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STRONGLY SEMI-θ-CONTINUITY IN THE BOX TOPOLOGY

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Abstract. In this paper, we characterized the concept of strongly semi-θ-continuous functions from an arbitrary topological space into the topological space with either the box or Tychonoff topology. Moreover, we introduced and characterized some versions of separation axioms related to semi-θ-open sets.

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1. Introduction

In 1963, Norman Levine [20] made the first approach to the study of open sets in topological spaces by introducing the concepts of semi-open sets, semi-closed sets, and semi-continuity. His groundbreaking work provided a foundation that inspired subsequent research into the characterization of open sets, leading to the development of numerous characterization of open sets.

In 1971, Crossley and Hildebrand [9] introduced the concept of semi-closure of a subset of a topological space. This concept was further explored by Das in 1973 [10], who investigated its applications through utilizing the concept of semi-open sets. Subsequently, in 2008, Navalagi and Gurushantanavar [26] contributed some characterizations for both semi-interior and semi-closure.

Let $(X, \mathfrak{T})$ be a topological space and $G \subseteq X$. Then G is semi-open if for every $x \in G$, there exists an open set O containing x such that $O \subseteq G \subseteq Cl(O)$. Equivalently, G is semi-open if $G \subseteq Cl(Int(G))$. A subset F of X is semi-closed if its complement $X \setminus F$ is semi-open in X. The semi-closure and semi-interior of G are, respectively, denoted and defined by $sCl(G) = \bigcap\{F\colon F \text{ is semi-closed and } G \subseteq F\}$ and $sInt(G) = \bigcup\{U\colon U \text{ is semi-open and } U \subseteq G\}$. Moreover, $sCl(G) = G \cup Int(Cl(G))$.

In 1968, Veličko [34] introduced the notions of θ-continuity, θ-closure, and θ-interior of a subset of a topological space in order to study the important class of **H**-closed spaces in terms of arbitrary

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filterbases. This paper extends Veličko's pioneering work [7,11,12,19,24], thereby contributing to a deeper understanding of θ-open set.

A subset G of a topological space X is said to be θ-open if for every $x \in G$, there exists an open set O such that $x \in O \subseteq Cl(O) \subseteq G$. The θ-closure and θ-interior of G are, respectively, denoted and defined by $Cl_\theta(G) = \{x \in X : Cl(U) \cap G \neq \varnothing$ for every open set U containing $x\}$ and $Int_\theta(G) = \{x \in X : Cl(U) \subseteq G$ for some open set U containing $x\}$. A subset A of X is θ-closed if $Cl_\theta(A) = A$ and θ-open if $Int_\theta(A) = A$. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X with $\mathcal{T}_\theta \subset \mathcal{T}$.

In 1986, Di Maio and Noiri [23] introduced the concept of semi-θ-open set and provided some characterizations and notable properties. Additionally, the concepts of semi-θ-interior and semi-θ-closure of a subset of a topological space are introduced.

A subset G of a topological space X is semi-θ-open [23] if for every $x \in G$, there exists a semi-open set U containing x such that $sCl(U) \subseteq G$. A subset F of X is semi-θ-closed if its complement is $X \setminus F$ is semi-θ-open. A point $p \in X$ is a semi-θ-interior [27] point of G if there exists a semi-open set U in X containing p such that $sCl(U) \subseteq G$. We denote by $sInt_\theta(G)$ the set of all semi-θ-interior point of G. A point $p \in X$ is a semi-θ-closure [23] point of G if for every semi-open set U in X containing p, $sCl(U) \cap G \neq \varnothing$. The set of all semi-$\theta$-closure point of G is denoted by $sCl_\theta(G)$.

In 2022, Singh and Gupta [32] introduced a new class of open set called semi-delta-open set (briefly δ_s-open set) and investigated its properties. Moreover, the notion of δ_s-open mappings and δ_s-continuity are introduced and investigated. In 2023, the concept of δ_s-closure, δ_s-interior, δ_s-open mappings and δ_s-continuity was further investigated in [33]. A subset G of a topological space X is δ_s*-open* [32] if for every $x \in G$, there exists an open set U containing x such that $Int(sCl(U)) \subseteq G$. A subset F of X is δ_s-closed if its complement $X \setminus F$ is δ_s-open.

It is known that $sInt(G)$ [26] (resp., $Int_\theta(G)$ [19], $sInt_\theta(G)$ [6]) is the largest semi-open (resp., θ-open, semi-θ-open) set contained in G and $sCl(G)$ [26] (resp., $Cl_\theta(G)$ [19], $sCl_\theta(G)$ [23]) is the smallest semi-closed (resp., θ-closed, semi-θ-closed) set containing G. Note that $x \in sInt(G)$ [26] (resp., $x \in Int_\theta(G)$ [34]) if and only if there exists a semi-open (resp., open) set U containing x such that $U \subseteq G$ (resp., $Cl(U) \subseteq G$). It is worth noting that $Int(G) \subseteq sInt(G)$ [26] (resp., $Int_\theta(G) \subseteq Int(G)$ [19]) and $sCl(G) \subseteq Cl(G)$ [26], (resp., $Cl(G) \subseteq Cl_\theta(G)$ [19], $sCl(G) \subseteq sCl_\theta(G)$ [6]). Moreover, a subset G of X is semi-closed [26] if $sCl(G) = G$ and semi-open [26] if $sInt(G) = G$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The box topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a basis consisting of all sets of the form $\prod\limits_{\alpha \in \mathcal{A}} U_\alpha$ where U_α is open in Y_α for each $\alpha \in \mathcal{A}$. In addition, the Tychonoff topology on $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $\langle U_\alpha \rangle = \mathrm{p}_\alpha^{-1}(U_\alpha)$, where $p_\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$, the αth coordinate projection map is defined by

$p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \cdots, \alpha_n$ and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \cdots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_\alpha^{-1}(U_{\alpha_1}) \cap p_\alpha^{-1}(U_{\alpha_2}) \cap \cdots \cap p_\alpha^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \cdots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \cdots, k\}$.

