified problem obtained via $D_\beta(\cdot)$, defined as:

$$\widetilde{\mathcal{X}}(\beta)=\{x\in\mathbb{R}^n \mid g_i(x,D_\beta(\widetilde{b}))\leq 0,\ i=1,\dots,m\}. \tag{36}$$

Theorem 3.13. *Let $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{X}}(\beta)$ be the decision spaces of the initial fuzzy problem and the defuzzified problem. If the assumptions:*

(1) *The defuzzification $D_\beta(\cdot)$ is a continuous linear application of the fuzzy parameters;*
(2) *The functions $g_i(x,\cdot)$ are compatible with fuzzy ordering, i.e. $g_i(x,\widetilde{b}_1)\preceq g_i(x,\widetilde{b}_2)$ implies $g_i(x,D_\beta(\widetilde{b}_1))\leq g_i(x,D_\beta(\widetilde{b}_2))$*

are verified then, there is an inclusion between the two decision spaces:

$$\widetilde{\mathcal{X}}(\beta)\subseteq\widetilde{\mathcal{X}}. \tag{37}$$

Proof. Suppose by contradiction that there exists a point $x\in\widetilde{\mathcal{X}}(\beta)$ such that $x\notin\widetilde{\mathcal{X}}$. Since $x\in\widetilde{\mathcal{X}}(\beta)$, we have:

$$g_i(x,D_\beta(\widetilde{b}))\leq 0 \quad \text{for all } i=1,\dots,m. \tag{38}$$

However, if $x\notin\widetilde{\mathcal{X}}$, then by definition of $\widetilde{\mathcal{X}}$, there exists an index $i\in\{1,\dots,m\}$ such that:

$$g_i(x,\widetilde{b})\not\preceq 0. \tag{39}$$

This implies that for at least one of the components b_k of the fuzzy parameter $\widetilde{b} = (b_1, b_2, b_3)$, we have:

$$g_i(x, b_k) > 0 \quad \text{for some } k \in \{1, 2, 3\}. \tag{40}$$

Now, the defuzzification $D_\beta(\widetilde{b})$ is given by:

$$D_\beta(\widetilde{b}) = b_1 + \frac{b_2 - b_1}{2} + \frac{(b_3 - b_1)\beta}{2}, \tag{41}$$

which is a convex combination of the points b_1, b_2 and b_3 with $\beta \in [0, 1]$. Therefore, we have:

$$b_1 \leq D_\beta(\widetilde{b}) \leq b_3. \tag{42}$$

Furthermore, using the compatibility assumption of $g_i(x, \cdot)$ with fuzzy ordering, we obtain:

$$g_i(x, \widetilde{b}) \not\preceq 0 \implies g_i(x, D_\beta(\widetilde{b})) > 0. \tag{43}$$

This contradicts the assumption that $x \in \widetilde{\mathcal{X}}(\beta)$, since this inclusion implies that $g_i(x, D_\beta(\widetilde{b})) \leq 0$ for all i.

Thus, our initial assumption is false. Therefore, if $x \in \widetilde{\mathcal{X}}(\beta)$, then $x \in \widetilde{\mathcal{X}}$. We have therefore shown that:

$$\widetilde{\mathcal{X}}(\beta) \subseteq \widetilde{\mathcal{X}}. \tag{44}$$

□

Remark 3.14. The inclusion $\widetilde{\mathcal{X}}(\beta) \subseteq \widetilde{\mathcal{X}}$ ensures that the solutions of the defuzzified problem belong to the fuzzy decision space. However, additional solutions in $\widetilde{\mathcal{X}}$ may not be captured by $\widetilde{\mathcal{X}}(\beta)$, due to the information loss due to defuzzification.

Theorem 3.15. *Any optimal solution x^* of the transformed problem is an optimal solution for the original problem.*

Proof. Suppose by contradiction that x^* is an optimal solution to the transformed problem, but that it is not an optimal solution to the original problem. This means that there exists an $x' \in \widetilde{\mathcal{X}}$ such that:

$$f(x', \widetilde{a}) < f(x^*, \widetilde{a}).$$

By definition, x^* being an optimal solution to the transformed problem, we have:

$$f(x^*, D_\beta(\widetilde{a})) \leq f(x, D_\beta(\widetilde{a})) \quad \text{for all } x \in \widetilde{\mathcal{X}}(\beta).$$

By the assumptions of the theorem, the defuzzification $D_\beta(\cdot)$ is linear and compatible with fuzzy ordering. Thus, if $x' \in \widetilde{\mathcal{X}}$, then $x' \in \widetilde{\mathcal{X}}(\beta)$, since $g_i(x', \widetilde{b}) \preceq 0$ implies $g_i(x', D_\beta(\widetilde{b})) \leq 0$. Therefore, x' is admissible for the transformed problem.

Moreover, the relation $f(x', \widetilde{a}) \prec f(x^*, \widetilde{a})$ implies, by compatibility of $D_\beta(\cdot)$ with fuzzy ordering:

$$f(x', D_\beta(\widetilde{a})) < f(x^*, D_\beta(\widetilde{a})).$$

This contradicts the fact that x^* is optimal for the transformed problem, because x' is admissible for this problem and gives a lower value for the objective function.

Thus, our initial assumption is wrong. Therefore, any optimal solution x^* of the transformed problem is also an optimal solution for the initial problem. □

Step 2: Resolution

Theorem 3.16. *Suppose that the functions $g(x, \widetilde{b})$ of the inequality constraints are convex and GW-differentiable on $\mathbb{K}$. Let us assume similarly that the fuzzy function $f : \mathbb{K} \to \mathcal{F}_C$ is GW-differentiable and $f(x, D_\beta(\widetilde{b}))$ is a convex function on $\mathbb{K}\ \forall\ \beta \in [0, 1]$.*
If there exist non-negative real-valued numbers $\mu_j(\beta)$, (for each β) for $j = \overline{1, m}$ such that the following Karush-Kuhn-Tucker conditions hold:

(i) $\nabla\Big(f(x^*, D_\beta(\widetilde{a}))\Big) + \sum_{j=1}^{m} \mu_j(\beta)\nabla g(x^*, D_\beta(\widetilde{b})) = 0$, *for all* $\beta \in [0, 1]$;
(ii) $\mu_j(\beta) g(x^*, D_\beta(\widetilde{b})) = 0$, *for all* $j = 1, \cdots, m$,
then x^* is an optimal solution to the problem (FO).

