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NUMERICAL ANALYSIS BY GODUNOV-TYPE SCHEME FOR KAZHIKHOV-SMAGULOV POLLUTION MODEL

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Abstract. The pollution model based on the Kazhikhov-Smagulov equations, a set of equations in fluid mechanics and environmental modeling, is thoroughly investigated numerically in this paper. We investigate the intricate behavior of pollution in a single spatial dimension using a Godunov-type method, which is renowned for its ability to capture discontinuities and shocks effectively. Our 1D simulations illustrate frequently overlooked facets of mixture phenomena, highlighting the model's complex and varied dynamics. Numerical experiments illustrate the relevance of the proposed numerical procedure.

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1. Introduction

The model analyzed in this paper is the pollution model based on Kazhikhov-Smagulov equations, which describe the behavior of a homogeneous mixture of two viscous and incompressible fluids, such as ethanol and ethylene glycol, oil and corn syrup, or water and glycerin or again fresh water over salt water. This study focuses on the evolution of the velocity of particles, whether at rest or in motion, while considering the common density of the mixture and the pressure.

Let Ω be a bounded open set in $\mathbb{R}^3$, and let $T > 0$ be a real number such that the interval $(0, T)$ is a subset of $\mathbb{R}$. The conservation of mass and momentum principles lead us to the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho W) = 0, \\ \\ \rho(\partial_t W + W.\nabla W) - \mu \Delta W - (\mu + \mu')\nabla \operatorname{div}(W) + \nabla p = \rho f, \end{cases} \tag{1.1}$$

where $\rho = \rho(x, y, z, t)$ is the common density of the mixture, $W = W(x, y, z, t)$ be the common velocity, and p be the pressure of the mixture, with μ and μ' representing the dynamic and kinematic viscosities,

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respectively. It is important to note that this system is subject to Fick's law formulated by:

$$\operatorname{div} V = -\operatorname{div}(\lambda \nabla \ln(\rho)), \tag{1.2}$$

with λ being the mass diffusion constant. By applying this law, we arrive at a system where the velocity has zero divergence, which is characteristic of these types of fluids. Thus, we obtain the following system while neglecting the λ^2 terms.

$$\begin{cases} \partial_t \rho + V.\nabla \rho = \lambda \Delta \rho, \\ \rho(\partial_t V + (V.\nabla)V) - \mu \Delta V - \lambda(\nabla \rho.\nabla)V - \lambda(V.\nabla)\nabla \rho + \nabla P = \rho f, \\ \operatorname{div}(V) = 0. \end{cases} \tag{1.3}$$

The above system is supplemented by the following boundary and initial conditions:

$$\begin{cases} \dfrac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \Sigma_T, \\ v = 0 \qquad \text{on} \quad \Sigma_T, \\ \rho_{|t=0} = \rho_0 \ ; \ v_{|t=0} = v_0 \qquad \text{in} \quad \Omega, \\ 0 < m \leq \rho_0 \leq M < +\infty, \end{cases} \tag{1.4}$$

where $\Sigma_T = (0;T) \times \partial\Omega$. By considering the system in one-dimensional space, (1.3) becomes:

$$\begin{cases} \partial_t \rho + V.\partial_x \rho = \lambda \partial_x^2 \rho, \\ \rho(\partial_t V + (V.\partial_x)V) - \mu \partial_x V - \lambda(\partial_x \rho.\partial_x)V - \lambda(V.\partial_x)\partial_x \rho + \partial_x P = \rho f, \\ \partial_x V = 0. \end{cases} \tag{1.5}$$

As a result, by simplifying and applying the properties of incompressible fluids, we obtain:

$$\begin{cases} \partial_t \rho + \partial_x(\rho V) = \lambda \partial_x^2 \rho, \\ \partial_t V + \partial_x(\rho V^2) - \lambda \partial_x^2(\rho V) + \partial_x P(\rho) = \rho \mathbf{g}, \\ \partial_x V = 0, \end{cases} \tag{1.6}$$

where the gravitational field replaces the resultant of the external forces f, and the pressure is a function of the density ρ.

In the literature, the numerical study of Kazhikhov-Smagulov-type models remains relatively rare. However, a few references deserve our attention. Authors who have focused on this topic primarily concentrate on the stability and convergence of numerical solutions obtained using finite element methods, finite volume methods, or a hybridization of the two. Among these works, we can cite those of R.C. Cabrales, F. Guillèn-Gonzàlez and J.V. Gutiérrez-Santacreu [8,19,20], who analyze the conditional and unconditional stability of a finite element-based scheme. Noteworthy as well are the contributions of Catarina Calgaro et al. [9,12], who developed a hybrid scheme establishing the stability and convergence of the discrete solution to the weak solution. For further exploration, readers may consult additional studies [6,10,11,13–15,18,21], which analyze, among other aspects, the behavior of approximation errors for discrete solutions.