Every projection map p_α is a continuous open surjection. It is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the αth coordinate projection map.

Prior research [18] have provided significant insights into the characterization of strongly semi-θ-continuous functions. However, the characterization of strongly semi-θ-continuous functions from an arbitrary topological space into the topological space with either the box topology or the Tychonoff topology remains unexplored. Furthermore, although various notions of semi-continuity have been introduced in [2,5,20], the connections between these concepts have not been established.

In this study, we characterize the concept of strongly semi-θ-continuous functions from an arbitrary topological space to the topological space with either the box or Tychonoff topology. Moreover, we introduce and characterize some versions of separation axioms related to semi-θ-open sets.

2. Strongly Semi-Theta-Continuity and other Versions of Continuity

In this section, we revisit the concept of strongly semi-θ-continuous functions and investigate its relationship to the other versions of continuity.

The following remark is an immediate consequence of [28, Lemma 7.1 and Diagram I], [29, Lemma 6.1], [30, Theorem 2.1] and [33, Remark 19].

Remark 2.1. The following diagram holds for any subset of a topological space.

$$\begin{tikzcd}
\theta\text{-open} \arrow[r] \arrow[d] & \delta_s\text{-open} \arrow[r] \arrow[d] & \text{open} \arrow[d] \\
\theta\text{-semi-open} \arrow[r] & \text{semi-}\theta\text{-open} \arrow[r] & \text{semi-open}
\end{tikzcd}$$

We note that the above diagram is also true for their respective closed sets. Moreover, none of the reverse implications above is true, as shown in the following examples.

Example 2.2. Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Consider the open set $A = \{a, b\}$. Note that $Int(sCl(\{a\})) = \{a\} \subseteq A$ and $Int(sCl(\{b\})) = \{b\} \subseteq A$. Hence, A is δ_s-open. To show that A is not θ-open, observe that the only open sets containing a are $\{a\}$, $\{a, b\}$ and X. However, $Cl(X) = Cl(\{a, b\}) = X \nsubseteq \{a, b\}$ and $Cl(\{a\}) = \{a, c\} \nsubseteq \{a, b\}$. Thus, A is not θ-open.

Example 2.3. Let $X = \{a, b, c, d\}$ with topology

$$\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

Consider the open set $B = \{a, c, d\}$. Observe that X and $\{a, c, d\}$ are the only open sets that contain d. However, $Int(sCl(\{a, c, d\})) = Int(sCl(X)) = X \nsubseteq B$. Then B is not δ_s-open.

Example 2.4. Consider the set of all real numbers $\mathbb{R}$ with the usual topology. Note that the closed and bounded interval $[1, 3]$, which is not open $\mathbb{R}$, is semi-open since $[1, 3] \subseteq Cl(Int([1, 3])) = [1, 3]$. Note also that $Cl([1, 3]) = [1, 3] \subseteq [1, 3]$. This means that $[1, 3]$ is θ-semi-open. This also follows that $[1, 3]$ is semi-θ-open. Since $[1, 3]$ is not open, it also not δ_s-open and not θ-open.

Example 2.5. Consider again the set $A = \{a, b\}$ in Example 2.2, which is a δ_s-open set. It follows that A is semi-θ-open. However, A is not θ-semi-open, since the only semi-open sets containing a are $\{a\}$, $\{a, b\}$, $\{a, c\}$, and X, but $Cl(\{a\}) = Cl(\{a, c\}) = \{a, c\} \nsubseteq A$ and $Cl(\{a, b\}) = Cl(X) = X \nsubseteq A$.

Example 2.6. Let $X = \{a, b\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}\}$. Cosider the open set $B = \{a\}$, which is also a semi-open set. Note that the only semi-open sets containing a are $\{a\}$ and X, but $sCl(\{a\}) = sCl(X) = X \nsubseteq B$. It follows that B is not a semi-θ-open set.

Remark 2.7. The following statements are true:

(i) θ-semi-open and δ_s-open sets are independent notions.
(ii) semi-θ-open and open sets are independent notions.
(iii) θ-semi-open and open sets are independent notions.

To verify (i), consider again the interval $[1, 3]$ in Example 2.4, which is a θ-semi-open set but not δ_s-open. From Example 2.2, the set $A = \{a, b\}$ is δ_s-open. It has been shown in Example 2.5, that A is not θ-semi-open.

To verify (ii), consider again the interval $[1, 3]$ in Example 2.4, which is a semi-θ-open set but not open. From Example 2.6, the set $B = \{a\}$ is open but not semi-θ-open.