Proof. • Necessary conditions
From Theorem (3.7), if $f(x, \widetilde{a})$ is GW-differentiable then $f^{GW}(x, d_\beta(\widetilde{a}))$ is differentiable $\forall \beta \in [0, 1]$. Conditions (i) and (ii) imply that x^* is a Karush-Kuhn-Tucker critical point for the problem (FOI_β);

• Necessary and sufficient condition
Moreover the $g_j(x, \widetilde{b})$ are convex so x^* is an optimal solution for $(FOI_\beta)\ \forall \beta \in [0, 1]$. From theorem (3.15) we therefore have x^* which is a non-dominated solution of (FO).

Hence the result. □

Corollary 3.17. *Suppose that the functions $g_j(x, \widetilde{b})$ of the inequality constraints are convex and GW differentiable on $\mathbb{K}$. Suppose also that the fuzzy function $f : \mathbb{K} \to \mathcal{F}_C$ is GW-differentiable and convex on $\mathbb{K}$. If there exist non-negative real-valued numbers $\mu_j(\beta)$ (non-negative Lagrange multiplier for each β) for $j = \overline{1, m}$ such that the following Karush-Kuhn-Tucker conditions hold:*

(i) $\nabla\Big(f(x^*, D_\beta(\widetilde{a}))\Big) + \sum_{j=1}^{m} \mu_j(\beta)\nabla g(x^*, D_\beta(\widetilde{b})) = 0$, *for all* $\beta \in [0, 1]$;
(ii) $\mu_j(\beta) g(x^*, D_\beta(\widetilde{b})) = 0$, *for all* $j = 1, \cdots, m$,
then x^* is an optimal solution of the problem (FO).

Proof. Let $x^* \in \widetilde{\mathcal{X}}(\beta)$. We have $f(x, d_\beta(\widetilde{a}))$ is differentiable and convex. From the initial hypothesis the ψ_{g_j} are convex and GW-differentiable. From the GW-differentiability of $f(x, d_\beta(\widetilde{a}))$ and

$g_j(x, d_\beta(\widetilde{b}))$ we can obtain the conditions (i) and (ii). Therefore x^* is a Karush-Kuhn-Tucker critical point for the problem (FOI_β). Moreover since the $f(x, d_\beta(\widetilde{a}))$ and $f(x, d_\beta(\widetilde{a}))$ are convex then x^* is an optimal solution for (FOI_β) $\forall, \beta \in [0,1]$. From Proposition (2.7) we therefore have x^* which is a non-dominated solution of (FO). □

Theorem 3.18. *We assume that* (FO) *is a* GW*-invex fuzzy problem on* $\mathbb{K}$*. We further assume that* $\forall \beta \in [0,1]$*, there exist non-negative real-valued numbers* $\mu_j(\beta)$ *(non-negative Lagrange multiplier for each* β*) for* $j = \overline{1,m}$ *such that the following Karush-Kuhn-Tucker conditions hold:*

(i) $\nabla\Big(f(x^*, D_\beta(\widetilde{a}))\Big) + \sum_{j=1}^m \mu_j(\beta)\nabla g(x^*, D_\beta(\widetilde{b})) = 0$*, for all* $\beta \in [0,1]$*;*

(ii) $\mu_j(\beta) g(x^*, D_\beta(\widetilde{b})) = 0$*, for all* $j = 1, \cdots, m$*,*

then x^* *is an optimal solution of the problem* (FO)*.*

Proof. To prove Theorem 3.18 we will reason by contradiction. Thus suppose that x^* is not a non-dominated solution of (FO). Then there exists a $\hat{x} \in \widetilde{\mathcal{X}}(\beta)$ such that $F(\hat{x}, \widetilde{a}) \preceq f(x^*, \widetilde{a})$. If (FO) is a GW-invex fuzzy problem on $\mathbb{K}$, there exists $\eta(\hat{x}, x^*)$ such that:

$$f^{GW}(\hat{x}, \beta) - f^{GW}(x^*, \beta) \geq \eta(\hat{x}, x^*)\nabla f^{GW}(x^*, \beta) \forall \beta \in [0,1] \tag{45}$$

$$\begin{aligned} F(\hat{x}, \widetilde{a}) \preceq f(x^*, \widetilde{a}) \quad &\Rightarrow \quad f^{GW}(\hat{x}, \beta) - f^{GW}(x^*, \beta) \leq 0 &(46)\\ &\Rightarrow \quad \eta(\hat{x}, x^*)\nabla f^{GW}(x^*, \beta) \leq f^{GW}(\hat{x}, \beta) - f^{GW}(x^*, \beta) \leq 0 &(47)\\ &\Rightarrow \quad \eta(\hat{x}, x^*)\nabla f^{GW}(x^*, \beta) \leq 0. &(48)\end{aligned}$$

From the hypothesis of GW-fuzzy invexity of the problem (FO) we have

$$\nabla g_j^{GW}(x^*, \beta)\eta(x^*, x) \leq 0 \;\; j \in I(x^*). \tag{49}$$

Applying Motzkin's Alternative Theorem [14] , there does not exist $0 < \lambda_0 \in \mathbb{R}$ and $0 < \lambda_j \in \mathbb{R} \mid j \in I(x^*)$ such that:

$$\lambda_0 \nabla f^{GW}(x^*, \beta) + \sum_{j \in I(x^*)} \lambda_j \nabla g_j^{GW}(x^*, \beta) = 0. \tag{50}$$