Regarding the use of Godunov-type schemes for approximating weak solutions of Navier-Stokes models, it is important to note that the design and numerical implementation of finite volume schemes using Godunov-type solvers were first proposed by Sergey Godunov in 1959 for the numerical resolution of partial differential equations [17]. These works were followed by those of Berthon et al. [2–5], D. Hoff [23], and J. Ouya and A. Ouedraogo in [24], which provide the motivation for the present work.

The objective of this study is to use a Godunov-type scheme to design a finite volume method capable of capturing discontinuities and shocks. We also propose numerical simulations that highlight the complex and diverse dynamics of the models, revealing often hidden aspects of mixture phenomena. It should be noted that the existence of a weak solution was demonstrated by M. Sy in [25].

Following this introduction, Section 2 presents some preliminaries, while Section 3 describes the proposed scheme and its properties, and Section 4 details the numerical results obtained.

2. PRELIMINARY

In the context of using a Godunov-type scheme, we reformulate problem (1.6) as follows;
we set:

$$\mathcal{W} = \begin{pmatrix} \rho \\ \rho V \end{pmatrix}, \quad \mathcal{F}(\mathcal{W}) = \begin{pmatrix} \rho V \\ \rho V^2 + P(\rho) \end{pmatrix};$$

$$\mathcal{D}(\mathcal{W})\partial_x \mathcal{W} = \begin{pmatrix} \lambda \partial_x \rho \\ \lambda \partial_x (\rho V) \end{pmatrix} \text{ and } \mathcal{S}(\mathcal{W}) = \begin{pmatrix} 0 \\ \rho \mathbf{g} \end{pmatrix}.$$

Thus, we reformulate as follows:

$$\partial_t \mathcal{W} + \partial_x \mathcal{F}(\mathcal{W}) = \partial_x \left(\mathcal{D}(\mathcal{W})\partial_x \mathcal{W}\right) + \mathcal{S}(\mathcal{W}), \tag{2.1}$$

where the vector $\mathcal{W}$ represents the conservative variables, $\mathcal{F}(\mathcal{W})$ the convective flux term, $\mathcal{D}(\mathcal{W})\partial_x \mathcal{W}$ denotes the viscosity term, and $\mathcal{S}(\mathcal{W})$ the source term; $\mathcal{W}$ belongs to the admissible states $\mathit{\Omega}$ defined by:

$$\mathit{\Omega} = \left\{ \mathcal{W} = (\rho, \rho V)^t \in \mathbb{R}^2 \; \rho > 0, \; \rho V \in \mathbb{R} \right\} \subset \mathbb{R}^2.$$

The stationary solutions of the system (1.6) are characterized by the following lemma:

Lemma 2.1. *Let P be an affine function of the density, defined by $P = a\rho + k$ with $(a, k) \in \mathbb{R}^* \times \mathbb{R}$. The stationary solutions of the system (1.6) are then given by:*

$$\begin{cases} V(x,t) = V^0(t), \\ \rho(x,t) = \dfrac{\beta}{\gamma} e^{\mu x}, \end{cases} \tag{2.2}$$

where $\gamma = \dfrac{V^0(t)}{\lambda}$ *and* β *is a function of* $(V^0(t), \mathbf{g}, \lambda, a)$*.*

Proof. At the stationary state, i.e., at $t = 0$, the system (1.6) becomes:

$$\begin{cases} \partial_x(\rho V) = \lambda \partial_x^2 \rho, \\ \partial_x(\rho V^2) - \lambda \partial_x^2(\rho V) + \partial_x P(\rho) = \rho \mathbf{g}, \\ \partial_x V = 0. \end{cases} \tag{2.3}$$

From $(2.3)_3$, we deduce that $V = V^0(t)$. Substituting this into $(2.3)_1$, we obtain:

$$\partial_x^2 \rho - \mu \partial_x \rho = 0.$$

Solving this differential equation gives:

$$\rho(x,t) = \frac{\beta}{\gamma} e^{\mu x} + \nu, \quad (\beta, \nu) \in \mathbb{R}^2.$$

Substituting $\rho(x,t)$ into $(2.3)_2$, we derive:

$$(V^0)^2 \partial_x \rho - \lambda V^0 \partial_x^2 \rho + a \partial_x \rho = \rho \mathbf{g}.$$

Then, we can identify same relation between β and ν:

$$\beta = \frac{\nu V^0 \mathbf{g}}{a V^0 - \lambda \mathbf{g}}, \qquad \text{with } \nu \in \mathbb{R}.$$

□

The primary goal of this study is to develop a finite volumes scheme capable of effectively approximating the weak solutions of equation (1.6), addressing both steady states and dynamic systems. Steady states, where the velocity V is zero, are crucial as they represent the long-term stable conditions that many real-world experiments are expected to reach over time. On the other hand, dynamic systems, characterized by non-zero velocities, are equally significant as they capture transient behaviors and evolving patterns observed in real-world scenarios.