For item (iii), if open implies θ-semi-open, then open implies semi-θ-open, which is a contradiction from (ii). Moreover, every θ-semi-open set is not necessarily open since the interval $[1, 3]$ in Example 2.4 is a θ-semi-open set but not open.

Theorem 2.8. *Let X be a topological space and $A \subseteq X$. Then the following holds:*

(i) *If A is both open and semi-closed, then A is δ_s-open.*
(ii) *If A is both semi-open and semi-closed, then A is semi-θ-open.*

Proof. (i) Suppose that A is both open and semi-closed and let $x \in A$. Using [26, Theorem 4.21 (ii)], $A = sCl(A)$. Hence, A is an open set containing x with $Int(sCl(A)) = Int(A) \subseteq A$. Therefore, A is δ_s-open.
(ii) Suppose that A is both semi-open and semi-closed and let $x \in A$. It follows that $A \subseteq sCl(A) = A \subseteq A$. Since A is semi-open and $sCl(A) \subseteq A$, A is semi-θ-open. $\square$

Definition 2.9. Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be *semi-continuous* [20] on X if for every $p \in X$ and each open set A in Y containing $f(p)$, there exists a semi-open set U in X containing p such that $f(U) \subseteq A$. Equivalently, f is semi-continuous on X if $f^{-1}(A)$ is semi-open in X for every open set A in Y.

Definition 2.10. Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be *strongly semi-θ-continuous* [18] (resp., *almost strongly semi-θ-continuous* [5]) on X if for every $x \in X$ and open set A in Y containing $f(x)$, there exists a semi-open set U in X containing x such that $f(sCl(U)) \subseteq A$ (resp., $f(sCl(U)) \subseteq sCl(A)$).

Definition 2.11. [2] Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be *semi-θ-continuous* (resp., *weakly semi-continuous*, *almost semi-continuous*) on X if for every $x \in X$ and open set A in Y containing $f(x)$, there exists a semi-open set U in X containing x such that $f(sCl(U)) \subseteq Cl(A)$ (resp., $f(U) \subseteq Cl(A)$, $f(U) \subseteq Int(Cl(A))$).

Remark 2.12. [16] The following diagram holds for any subset of a topological space.

$$\begin{tikzcd}
\text{strongly semi-}\theta\text{-continuous} \arrow[r] \arrow[d] & \text{semi-continuous} \arrow[d] \\
\text{almost strongly semi-}\theta\text{-continuous} \arrow[r] \arrow[d] & \text{almost semi-continuous} \arrow[d] \\
\text{semi-}\theta\text{-continuous} \arrow[r] & \text{weakly semi-continuous}
\end{tikzcd}$$

It is known [2] that every almost semi-continuous function is semi-θ-continuous function.

Theorem 2.13. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. If Y is regular, then the following statements are equivalent:*

(i) *f is strongly semi-θ-continuous on X;*
(ii) *f is weakly semi-continuous on X;*
(iii) *f is semi-continuous on X.*

Proof. (i)⇒(ii) Suppose that f is strongly semi-θ-continuous on X. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then there exists an open set W containing $f(x)$ in Y such that $W \subseteq Cl(W) \subseteq V$.

By assumption, there exists a semi-open set U containing x in X such that $f(x) \in f(U) \subseteq f(sCl(U)) \subseteq W \subseteq Cl(W) \subseteq V$. Hence, f is weakly semi-continuous on X.

(ii)⇒(iii) Assume that f is weakly semi-continuous on X. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then there exists an open set W containing $f(x)$ in Y such that $W \subseteq Cl(W) \subseteq V$. By assumption, there exists a semi-open set U containing x in X such that $f(x) \in f(U) \subseteq Cl(W) \subseteq V$. Therefore, f is semi-continuous on X.

(iii)⇒(i) Assume that f is semi-continuous on X. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then there exists an open set W containing $f(x)$ in Y such that $W \subseteq Cl(W) \subseteq V$. By assumption, there exists a semi-open set U in X containing x such that $f(x) \in f(U) \subseteq W$. Note that $f(sCl(U)) \subseteq Cl(W)$. To see this, let $y \in f(sCl(U))$. Then there exists $x \in sCl(U)$ such that $f(x) = y$. Let G be an open set in Y containing y. Then by assumption, $f^{-1}(G)$ is semi-open in Y. Since $x \in sCl(U)$ and $f^{-1}(G)$ is semi-open in X, it follows that $f^{-1}(G) \cap U \neq \varnothing$. Hence, $\varnothing \neq f(f^{-1}(G) \cap U) = G \cap f(U) \subseteq G \cap W$ so that $y \in Cl(W)$. Thus, $f(sCl(U)) \subseteq Cl(W) \subseteq V$. Therefore, f is strongly semi-θ-continuous on X. □

In view of [16, Remark 2.1] and Theorem 2.13, we have the following corollary.

Corollary 2.14. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. If Y is regular, then the following statements are equivalent:*

(i) *f is strongly semi-θ-continuous on X;*
(ii) *f is weakly semi-continuous on X;*
(iii) *f is semi-continuous on X;*
(iv) *f is almost strongly semi-θ-continuous on X;*
(v) *f almost semi-continuous on X;*
(vi) *f is semi-θ-continuous on X.*

Theorem 2.15. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of a subset A of X, where $\mathcal{D}$ is the set $\{0, 1\}$ with the discrete topology. Then f_A is strongly θ-semi-continuous if and only if A is both semi-θ-open and semi-θ-closed.*

Proof. Suppose that f_A is strongly semi-θ-continuous on X. Let $O_1 = \{1\}$ and $O_2 = \{2\}$. Then O_1 and O_2 are both open in $\mathcal{D}$. By assumption, $A = f^{-1}(O_1)$ and $X \setminus A = f^{-1}(O_2)$ are semi-θ-open in X. Hence, A is both semi-θ-open and semi-θ-closed.