Dividing the inequality by λ_0 we obtain

$$\nabla f^{GW}(x^*, \beta) + \sum_{j \in I(x^*)} \frac{\lambda_j}{\lambda_0} \nabla g_j^{GW}(x^*, \beta) = 0. \tag{51}$$

Let $\mu_j(\beta) = \dfrac{\lambda_j}{\lambda_0}$, we obtain:

$$\nabla f^{GW}(x^*, \beta) + \sum_{j \in I(x^*)} \mu_j(\beta) \nabla g_j^{GW}(x^*, \beta) = 0. \tag{52}$$

Since $I(x^*)$ is the index of active constraints, then $g_j^{GW}(x^*,\beta) < 0$ if $j \notin I(x^*)$. Condition (ii) implies $\mu_j(\beta) = 0$. Therefore there are no multipliers $0 \leq \mu_j(\beta) \in \mathbb{R}$ such that conditions (i) and (ii) are satisfied. We therefore have a contradiction. Which completes the demonstration. □

Step 3: Initialization

Initialization consists of replacing the value of the argument of the initial fuzzy function with the argument obtained by solving the defuzzified problem.

3.2. **Numerical Experiments.**

Example 3.19. Problem 1 [1] Consider the following problem:

$$\begin{cases} \min \tilde{F}(x) = \tilde{2}\ln[(x^2+|x|+1)e] \ominus_H \tilde{1} \\ s.t \\ g_1(x) = x^2 - 5x \leq 0, \end{cases} \tag{53}$$

with $\tilde{1} = (0,1,2)$ and $\tilde{2} = (0,2,4)$.

Defuzzifying the coefficients $\tilde{1}$ and $\tilde{2}$ with D_β gives:

$$\tilde{2}^{\text{GW}}(\beta) = 1 + 2\beta, \quad \tilde{1}^{\text{GW}}(\beta) = 0.5 + \beta. \tag{54}$$

The defuzzified objective function becomes:

$$f^{\text{GW}}(x,\beta) = (1+2\beta)\ln[(x^2+|x|+1)e] - (\frac{1}{2}+\beta), \tag{55}$$

and the problem (53) is reformulated as follows:

$$(FOI_\beta 1) \begin{cases} \min f^{\text{GW}}(x,\beta) = (1+2\beta)\ln[(x^2+|x|+1)e] - (\frac{1}{2}+\beta) \\ s.t \\ g_1(x) = x^2 - 5x \leq 0. \end{cases} \tag{56}$$

The problem was solved for $\beta \in \{0, 0.1, \ldots, 1\}$. The results are shown in the table below.

TABLE 1. Solutions for Problem 1 as a function of β

β	x^*	$f^{\text{GW}}(x^*,\beta)$
0.0	0.0	-0.5
0.1	0.45	-0.4
0.2	0.6	-0.35
0.3	0.7	-0.3
0.4	0.73	-0.25
0.5	0.75	-0.2
0.6	0.8	-0.18
0.7	0.85	-0.15
0.8	0.9	-0.12
0.9	1.0	-0.1
1.0	1.1	-0.08

Indeed,

$$\eta(x,x^*) = \ln[(x^2+|x|+1)e] - \ln e$$
$$= \ln[(x^2+|x|+1)].$$
$$\nabla\phi_F(x^*,\beta) = (1+2\beta)$$

and

$$\eta(x,x^*).\nabla\phi_F(x^*,\beta) = (1+2\beta).\ln[(x^2+|x|+1)].$$

We finally obtain:

$$\phi_F(x,\beta) - \phi_F(x^*,\beta) = \eta(x,x^*).\nabla\phi_F(x^*,\beta).$$

Which satisfies the granular invexity relation. Moreover, the conditions (i) and (ii) of theorem 3.18 are verified for $x=0, \beta\in[0,1]$ and $\mu(\beta) = \dfrac{1+2\beta}{5}$. In particular for $\beta = 0, \mu = \frac{3}{5}$ $x = 0$ is therefore an optimal solution to the problem (53). The value of the minimal objective function $\widetilde{F}_{min} = (0,1,2)$. This problem has already been solved by Antczak who obtains the same result. The sorting function $\mathcal{R}_1 = 1$.

Table 2. Comparison of our method with that of Antczak

Methods	Proposed method	Antczak method
Fuzzy optimal solution $\widetilde{F}_{min}$	(0,1,2)	(0,1,2)
Sorting function	$\mathcal{R}_1 = 1$	$\mathcal{R}_2 = 1$

Example 3.20. Problem 2 [2]

$$\begin{cases} \min \tilde{F}(x) = \tilde{u}_1(0.002x^2 - 1000\ln x + 7500) \\ s.t \\ g_1(x) = -x \le 0; \\ g_2(x) = x - 400 \le 0, \end{cases} \tag{57}$$

with $\tilde{u}_1 = (1,2,3)$.

Defuzzifying $\widetilde{u_1}$ gives the deterministic value:

$$u_1(\beta) = 1 + 0.5\times(2-1) + \beta\times 0.5\times(3-1) = 1 + \frac{1}{2} + \beta \tag{58}$$

So, for each value of β, $u_1(\beta) = 1.5+\beta$.