This dual focus allows us to evaluate the proposed scheme's robustness and accuracy across a broad range of phenomena. To achieve this, we employ a Godunov-type scheme (see [16,22,26]) adapted to handle both static and dynamic situations effectively. The following sections detail our approach to approximating weak solutions of equation (1.6) in these two distinct contexts. Consequently the second forms of states are defined by following:

$$\begin{cases} V = 0 \\ \\ \partial_x \rho = 0, \end{cases} \quad \text{and} \quad \begin{cases} V(x,t) = V^0(t) \\ \\ \rho(x,t) = \dfrac{\beta}{\gamma} e^{\gamma x}, \end{cases} \tag{2.4}$$

where $\gamma = \dfrac{V^0(t)}{\lambda}$ and β is a function of $(V^0(t), \mathbf{g}, \lambda, a)$.

3. Finite Volumes Scheme of Homogeneous Problem

The goal of finite volume schemes is to approximate the weak solutions of the problem:

$$\begin{cases} \partial_t \mathcal{W} + \partial_x \mathcal{F}(\mathcal{W}) = 0, \\ \mathcal{W}(x, t = 0) = \mathcal{W}_0(x), \end{cases} \tag{3.1}$$

with $\mathcal{W}_0 \in \Omega$. This approach is appropriate for the hyperbolic Navier-Stokes system (also the Kazhikhov-Smagulov system) and is used in this study. The Godunov scheme, finite volume schemes, and schemes of a Godunov type are briefly described. The reference for further information is [22,24,26,27]. The initial phase is the spatial discretization shown in Figure 1. This phase takes into account a uniform grid that is defined by a series of points $(x_{i+1/2})_{i\in\mathbb{Z}}$, with the following equation:

$$x_{i+1/2} = x_{i-1/2} + \delta x,$$

where δx is the spatial step size, assumed constant.

We define the cells $c_i = (x_{i-1/2}, x_{i+1/2})$ centered at x_i. Similarly, the sequence $(t^n)_{n\in\mathbb{N}}$ is defined by $t^0 = 0$ and $t^{n+1} = t^n + \delta t$, where δt is the time step, constrained by a CFL (Courant-Friedrichs-Lewy) condition.

Let $\mathcal{W}_i^n$ denote the approximation of the solution $\mathcal{W}(x, t^n)$ at time t^n. Specifically, this value corresponds to the approximation of the average of the exact solution $\mathcal{W}(x, t^n)$ over the cell c_i, expressed as:

$$\begin{cases} \mathcal{W}_i^n \simeq \dfrac{1}{\delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{W}(x, t^n) dx, & n \geq 1, \\ \\ \mathcal{W}_i^0 \simeq \dfrac{1}{\delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{W}(x, t^0) dx, & n = 0. \end{cases} \tag{3.2}$$

By integrating (3.1) over $c_i \times [t^n, t^{n+1})$, we obtain:

$$\begin{aligned} &\int_{x_{i-1/2}}^{x_{i+1/2}} \left(\mathcal{W}(x, t^{n+1}) - \mathcal{W}(x, t^n)\right) dx \\ &\qquad + \int_{t^n}^{t^{n+1}} \left(\mathcal{F}(\mathcal{W}(x_{i+1/2}, t)) - \mathcal{F}(\mathcal{W}(x_{i-1/2}, t))\right) dt = 0. \end{aligned} \tag{3.3}$$

Using (3.2) and approximating the numerical flux as:

$$\mathcal{F}_\Delta\left(\mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) \simeq \frac{1}{\delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}(\mathcal{W}(x_{i+1/2}, t)) dt,$$

the finite volumes scheme can be expressed as:

$$\mathcal{W}_i^{n+1} = \mathcal{W}_i^n - \frac{\delta t}{\delta x}\left[\mathcal{F}_\Delta\left(\mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) - \mathcal{F}_\Delta\left(\mathcal{W}_{i-1}^n, \mathcal{W}_i^n\right)\right], \tag{3.4}$$

where the numerical flux $\mathcal{F}_\Delta$ provides an approximation of the physical fluxes at the interfaces.

3.1. **Godunov Scheme of homogeneous problem.** In this subsection, we present the application of the Godunov scheme [16, 24, 26] for a homogeneous mixture of incompressible fluids. The system is discretized using a finite volume method. The aim is to capture interactions between shear layers, gravity waves, and vortex instabilities while ensuring numerical stability through an appropriate CFL condition.