Conversely, suppose that A is both semi-θ-open and semi-θ-closed. Let O be an open set in $\mathcal{D}$. Then, we have the following

$$f_A^{-1}(O) = \begin{cases} \varnothing & \text{if } O = \varnothing \\ X & \text{if } O = \{0,1\} \\ A & \text{if } O = \{1\} \\ X \setminus A & \text{if } O = \{0\}. \end{cases}$$

Hence, $f_A^{-1}(O)$ is semi-θ-open in X. Therefore, f_A is strongly semi-θ-continuous on X. $\square$

Definition 2.16. [21] A topological space X is *semi-Hausdorff* if given any pair of distinct points $p, q \in X$, there exist disjoint semi-open sets U and V such that $p \in U$ and $q \in V$.

Definition 2.17. [8, p.162] A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 2.18. *Let X and Y be topological spaces and $f\colon X \to Y$ be a strongly semi-θ-continuous injective function. If Y is a T_1-space, then X is semi-Hausdorff.*

Proof. Let x and y be distinct points in X. Then $f(x) \neq f(y)$. Since Y is a T_1-space, there exist disjoint open sets U and V such that $f(x) \in U$, $f(y) \notin U$ and $f(y) \in V$, $f(x) \notin V$. Since f is strongly semi-θ-continuous, there exists a semi-open set W in X containing x such that $f(sCl(W)) \subseteq U$. Hence, $y \notin sCl(W)$. It follows that $y \in X \setminus sCl(W)$. Thus, there exist disjoint semi-open sets W and $X \setminus sCl(W)$ containing x and y, respectively. Therefore, X is semi-Hausdorff. Similarly, for the case that $f(y) \in V$ and $f(x) \notin V$. Consequently, X is semi-Hausdorff. $\square$

Theorem 2.19. *Let X and Y be topological spaces and $f\colon X \to Y$ be a continuous function. If Y is regular and a T_1-space, then f is strongly semi-θ-continuous.*

Proof. Suppose that f is continuous on X. Let $x \in X$ and V be an open set in Y containing $f(x)$. Then there exists an open set $W \ni f(x)$ in Y such that $W \subseteq Cl(W) \subseteq V$. Note that $f(x) \subseteq sCl(W) \subseteq Cl(W) \subseteq V$. By assumption,

$$x \in f^{-1}(W) \subseteq sCl(f^{-1}(W)) \subseteq f^{-1}(sCl(W)) \subseteq f^{-1}(V).$$

Let $U = f^{-1}(W)$. Then $f(x) \in f(U) \subseteq f(sCl(U)) \subseteq sCl(W) \subseteq V$. By assumption, U is open in X and since every open is semi-open, it follows that U is semi-open with $f(sCl(U)) \subseteq V$. Therefore, f is strongly semi-θ-continuous on X. $\square$

3. SEMI-θ-CLOSED GRAPH

In this section, we introduce and characterize the concept of a semi-θ-closed graph function.

Definition 3.1. Let X and Y be topological spaces, $f\colon X \to Y$ be a function, and $g\colon X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. Usually, the graph function of f is denoted by $G(f)$. We may also consider the graph function $G(f)$ as a collection of ordered pairs $(x, (fx))$. Then $G(f)$ is said to be *semi-θ-closed* in $X \times Y$, if for every $(x, w) \notin G(f)$, there exist semi-open sets U and V containing x and w respectively, such that $(sCl(U) \times sCl(V)) \cap G(f) = \varnothing$.

Theorem 3.2. *Let X and Y be topological spaces, $f\colon X \to Y$ be a function, and $G(f)$ be the graph function of f. Then $G(f)$ is semi-θ-closed in $X \times Y$ if and only if for each $x \in X$ and $w \in Y$ such that $w \neq f(x)$, there exist semi-open sets U and V with $U \ni x$ and $V \ni w$ such that $f(sCl(U)) \cap sCl(V) = \varnothing$.*

Proof. Let $x \in X$ and $w \in Y$ such that $w \neq f(x)$. Then $(x, w) \notin G(f)$. By assumption, there exist semi-open sets U and V with $x \in U$ and $w \in V$ such that $(sCl(U) \times sCl(V)) \cap G(f) = \varnothing$. Hence, $f(sCl(U)) \cap sCl(V) = \varnothing$.

Conversely, suppose that $(x, w) \notin G(f)$. Then $w \neq f(x)$. By assumption, there exist semi-open sets U and V containing x and w respectively, such that $f(sCl(U)) \cap sCl(V) = \varnothing$. Therefore, $(sCl(U) \times sCl(V)) \cap G(f) = \varnothing$. □

Definition 3.3. Let $G(f)$ be a graph function of a function $f\colon X \to Y$. Then $G(f)$ is said to be *semi-θ-closed* in X, if for every $(x, w) \notin G(f)$, there exist semi-open sets U and V containing x and w, respectively, such that $(sCl(U) \times V) \cap G(f) = \varnothing$.