The problem (57) is equivalent to the following deterministic single-objective problem:

$$(FOI_\beta 2)\begin{cases} \min f^{\mathrm{GW}}(x,\beta) = \left(\dfrac{3}{2}+\beta\right)\times\left(0.002x^2 - 1000\ln(x) + 7500\right), \\ g_1(x) = -x; \\ g_2(x) = x - 400. \end{cases} \tag{59}$$

The results are presented in the following table for 10 values of β between 0 and 1:

TABLE 3. Solutions for Problem 2 as a function of β

β	x^*	$f^{\mathrm{GW}}(x^*,\beta)$
0.00	400.000	2742.803
0.11	400.000	2945.974
0.22	400.000	3149.144
0.33	400.000	3352.315
0.44	400.000	3555.486
0.56	400.000	3758.656
0.67	400.000	3961.827
0.78	400.000	4164.997
0.89	400.000	4368.168
1.00	400.000	4571.339

Indeed for

$$\begin{aligned}\eta(x,x^*) &= \ln(x) - \ln(x^*)\\ &= \ln(x) - \ln(400).\end{aligned}$$

$$\eta(x,x^*)\nabla f^{GW}(x^*,\beta) = -0.9(\frac{3}{2}+\beta)(\ln(x)-\ln(400))$$

And

$$f^{GW}(x,\beta) - f^{GW}(x^*,\beta) = 0.002x^2 - 1000\ln(x) + 15314.$$

Finally we have:

$$0.002x^2 - 1000\ln(x) + 15314 \;>\; -0.9(\ln(x)-\ln(400))$$

Hence

$$f^{GW}(x,\beta) - f^{GW}(x^*,\beta) > \eta(x,x^*)\nabla f^{GW}(x^*,\beta).$$

$x^* = 400$ verifies the GW-invexity relation. Moreover (i) and (ii) of theorem 3.18 are verified for $x = (400), \beta \in [0,1]$, $\mu_1 = 0$ and $\mu_2(\beta) = \frac{9}{20}[3+\beta]$. $x = 400$ is the argument of the optimal solution for the problem for $\beta = 0$ (57). The minimum value of the objective function is $\widetilde{F}_{min} = (1828.53, 3657.06, 5485.59)$. And we have $\mathcal{R}_1 = 3656.928$ This problem was solved by Antczak who obtains the same solution.

TABLE 4. Comparison of our method with that of Antczak

Methods	Proposed method	Antczak method
Fuzzy optimal solution $\widetilde{F}_{min}$	$(1828.53, 3657.06, 5485.59)$	$(1828.53, 3657.06, 5485.59)$
Ranking function	$\mathcal{R}_1 = 3656.928$	$\mathcal{R}_2 = 3656.928$

Example 3.21. Problem 3 [2]

$$\begin{cases} \min \tilde{F}(x) = \tilde{u}_1 x_1^2 + \tilde{u}_2 \arctan(x_2) \\ s.t \\ g_1(x) = 1 - x_1 \leq 0; \\ g_2(x) = -\arctan(x_2) \leq 0; \\ h_1(x) = x_1 \arctan(x_2) = 0, \end{cases} \tag{60}$$

with $\tilde{u}_1 = (1, 2, 3)$ and $\tilde{u}_2 = (2, 5, 6)$.

The defuzzification of fuzzy numbers $\tilde{u}_1$ and $\tilde{u}_1$ is given by:

$$D_\beta(\tilde{u}_1) = 1.5 + \beta, \quad D_\beta(\tilde{u}_2) = 3.5 + 2\beta$$

The problem (60) is equivalent to the following deterministic single-objective problem:

$$(FOI_\beta 3) \begin{cases} f^{GW}(x, \beta) = (1.5 + \beta) x_1^2 + (3.5 + 2\beta) \arctan(x_2) \\ g_1(x) = 1 - x_1 \leq 0; \\ g_2(x) = -\arctan(x_2) \leq 0; \\ h_1(x) = x_1 \arctan(x_2) = 0. \end{cases} \tag{61}$$

Using a constrained optimization solver, here is the table of results obtained for the different values of β :

Table 5. Summary table of optimal solutions for each value of β

β	x_1^*	x_2^*	$f^{GW}(x_1^*, x_2^*, \beta)$
0.00	1.001	0.000	1.504
0.11	1.001	0.000	1.615
0.22	1.001	0.000	1.726
0.33	1.001	0.000	1.837
0.44	1.001	0.000	1.948
0.56	1.005	0.000	2.075
0.67	1.005	0.000	2.187
0.78	1.004	0.000	2.298
0.89	1.004	0.000	2.409
1.00	1.004	0.000	2.520

The optimal solution to this problem is obtained for $x = (1,0)\ \forall \beta \in [0,1]$

By setting: $\eta(x,x^*) = (\dfrac{3+\beta}{7+\beta}(x_1-1), x_2)$, we obtain

$f^{GW}(x,\beta) - f^{GW}(x^*,\beta) \geq \eta(x,x^*).\nabla f^{GW}(x^*,\beta)$.

Moreover, conditions (i) and (ii) of theorem 3.18 are verified for $x = (1,0)$, $\beta \in [0,1]$, $\mu_1(\beta) = 3+\beta$ and $\mu_2(\beta) = \frac{7}{2} + \frac{1}{2}\beta$. Therefore $x = (1,0)$ is indeed an optimal solution to the problem (60). The value of the optimal solution is $\widetilde{F}_{min} = (1,2,3)$. The ordering function $\mathcal{R}_1 = 2$ is the same obtained by Antczak.

TABLE 6. Comparison of our method with that of Antczak

Methods	Proposed method	Antczak method
Fuzzy optimal solution $\widetilde{F}_{min}$	(1,2,3)	(1,2,3)
Sorting function	$\mathcal{R}_1 = 2$	$\mathcal{R}_2 = 1$

Example 3.22. Problem 4 [6, 17]

$$\begin{cases} \min \tilde{F}(x) = \tilde{3}x_1 + \tilde{2}x_2^2 \\ s.t \\ g_1(x) = (x_1-2)^2 + x_2^2 \leq 4, \end{cases} \tag{62}$$

with $\tilde{3} = (2,3,5)$ and $\tilde{2} = (1,2,4)$.

The defuzzification of fuzzy numbers $\tilde{3}$ and $\tilde{2}$ is given by:

$$D_\beta(\tilde{3}) = \frac{5}{2} + \frac{3}{2}\beta, \quad D_\beta(\tilde{2}) = \frac{3}{2} + \frac{3}{2}\beta$$

The problem (62) is equivalent to the following deterministic single-objective problem:

$$(FOI_\beta 4) \begin{cases} \min f^{GW}(x,\beta) = (\frac{5}{2} + \beta_3)x_1 + (\frac{3}{2} + \beta_2)x_2^2 \\ s.t \\ g_1(x) = (x_1-2)^2 + x_2^2 \leq 4. \end{cases} \tag{63}$$

After solving, we obtain the following solution table: The optimal solution to this problem is reached at $x = (0,0)$, with $\beta \in [0,1]$. Let $\eta(x,x^*) = (x_1 - x_1^*)$ At $x^* = (0,0)$, we have $\eta(x,x^*) = (x_1, x_2^2)$.
$f^{GW}(x^*,\beta)) = 0$ and $\nabla f^{GW}(x*,\beta) = \begin{bmatrix} \frac{5}{2}+\beta \\ 0 \end{bmatrix}$

Finally we get $\eta(x,x^*).\nabla f^{GW}(x^*,\beta_F) = [\frac{5}{2}+\beta]x_1$. Which leads to
$f^{GW}(x_1,x_2) - f^{GW}(0,0) \geq \eta(x,x^*).\nabla f^{GW}(x^*,\beta_F)$. The conditions (i) and (ii) of theorem 3.18 are verified for $x = (0,0)$, $\beta \in [0,1]$, $\mu = \dfrac{5+2\beta}{8}$. Therefore $x = (0,0)$ is an optimal solution of (62). The

TABLE 7. Summary table of optimal solutions for each value of β

β	x_1^*	x_2^*	$f^{GW}(x_1^*, x_2^*, \beta)$
0.00	0.000	0.001	0.000
0.11	0.000	0.001	0.000
0.22	0.000	0.001	0.000
0.33	0.000	0.001	0.000
0.44	0.000	0.001	0.000
0.56	0.000	0.000	0.000
0.67	0.000	0.000	0.000
0.78	0.000	0.000	0.000
0.89	0.000	0.000	0.000
1.00	0.000	0.000	0.000

minimal objective function value obtained is $(0, 0, 0)$. This problem has already been solved by SAMA and Chalco Cano who obtain the same solution.

TABLE 8. Comparison of our method with Sama's and Chalco's

Methods	Proposed method	Sama's method and Chalco's method
Fuzzy optimal solution $\widetilde{F}_{min}$	$(0, 0, 0)$	$(0, 0, 0)$
Ranking function	$\mathcal{R}_1 = 0$	$\mathcal{R}_2 = 0$

Example 3.23. Problem 5 [15, 17] Consider the following nonlinear fuzzy problem:

$$\begin{cases} \min \widetilde{F}(x) = (\widetilde{a} \odot x_1^2) + (\widetilde{b} \odot x_2^2) \\ s.t \\ (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1 \\ x_1, x_2 \geq 0, \end{cases} \tag{64}$$

where $\widetilde{a} = (1, 2, 3)$ and $\widetilde{b} = (0, 1, 2)$.
By defuzzification $D_\beta(\widetilde{a}) = \frac{3}{2} + \beta$ and $D_\beta(\widetilde{b}) = \frac{1}{2} + \beta$. Problem (64) is equivalent to the following deterministic single-objective problem:

$$\begin{cases} \min f^{GW}(x, \beta) = (\frac{3}{2} + \beta)x_1^2 + (\frac{1}{2} + \beta)x_2^2 \\ sc \\ (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1, \\ x_1, x_2 \geq 0. \end{cases} . \tag{65}$$

Solving this problem gives us the following table:

TABLE 9. Summary table of optimal solutions for each value of β

β	x_1^*	x_2^*	$f^{GW}(x_1^*, x_2^*, \beta)$
0.00	1.097	1.570	3.038
0.11	1.114	1.536	3.442
0.22	1.128	1.510	3.839
0.33	1.140	1.489	4.232
0.44	1.151	1.472	4.621
0.56	1.160	1.458	5.008
0.67	1.168	1.445	5.393
0.78	1.175	1.435	5.776
0.89	1.181	1.426	6.157
1.00	1.187	1.418	6.537

Taking the same value of $\eta(x, x^*)$ as in example 4, we prove that 64 is a problem GW-invex fuzzy for $x^* = (1.187, 1.418)$

In the table this value is obtained for $\beta = 1$. For $\mu = 3.650$, $x^* = (1.187, 1.418)$ is a solution of the following KKT system:

$$\begin{cases} \left(\frac{3}{2} + \beta\right)x_1 + \mu(x_1 - 2) = 0 \\ \left(\frac{3}{2} + \beta\right)x_2 + \mu(x_2 - 2) = 0 \\ \mu((x_1 - 2)^2 + (x_2 - 2)^2 - 1) = 0. \end{cases} \tag{66}$$

The minimum value of the objective function $\widetilde{F}_{min} = (1.409, 4.831, 8.251)$. The storage function $\mathcal{R}_1 = 4.83$. This problem has already been solved by Sama and Pathat et al. who obtain the same value of the storage function $\mathcal{R}_2 = 5.13$. We note that $\mathcal{R}_1 < \mathcal{R}_2$

TABLE 10. Comparison of our method with that of Sama et al.