Theorem 3.4. *Let X and Y be topological spaces, $f\colon X \to Y$ be a function, and $G(f)$ be the graph function of f. Then $G(f)$ is semi-θ-closed in X if and only if for each $x \in X$ and $w \in Y$ such that $w \neq f(x)$, there exist semi-open sets U and V with $U \ni x$ and $V \ni w$ such that $f(sCl(U)) \cap V = \varnothing$.*

Corollary 3.5. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. Then $G(f)$ is semi-θ-closed in $X \times Y$ if for each $x \in X$ and $w \in Y$ such that $w \neq f(x)$, there exist semi-θ-open sets U and V with $U \ni x$ and $V \ni w$ such that $(U \times V) \cap G(f) = \varnothing$.*

Corollary 3.6. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. Then $G(f)$ is semi-θ-closed in X if for each $x \in X$ and $w \in Y$ such that $w \neq f(x)$, there exists a semi-θ-open set U and a semi-open set V with $U \ni x$ and $V \ni w$ such that $(U \times V) \cap G(f) = \varnothing$.*

Theorem 3.7. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. If f is strongly semi-θ-continuous on X and Y is Hausdorff, then $G(f)$ is semi-θ-closed in X.*

Proof. Assume that f is strongly semi-θ-continuous on X. Let $x \in X$ and $f(x) \neq y$. Then there exist disjoint open sets W and V such that $f(x) \in W$ and $y \in V$. By assumption, there exists a semi-open set $U \ni x$ in X such that $f(sCl(U)) \subseteq W$. Hence, $f(sCl(U)) \cap V = \varnothing$. Thus, $(sCl(U) \times V) \cap G(f) = \varnothing$. Therefore, $G(f)$ is semi-θ-closed in X. □

4. STRONGLY SEMI-θ-CONTINUOUS FUNCTIONS IN THE BOX TOPOLOGY

In this section, we provide a characterization of strongly semi-θ-continuous functions from an arbitrary topological space into the topological space with either the box or Tychonoff topology.

Theorem 4.1. *Let X be a topological space and $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a topological space with either the box or Tychonoff topology. A function $f \colon X \to Y$ is strongly semi-θ-continuous on X if and only if $p_\alpha \circ f$ is strongly semi-θ-continuous on X for every $\alpha \in \mathcal{A}$.*

Proof. Suppose that f is strongly semi-θ-continuous on X. Let $\alpha \in \mathcal{A}$ and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. By assumption, $f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$ is semi-θ-open in X. Thus, $p_\alpha \circ f$ is strongly semi-θ-continuous on X for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is strongly semi-θ-continuous on X for every $\alpha \in \mathcal{A}$. Let B_α be open in Y_α for each $\alpha \in \mathcal{A}$. Then $\langle B_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(B_\alpha) = f^{-1}(p_\alpha^{-1}(B_\alpha)) = f^{-1}(\langle B_\alpha \rangle)$ is semi-θ-open in X. Therefore, f is strongly semi-θ-continuous on X. ☐

Corollary 4.2. *Let X be a topological space and $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a topological space with either the box or Tychonoff topology and $f_\alpha \colon X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f \colon X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is strongly semi-θ-continuous on X if and only if each f_α is strongly semi-θ-continuous on X for each $\alpha \in \mathcal{A}$.*

Proof. By Theorem 4.1, it is enough to show that $p_\alpha \circ f = f_\alpha$. Let $x \in X$. Then, we have $(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\beta(x) \rangle) = f_\alpha(x)$ for each $\alpha \in \mathcal{A}$. Therefore, $p_\alpha \circ f = f_\alpha$. ☐

Before we proceed with proving the following theorem, we shall consider first the following lemma related to semi-open sets.

Lemma 4.3. *Let $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a topological space with the box topology and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$.*

(i) *O_α is semi-open in Y_α if and only if $O = \prod\{O_\alpha \colon \alpha \in \mathcal{A}\}$ is semi-open in Y.*
(ii) *If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $sCl(\prod\{A_\alpha \colon \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl(A_\alpha) \colon \alpha \in \mathcal{A}\}$.*

Proof. Following the proof of [15, Theorem 3.4], $O = \prod\{O_\alpha \colon \alpha \in \mathcal{A}\}$ is semi-open in Y if each O_α is semi-open in Y_α.

Conversely, suppose that $\prod\{O_\alpha \colon \alpha \in \mathcal{A}\}$ is semi-open in Y. Then there exists a basic open set $G = \prod\{G_\alpha \colon \alpha \in \mathcal{A}\}$ in Y such that $G \subseteq O \subseteq Cl(G) = \prod\{Cl(G_\alpha \colon \alpha \in \mathcal{A})\}$. Hence, for all $\alpha \in \mathcal{A}$, G_α is open in Y_α and $G_\alpha \subseteq O_\alpha \subseteq Cl(G_\alpha)$. Thus, each O_α is semi-open in Y_α.