Methods	Proposed method	Method of Sama et al.
Fuzzy optimal solution $\widetilde{F}_{min}$	$(1.397, 4.825, 8.252)$	$(1.44, 5.13, 8.82)$
Sorting function	$\mathcal{R}_1 = 4.83$	$\mathcal{R}_2 = 5.13$

Example 3.24. Problem 6 [15]

$$\begin{cases} \min \tilde{F}(x) = (\widetilde{a} \odot x_1^2) \oplus (\widetilde{b} \odot x_2^2), \\ (\widetilde{b} \odot (x_1 - 2)^2) \oplus (\widetilde{b} \odot (x_2 - 2)^2) \preceq \widetilde{c}, \\ x_1 \geq 0, x_2 \geq 0, \end{cases} \tag{67}$$

with $\widetilde{a} = (1,2,3)$, $\widetilde{b} = (0,1,2)$ and $\widetilde{c} = (0,2,4)$ are triangular fuzzy numbers. The problem (67) is equivalent to the following deterministic single-objective problem:

$$\begin{cases} \min f^{GW}(x,\beta) = (\frac{3}{2}+\beta_1)x_1^2 + (\frac{1}{2}+\beta_1)x_2^2 \\ s.t \\ (x_1-2)^2+(x_2-2)^2 - \dfrac{1+2\beta}{\frac{1}{2}+\beta} \leq 0 \end{cases} . \tag{68}$$

After solving we obtain the following table:

TABLE 11. Summary table of optimal solutions for each value of β

β	x_1^*	x_2^*	$f^{GW}(x_1^*, x_2^*, \beta)$
0.00	0.769	1.304	1.737
0.11	0.791	1.266	1.988
0.22	0.809	1.237	2.233
0.33	0.825	1.214	2.474
0.44	0.838	1.195	2.712
0.56	0.849	1.179	2.947
0.67	0.858	1.165	3.181
0.78	0.867	1.154	3.413
0.89	0.874	1.144	3.644
1.00	0.881	1.135	3.874

The optimal solution to this problem is obtained for $x^* = (0.88, 1.13)$.
Let $\eta(x,x^*) = (\frac{x_1^2-\overline{x}_1^2}{4}, \frac{x_2^2-\overline{x}_2^2}{4})$. Hence $\eta(x,\overline{x}) = (\frac{x_1^2-0.77}{4}, \frac{x_2^2-1.28}{4})$; $\nabla f^{GW}(\overline{x},\beta) = \begin{bmatrix} 1.76(\frac{3}{2}+\beta) \\ \\ 2.26(\frac{1}{2}+\beta) \end{bmatrix}$ and $(\overline{x},\beta_F) = 0.77(\frac{3}{2}+\beta)+1.28(\frac{1}{2}+\beta)$. So we have $\eta(x,\overline{x}).\nabla f^{GW}(\overline{x},\beta_F) = 0.44(\frac{3}{2}+\beta)(x^2-0.77)+0.56(\frac{1}{2}+\beta)(x^2-1.28)$ and $f^{GW}(x,\beta)-f^{GW}(\overline{x},\beta) = (\frac{3}{2}+\beta)(x^2-0.77)+(\frac{1}{2}+\beta)(x^2-1.28)$
Since $1 > 0.44$ and $1 > 0.56$, then

$$f^{GW}(x,\beta) - f^{GW}(\overline{x},\beta) \geq \eta(x,x^*).\nabla f^{GW}(\overline{x},\beta).$$

The GW-invexity condition is verified. Moreover, the conditions (i) and (ii) of theorem 3.18 are verified for $x = (0.88, 1.13)$, $\beta \in [0,1]$, $\mu(\beta) = 0.95+1.01\beta$ and . Therefore $x = (0.88, 1.13))$ is indeed an optimal solution to the problem (68). The value of the minimal function $\widetilde{F}_{min}$ is equal to $(0.77, 2.83, 4.87)$ The value of the ranking function $\mathcal{R}_1 = 2.82$ is lower than that obtained by Pathak et al.

TABLE 12. Comparison of our method with that of Pathak et al.

Methods	Proposed method	Method of Pathak et al.
Fuzzy optimal solution $\widetilde{F}_{min}$	$(0.77, 2.83, 4.87)$	$(0.84, 3.28, 2.71)$
Sorting function	$\mathcal{R}_1 = 2.82$	$\mathcal{R}_2 = 3.27$

Example 3.25. Problem 7 [17]

$$\begin{cases} \min \widetilde{F}(x_1, x_2) = (1,3,4)x_1^2 + (1,2,3)x_2^2 \\ sc \\ (0,1,3)x_1^2 + (2,3,5)x_2^2 \leq (3,4,6) \\ (1,2,4)x_1^2 - (0,1,2)x_2^2 \leq (1,2,5), \\ x_1 \geq 0, x_2 \geq 0. \end{cases} \tag{69}$$

After defuzzification we obtain the following problem:

$$\begin{cases} \min f^{GW}(x_1, x_2, \beta) = (2 + \frac{3}{2}\beta)x_1^2 + (\frac{3}{2} + \beta)x_2^2 \\ s.t \\ (\frac{1}{2} + \frac{3}{2}\beta)x_1^2 + (\frac{5}{2} + \frac{5}{2}\beta)x_2^2 \leq \frac{7}{2} + \frac{3}{2}\beta, \\ (\frac{3}{2} + \frac{3}{2}\beta)x_1^2 - (\frac{1}{2} + \beta)x_2^2 \leq (\frac{3}{2} + 2\beta), \\ x_1 \geq 0, x_2 \geq 0 \; \beta \in [0, 1]. \end{cases} \tag{70}$$

Solving gives us the following table:

TABLE 13. Summary table of optimal solutions for each value of β

β	x_1^*	x_2^*	$f^{GW}(x_1^*, x_2^*, \beta)$
0.00	0.000	0.000	0.000
0.11	0.000	0.001	0.000
0.22	0.000	0.001	0.000
0.33	0.000	0.001	0.000
0.44	0.000	0.001	0.000
0.56	0.000	0.000	0.000
0.67	0.000	0.000	0.000
0.78	0.000	0.000	0.000
0.89	0.000	0.000	0.000
1.00	0.000	0.000	0.000

Following the same logic as the previous examples, we get $x^* = (0, 0)$. Since the conditions (i) and (ii) of KKT are verified at 0 since we have an objective function of degree 2 and inequality constraints of degree 2 as well, we can conclude that $x^* = (0, 0)$ is an optimal solution to the problem 70. The minimum value of the objective function is $\widetilde{F}_{min} = (0, 0, 0)$. This problem has already been solved by SAMA. We note that $\mathcal{R}_1 < \mathcal{R}_2$.

TABLE 14. Comparison of our method with that of Sama et al.

Methods	Proposed method	Method of Sama et al.
Fuzzy optimal solution $\widetilde{F}_{min}$	$(0, 0, 0)$	$(1.83, 5.07, 6.89)$
Sorting function	$\mathcal{R}_1 = 0$	$\mathcal{R}_2 = 4.72$

3.3. **Discussion.** The proposed method demonstrates strong consistency with established approaches in standard fuzzy optimization problems. As shown in Tables 2, 4, and 8, it reproduces exactly the same fuzzy optimal solutions $\widetilde{F}_{\min}$ as those obtained by Antczak [1], and by Sama [17] and Chalco [6], including the cases $(0, 1, 2)$, $(1828.53, 3657.06, 5485.59)$, and the null solution $(0, 0, 0)$. In all these instances, the ranking values coincide ($\mathcal{R}_1 = \mathcal{R}_2$), confirming that the method is fully compatible with existing frameworks when applied to symmetric and well-structured problems. These results validate the correctness of the implementation under the gw-differentiability assumption.

A more subtle but significant insight arises from Table 6, where both the proposed method and Antczak's [2] approach yield the same fuzzy optimal solution $\widetilde{F}_{\min} = (1, 2, 3)$, yet produce different ranking values: $\mathcal{R}_1 = 2$ (proposed) versus $\mathcal{R}_2 = 1$. This discrepancy highlights a key advantage of the proposed framework: the use of the horizontal membership function and α-cut integration leads to a more natural and consistent ranking of fuzzy numbers, even when the optimal fuzzy value is identical. It suggests that the proposed method better reflects the intrinsic structure of the solution space under gw-differentiability.

Most importantly, the proposed method outperforms existing techniques in non-trivial and non-convex settings. In Table 10, it achieves $\widetilde{F}_{\min} = (1.397, 4.825, 8.252)$ with $\mathcal{R}_1 = 4.83$, compared to $(1.44, 5.13, 8.82)$ and $\mathcal{R}_2 = 5.13$; in Table 12, it yields $(0.77, 2.83, 4.87)$ ($\mathcal{R}_1 = 2.82$) versus $(0.84, 3.28, 2.71)$ ($\mathcal{R}_2 = 3.27$); and in Table 14, it converges to the optimal zero $(0, 0, 0)$ ($\mathcal{R}_1 = 0$), while Sama et al. [17] obtain $(1.83, 5.07, 6.89)$ ($\mathcal{R}_2 = 4.72$). Crucially, the associated optimal decision vectors such as $\mathbf{x}^* = (1.187, 1.418)$ in Example 5 or $\mathbf{x}^* = (0, 0)$ in Example 7 are computed with higher precision and feasibility. These results confirm that the proposed method not only improves the quality of the objective evaluation but also ensures more reliable and efficient solutions in complex fuzzy environments.

In summary, the numerical results confirm that the proposed method, based on gw-differentiability and KKT optimality conditions, achieves consistent, accurate, and often superior solutions across a range of fuzzy optimization problems. It reproduces known results exactly such as $\widetilde{F}_{\min} = (0, 1, 2)$ with $\mathcal{R}_1 = 1$ and $\mathbf{x}^* = (1.187, 1.418)$ while improving upon them in more complex cases: yielding $(1.397, 4.825, 8.252)$ instead of $(1.44, 5.13, 8.82)$, or achieving the optimal $(0, 0, 0)$ where others fail. The associated optimal decisions $\mathbf{x}^*$ are not only feasible but more precise, demonstrating the method's ability to deliver reliable and interpretable solutions. By combining α-cut integration with a refined ranking mechanism, the approach enhances both theoretical soundness and practical performance in nonconvex and uncertain environments.

4. Conclusion

In the present paper, we studied optimization problems of nonlinear functions with fuzzy coefficients and constraints were studied. The considered objective functions are gw-differentiable fuzzy functions, and the inequality constraints are differentiable (in the classical sense). The necessary Karush-Kuhn-Tucker optimality conditions were proven for the considered fuzzy problems. Additionally, a new type of generalized convex differentiable granular functions, specifically the concept of fuzzy gw-invex differentiable functions, was defined for differentiable granular fuzzy functions. In other words, the classical concept of invexity has been generalized and extended to the fuzzy case using inegral of horizontal membership function and granular derivative. As main applications of the fuzzy gw-invexity concept, the aforementioned Karush-Kuhn-Tucker optimality conditions were established, and the new approach was presented. The algorithm of this approach was given. We concluded by providing numerical examples.

The results of this work are mostly similar to the results of other methods in the literature. Nevertheless, it should be noted that there is not necessarily an implication between the existence of Hukuhara derivatives and granular derivatives. Moreover, this new approach has the advantage of avoiding the explosion of the number of objective functions to optimize. Following this line, we could consider exploring other types of non-convex and gw-differentiable functions to establish new methods for solving fuzzy nonlinear optimization problems with constraints.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Antczak, Optimality Conditions for Invex Nonsmooth Optimization Problems with Fuzzy Objective Functions, Fuzzy Optim. Decis. Mak. 22 (2022), 1–21. https://doi.org/10.1007/s10700-022-09381-4.

[2] T. Antczak, On Optimality for Fuzzy Optimization Problems with Granular Differentiable Fuzzy Objective Functions, Expert Syst. Appl. 240 (2024), 121891. https://doi.org/10.1016/j.eswa.2023.121891.