The proof of (ii) is similar to [15, Theorem 3.5], hence omitted. ☐

Theorem 4.4. *Let $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a product space, and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. If each O_α is semi-θ-open in Y_α, then $O = \prod\{O_\alpha \colon \alpha \in \mathcal{A}\}$ is semi-θ-open in Y.*

Proof. Suppose that O_α is semi-θ-open in Y_α for each $\alpha \in \mathcal{A}$. Then there exists a semi-open set V_α containing x_α such that $V_\alpha \subseteq sCl(V_\alpha) \subseteq O_\alpha$ for all $\alpha \in \mathcal{A}$. Let $V = \prod\{V_\alpha \colon \alpha \in \mathcal{A}\}$. By Lemma 4.3 (i), V is semi-open set in Y. Hence, by Lemma 4.3 (ii) we have

$$V \subseteq sCl(\prod\{V_\alpha \colon \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl(V_\alpha) \colon \alpha \in \mathcal{A}\} \subseteq \prod\{O_\alpha \colon \alpha \in \mathcal{A}\} = O.$$

Therefore, $O = \prod\{O_\alpha \colon \alpha \in \mathcal{A}\}$. □

Corollary 4.5. *Let $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a finite product space, $\varnothing \neq O_i \subseteq Y_i$ for each $i \in \{1, 2, \ldots, n\}$. If each O_i is semi-θ-open in Y_i, then $O = \langle O_1, \ldots, O_n\rangle$ is semi-θ-open in Y.*

Theorem 4.6. *Let $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a topological space with the box topology. If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $\prod\{sInt_\theta(A_\alpha) \colon \alpha \in \mathcal{A}\} \subseteq sInt_\theta(\prod\{A_\alpha \colon \alpha \in \mathcal{A}\})$.*

Proof. Let $x = \langle x_\alpha\rangle \in \prod\{sInt_\theta(A_\alpha) \colon \alpha \in \mathcal{A}\}$. Then $x_\alpha \in sInt_\theta(A_\alpha)$ for all $\alpha \in \mathcal{A}$. This means that for all $\alpha \in \mathcal{A}$ there exists a semi-open set V_α containing x_α such that $sCl(V_\alpha) \subseteq A_\alpha$ for each $\alpha \in \mathcal{A}$. Let $V = \prod\{V_\alpha \colon \alpha \in \mathcal{A}\}$. Then V is semi-open in Y that contains x. By Lemma 4.3 (ii), we have $sCl(\prod\{V_\alpha \colon \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl(V_\alpha) \colon \alpha \in \mathcal{A}\} \subseteq \prod\{A_\alpha \colon \alpha \in \mathcal{A}\}$. Thus, $\langle x_\alpha\rangle \in sInt_\theta(\prod\{A_\alpha \colon \alpha \in \mathcal{A}\})$. □

Corollary 4.7. *Let $Y = \prod\{Y_i \colon 1 \leq i \leq n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i \in \{1, 2, \ldots, n\}$. Then $\langle sInt_\theta(A_i), \ldots, sInt_\theta(A_n)\rangle \subseteq sInt_\theta(\langle A_1, \ldots, A_n\rangle)$.*

Before proving the next theorem, we shall consider first the following lemma.

Lemma 4.8. *Let X be a topological space and $A \subseteq X$. Then the following statements are equivalent:*

(i) $x \in sCl_\theta(A)$.

(ii) *For every semi-θ-open set U containing x, $U \cap A \neq \varnothing$.*

(iii) *For every semi-open set U containing x, $sCl(U) \cap A \neq \varnothing$.*

Proof. The proofs of (i)⇒(ii) and (ii)⇒(i) are straightforward, hence omitted.

(i)⇒(iii) Suppose that there exists a semi-open set U containing x such that $sCl(U) \cap A = \varnothing$. This means that $sCl(U) \subseteq X \setminus A$. Using [6, Theorem 2.7], $x \in sInt_\theta(X \setminus A) = X \setminus sCl_\theta(A)$. Therefore, $x \notin sCl_\theta(A)$.

(iii)⇒(ii) Suppose that there exists a semi-θ-open set V containing x such that $V \cap A = \varnothing$. Since V is semi-θ-open, there exists a semi-open set U containing x such that $sCl(U) \subseteq V$. Hence, $sCl(U) \cap A \subseteq V \cap A = \varnothing$. Therefore, $sCl(U) \cap A = \varnothing$. □

Theorem 4.9. *Let $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ be a topological space with the box topology. If $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, then $sCl_\theta(\prod\{A_\alpha \colon \alpha \in \mathcal{A}\}) \subseteq \prod\{sCl_\theta(A_\alpha) \colon \alpha \in \mathcal{A}\}$.*

Proof. Assume that $\langle x_\alpha \rangle \in sCl_\theta(\prod\{A_\alpha \colon \alpha \in \mathcal{A}\})$. Suppose on that contrary that $\langle x_\alpha \rangle \notin \prod\{sCl_\theta(A_\alpha) \colon \alpha \in \mathcal{A}\}$. This means $x_\beta \notin sCl_\theta(A_\beta)$ for some $\beta \in \mathcal{A}$. By Lemma 4.8, there exists a semi-open set V_β containing x_β such that $sCl(V_\beta) \cap A_\beta = \varnothing$. Let $\langle V_\beta \rangle -\colon V_\beta \times \prod\{Y_\alpha \colon \alpha \neq \beta\}$. Then $\langle V_\beta \rangle$ is semi-open in Y that contains $\langle x_\alpha \rangle$. Using Lemma 4.3, we have $sCl(\langle V_\beta \rangle) \subseteq \langle sCl(V_\beta) \rangle$. Thus,

$$sCl(\langle V_\beta \rangle) \cap \prod\{A_\alpha \colon \alpha \in \mathcal{A}\} \subseteq \langle sCl(V_\beta) \rangle \cap \prod\{A_\alpha \colon \alpha \in \mathcal{A}\} = (sCl(V_\beta) \cap A_\beta) \times \prod_{\alpha \neq \beta} A_\alpha = \varnothing,$$

a contradiction. Therefore, $\langle x_\alpha \rangle \in \prod\{sCl_\theta(A_\alpha) \colon \alpha \in \mathcal{A}\}$. □

Corollary 4.10. *Let* $Y = \prod\{Y_i \colon 1 \leq i \leq n\}$ *be a finite product space and* $A_i \subseteq Y_i$ *for each* $i \in \{1, 2, \ldots, n\}$. *Then* $sCl_\theta(\langle A_1, \ldots, A_n \rangle) \subseteq \langle sCl_\theta(A_i), \ldots, sCl_\theta(A_n) \rangle$.

Theorem 4.11. *Let* $X = \prod\{X_\alpha \colon \alpha \in \mathcal{A}\}$ *and* $Y = \prod\{Y_\alpha \colon \alpha \in \mathcal{A}\}$ *be product spaces, let* $f_\alpha \colon X_\alpha \to Y_\alpha$ *be a function. If each* f_α *is strongly semi-*θ*-continuous on* X_α*, then the function* $f \colon X \to Y$ *defined by* $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ *is strongly semi-*θ*-continuous on* X.

Proof. Assume that f_α is strongly semi-θ-continuous on X_α for each $\alpha \in \mathcal{A}$. Let $\langle x_\alpha \rangle \in X$ and V be an open set in Y containing $f(\langle x_\alpha \rangle)$. Since V is open in Y, there exists a basic open set $\langle W_{\alpha_1}, \ldots, W_{\alpha_n} \rangle$ containing $f(\langle x_\alpha \rangle) = \langle f(x_\alpha) \rangle$ such that $\langle W_{\alpha_1}, \ldots, W_{\alpha_n} \rangle \subseteq V$. This means that $f_{\alpha_i}(x_{\alpha_i}) \in W_{\alpha_i}$ for all $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{A}$. By assumption, there exists a semi-open set U_{α_i} containing x_{α_i} such that $f_{\alpha_i}(sCl(U_{\alpha_i})) \subseteq W_{\alpha_i}$. Let $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$. Then U is semi-open that contains $\langle x_\alpha \rangle$. Note that $sCl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) \subseteq \langle sCl(U_{\alpha_1}), \ldots, sCl(U_{\alpha_n}) \rangle$. It follows that

$$\begin{aligned} f(sCl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle)) &\subseteq f(\langle sCl(U_{\alpha_1}), \ldots, sCl(U_{\alpha_n}) \rangle) \\ &= \langle f_{\alpha_1}(sCl(U_{\alpha_1})), \ldots, f_{\alpha_n}(sCl(U_{\alpha_n})) \rangle \\ &\subseteq \langle W_{\alpha_1}, \ldots, W_{\alpha_n} \rangle \\ &\subseteq V. \end{aligned}$$

Consequently, f is strongly semi-θ-continuous on X. □

5. SOME VERSION OF SEPARATION AXIOMS

In this section, we introduce and characterize some versions of separation axioms related to semi-θ-open sets.

Definition 5.1. A topological space X is said to be

(i) semi-θ-*Hausdorff* if given any pair of distinct points $p, q \in X$, there exist disjoint semi-θ-open sets U and V such that $p \in U$ and $q \in V$ [1];

(ii) *semi-*θ*-regular* if for each closed set F and each point $p \notin F$, there exist disjoint semi-θ-open sets U and V such that $x \in U$ and $F \subseteq V$;

(iii) X is *semi-θ-normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint semi-θ-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Before we proceed with the characterization, we shall consider first the following results related to semi-θ-connected space.

Theorem 5.2. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is semi-θ-connected;*

(ii) *The only subsets of X that are both semi-θ-open and semi-θ-closed are $\varnothing$ and X;*

(iii) *No strongly θ-semi-continuous function $f : X \to \mathcal{D}$ is surjective.*

Proof. (i)⇒(ii) Assume that X is semi-θ-connected and let $A \subseteq X$. Let A be both semi-θ-open and semi-θ-closed. Then $X \setminus A$ is both semi-θ-open and semi-θ-closed. Note that $X = A \cup (X \setminus A)$. Since X is semi-θ-connected, it follows that A is either $\varnothing$ or X.

(ii)⇒(iii) Suppose that $\varnothing$ and X are the only subsets of X that are both semi-θ-open and semi-θ-closed. Let $f : X \to \mathcal{D}$ be a strongly semi-θ-continuous surjection. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Note that every set in $\mathcal{D}$ is open. Since $\{0\}$ is both open and closed in $\mathcal{D}$, it follows that $f^{-1}(\{0\})$ is both semi-θ-open and semi-θ-closed in X, a contradiction.