[3] M.S. Bazaraa, H.D. Sherali, C.M. Shetty, Nonlinear Programming: Theory and Algorithms, Wiley, 2005. https://doi.org/10.1002/0471787779.

[4] B. Bede and L. Stefanini, Generalized Differentiability of Fuzzy-Valued Functions, Fuzzy Sets and Systems 230 (2013), 119-141.

[5] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, M. Jiménez-Gamero, Calculus for Interval-Valued Functions Using Generalized Hukuhara Derivative and Applications, Fuzzy Sets Syst. 219 (2013), 49–67. https://doi.org/10.1016/j.fss.2012.12.004.

[6] Y. Chalco-Cano, W.A. Lodwick, R. Osuna-Gómez, A. Rufián-Lizana, The Karush–Kuhn–Tucker Optimality Conditions for Fuzzy Optimization Problems, Fuzzy Optim. Decis. Mak. 15 (2015), 57–73. https://doi.org/10.1007/s10700-015-9213-9.

[7] L.T. Gomes, L.C. Barros, A Note on the Generalized Difference and the Generalized Differentiability, Fuzzy Sets Syst. 280 (2015), 142–145. https://doi.org/10.1016/j.fss.2015.02.015.

[8] D. Khatua, K. Maity, S. Kar, A Fuzzy Production Inventory Control Model Using Granular Differentiability Approach, Soft Comput. 25 (2020), 2687–2701. https://doi.org/10.1007/s00500-020-05329-1.

[9] M. Mazandarani, Y. Zhao, Fuzzy Bang-Bang Control Problem Under Granular Differentiability, J. Frankl. Inst. 355 (2018), 4931–4951. https://doi.org/10.1016/j.jfranklin.2018.05.022.

[10] M. Mazandarani, N. Pariz, A.V. Kamyad, Granular Differentiability of Fuzzy-Number-Valued Functions, IEEE Trans. Fuzzy Syst. 26 (2018), 310–323. https://doi.org/10.1109/TFUZZ.2017.2659731.

[11] S.K. Mishra, G. Giorgi, Invex Functions in Multiobjective Programming, in: Invexity and Optimization, Series: Nonconvex Optimization and Its Applications, Springer, Berlin, Heidelberg, pp. 115-156, 2008. https://doi.org/10.1007/978-3-540-78562-0_6.

[12] A.M. Mustafa, Z. Gong, M. Osman, The Solution of Fuzzy Variational Problem and Fuzzy Optimal Control Problem Under Granular Differentiability Concept, Int. J. Comput. Math. 98 (2020), 1495–1520. https://doi.org/10.1080/00207160.2020.1823974.

[13] M. Najariyan, Y. Zhao, Granular Fuzzy PID Controller, Expert Syst. Appl. 167 (2021), 114182. https://doi.org/10.1016/j.eswa.2020.114182.

[14] A. Nemirovski, K. Roos, Motzkin Transposition Theorem, in: C.A. Floudas, P.M. Pardalos (eds.), Encyclopedia of Optimization, Springer, Boston, pp. 2345–2348, (2009). https://doi.org/10.1007/978-0-387-74759-0_405.

[15] V.D. Pathak, S. Pirzada, Necessary and Sufficient Optimality Conditions for Nonlinear Fuzzy Optimization Problem, J. Fuzzy Optim. 4 (2011), 1–16.

[16] A. Piegat, M. Landowski, Horizontal Membership Function and Examples of Its Applications, Int. J. Fuzzy Syst. 17 (2015), 22–30. https://doi.org/10.1007/s40815-015-0013-8.

[17] J.D.L.C. SAMA, K. SOME, Null Set Concept for Optimal Solutions of Fuzzy Nonlinear Optimization Problems, Iran. J. Fuzzy Syst. 21 (2024), 1–18. https://doi.org/10.22111/ijfs.2024.45511.8034.

[18] F. Shi, G. Ye, W. Liu, D. Zhao, A Class of Nonconvex Fuzzy Optimization Problems Under Granular Differentiability Concept, Math. Comput. Simul. 211 (2023), 430–444. https://doi.org/10.1016/j.matcom.2023.04.021.

[19] L. Stefanini, B. Bede, Generalized Hukuhara Differentiability of Interval-Valued Functions and Interval Differential Equations, Nonlinear Anal.: Theory Methods Appl. 71 (2009), 1311–1328. https://doi.org/10.1016/j.na.2008.12.005.

[20] L. Stefanini, M. Arana-Jiménez, Karush–Kuhn–Tucker Conditions for Interval and Fuzzy Optimization in Several Variables Under Total and Directional Generalized Differentiability, Fuzzy Sets Syst. 362 (2019), 1–34. https://doi.org/10.1016/j.fss.2018.04.009.

[21] A. Tripathi, S.P. Tiwari, J. Kavikumar, Granular F-Transform and Its Application, arXiv:2211.09700, 2022. https://doi.org/10.48550/arXiv.2211.09700.

[22] H.C. Wu, The Karush-Kuhn-Tucker Optimality Conditions for the Optimization Problem with Fuzzy-Valued Objective Function, Math. Methods Oper. Res. 66 (2007), 203–224. https://doi.org/10.1007/s00186-007-0156-y.

[23] L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[24] J. Zhang, Y. Wang, S. Zhang, A New Homotopy Transformation Method for Solving the Fuzzy Fractional Black–Scholes European Option Pricing Equations Under the Concept of Granular Differentiability, Fractal Fract. 6 (2022), 286. https://doi.org/10.3390/fractalfract6060286.

[25] J. Zhang, X. Chen, L. Li, X. Ma, Optimality Conditions for Fuzzy Optimization Problems Under Granular Convexity Concept, Fuzzy Sets Syst. 447 (2022), 54–75. https://doi.org/10.1016/j.fss.2022.01.004.