(iii)⇒(i) Suppose no strongly semi-θ-continuous function $f : X \to \mathcal{D}$ is surjective. Let $X = A \cup B$, where A and B are disjoint nonempty semi-θ-open sets. Then A and B are also semi-θ-closed sets. Consider the characteristic function $f_A : X \to \mathcal{D}$ of $A \subseteq X$. By Theorem 2.15, f_A is strongly semi-θ-continuous, a contradiction. Thus, X is semi-θ-connected. □

The following three results can be proved using the same techniques employed in [4] and [15].

Theorem 5.3. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is semi-θ-Hausdorff;*

(ii) *Let $x \in X$. For $w \neq x$, there exists a semi-θ-open set U containing x such that $y \notin sCl_\theta(U)$;*

(iii) *For each $x \in X$, $V = \cap\{sCl_\theta(U) \colon U$ is semi-θ-open set containing $x\} = \{x\}$.*

Theorem 5.4. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is semi-θ-regular;*

(ii) *For each $x \in X$ and an open set U containing x, there exists a semi-θ-open set V such that $x \in V \subseteq sCl_\theta(V) \subseteq U$;*

(iii) *For each $x \in X$ and closed set F with $x \notin F$, there exists a semi-θ-open set V containing x such that $F \cap sCl_\theta(V) = \varnothing$.*

Theorem 5.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is semi-θ-normal;*

(ii) *For each closed set A and an open set U containing A, there exists a semi-θ-open set V containing A such that $sCl_{\theta}(V) \subseteq U$;*

(iii) *For each pair of disjoint closed sets A and B, there exists a semi-θ-open set V containing A such that $sCl_{\theta}(V) \cap B = \varnothing$.*

Theorem 5.6. *Let X be a T_1-space. Then the following statements hold:*

(i) *If X is semi-θ-normal, then X is semi-θ-regular;*

(ii) *If X is semi-θ-regular, then X is semi-θ-Hausdorff.*

Proof. (i) Let F be a closed set and let $x \notin F$. Since every finite point set is closed in a T_1-space, $\{x\}$ is closed in X since X. By assumption, there exist disjoint semi-θ-open sets U and V such that $\{x\} \subseteq U$ and $F \subseteq V$. Note that $x \in \{x\}$ so that $x \in U$. Hence, X is semi-θ-regular.

(ii) Let $x, w \in X$ with $x \neq w$. Then there exist open sets U and V such that $x \in U$, $w \notin U$ and $w \in V$, $x \notin V$. It follows that $X \setminus U$ is closed with $x \notin X \setminus U$ and $w \in X \setminus U$. By assumption, there exist disjoint semi-θ-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Clearly, $w \in B$. Therefore, X is semi-θ-Hausdorff. □

Definition 5.7. A topological space X is said to be

(i) *semi-closed* (resp., *semi-Lindelöf*) if every cover of X by semi-open sets has a finite (resp., countable) subcollection whose semi-closures cover X;

(ii) *countably semi-closed* (resp., *lightly compact* [3]) if every countable cover of X by semi-open (resp., open) sets has a finite subcollection whose semi-closures (resp., closures) cover X;

(iii) *quasi H-closed* [31] (resp., *almost Lindelöf* [13]) if every cover of X by open sets has a finite (resp., countable) subcollection whose closures cover X.

Definition 5.8. A subset A of a topological space X is said to be

(i) *semi-closed relative to X* if every cover of A by semi-open sets in X has a finite subcollection whose semi-closures cover A;

(ii) *quasi H-closed relative to X* [31] if every cover of A by open sets in X has a finite subcollection whose closures cover A.

Theorem 5.9. *Let X and Y be topological spaces and $f : X \to Y$ be semi-θ-continuous on X. If $A \subseteq X$ is semi-closed relative to X, then $f(A)$ is quasi H-closed relative to Y.*

Proof. Let $\{V_{\alpha} : \alpha \in \mathcal{A}\}$ be a cover of $f(A)$ by open sets in Y. Then $f(A) \subseteq \cup\{V_{\alpha} : \alpha \in \mathcal{A}\}$. Let $x \in A$. Then $f(x) \in V_{\beta(x)}$ for some $\beta(x) \in \mathcal{A}$. Since f is semi-θ-continuous on X, by Definition 2.11, there exists a semi-open set $U_{\beta(x)}$ containing x such that $f(sCl(U_{\beta(x)})) \subseteq Cl(V_{\beta(x)})$. Note that $\{U_{\beta(x)} : x \in A\}$ is a collection of semi-open sets in X that cover A. Since A is semi-closed relative to X, there exists a finite set $F \subseteq A$ such that $A \subseteq \cup\{sCl(U_{\beta(x)}) : x \in F\}$. This means that

$$f(A) \subseteq \cup\{f(sCl(U_{\beta(x)})) : x \in F\} \subseteq \cup\{Cl(V_{\beta(x)}) : x \in F\}.$$

Thus, $f(A)$ is quasi H-closed relative to Y. □

Corollary 5.10. *Let X and Y be topological spaces and $f : X \to Y$ be semi-θ-continuous and surjective. Then the following properties hold:*

(i) *If X is semi-closed, then Y is quasi H-closed.*
(ii) *If X is semi-Lindelöf, then Y is almost Lindelöf.*
(iii) *If X is countably semi-closed, then Y is lightly compact.*

Recommendations

Future research may focus on developing versions of Urysohn's lemma and the Tietze extension theorem for semi-θ-open sets.

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