At time t^n, the approximation of the solution as piecewise constant functions is given by

$$\mathcal{W}_\Delta(x,t^n) = \mathcal{W}_i^n \quad \text{if } x \in (x_{i-1/2}, x_{i+1/2}).$$

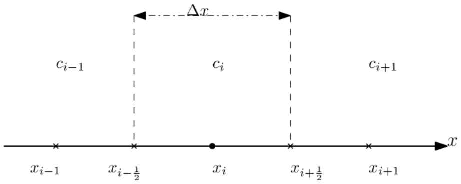

FIGURE 1. Meshing of the domain

The evolution to time $t^n + \delta t$ is performed by locally solving the Riemann problem at each interface $x_{i+1/2}$. This problem is written as follows:

$$\begin{cases} \partial_t \mathcal{W}(x,t) + \partial_x \mathcal{F}(\mathcal{W}(x,t)) = 0, \\ \mathcal{W}(x,t^n) = \mathcal{W}_\Delta(x,t). \end{cases} \tag{3.5}$$

The exact solution to this Riemann problem is denoted by $\mathcal{W}_\mathcal{R}\left(\dfrac{x - x_{i+1/2}}{t}, \mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right)$.

To ensure numerical stability and prevent interference between the solutions of the Riemann problems, a CFL condition is imposed. For a homogeneous mixture of incompressible fluids, this condition is written as:

$$\frac{\delta t}{\delta x} \max_{i\in\mathbb{Z}} \left(\left|\eta^-_{i+1/2}\right|, \left|\eta^+_{i+1/2}\right| \right) \leq \frac{1}{2}, \tag{3.6}$$

where $\eta^-_{i+1/2}$ and $\eta^+_{i+1/2}$ represent the minimal and maximal characteristic wave speeds derived from the exact solution of the Riemann problem.

Through the Jacobian matrix associated with equation (2.1), we obtain the eigenvalues defined as follows:

$$\eta \in \{0; V\}. \tag{3.7}$$

Then, for an incompressible fluid, these speeds are related to the local velocity V_i^n. So the CFL condition can be written by:

$$\frac{\delta t}{\delta x} \max_{i\in\mathbb{Z}, n\in\mathbb{N}} |V_i^n| \leq \frac{1}{2}. \tag{3.8}$$

This condition physically ensures that the information propagated by pressure waves and diffusive terms does not exceed the length of a cell δx during a time step δt.

Under the CFL condition (3.8) the approximate solution at time $t^n + \delta t$ is obtained by projecting the exact solution of the Riemann problems onto piecewise constant functions. The update of the average values in each cell is given by:

$$\mathcal{W}_i^{n+1} = \frac{1}{\delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{W}_{\mathcal{R}}(x, t^n + \delta t)\, dx. \tag{3.9}$$

This formulation ensures that the solution is conservative and captures the dynamics of the homogeneous mixture of incompressible fluids, including shock waves and transition zones smoothed by diffusion.

Lemma 3.1. [24] *At $\delta t > 0$, consider $\mathcal{W}_{\mathcal{R}}$ the solution of the Riemann problem* (3.6)*. Then, under the condition CFL* (3.8)*, we have*

$$\begin{aligned} &\frac{2}{\delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} \mathcal{W}_{\mathcal{R}} \left(\frac{x - x_{i-\frac{1}{2}}}{t}, \mathcal{W}_{i-1}, \mathcal{W}_i \right) dx \\ &= \mathcal{W}_i - \frac{2\delta t}{\delta x} \left(\mathcal{F}(\mathcal{W}_i) - \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_{i-1}, \mathcal{W}_i)) \right) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\frac{2}{\delta x} \int_{x_i}^{x_{i+\frac{1}{2}}} \mathcal{W}_{\mathcal{R}} \left(\frac{x - x_{i+\frac{1}{2}}}{t}, \mathcal{W}_i, \mathcal{W}_{i+1} \right) dx \\ &= \mathcal{W}_i - \frac{2\delta t}{\delta x} \left(\mathcal{F}(\mathcal{W}_i) - \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_i, \mathcal{W}_{i+1})) \right). \end{aligned} \tag{3.11}$$

Consequently, the relations (3.10) and (3.11) give us:

$$\begin{aligned} \mathcal{W}_i^{n+1} &= \frac{1}{2} \left(\mathcal{W}_i^n - \frac{\delta t}{\delta x/2} \left(\mathcal{F}(\mathcal{W}_i^n) \right) - \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_{i-1}, \mathcal{W}_i)) \right) \\ &+ \frac{1}{2} \left(\mathcal{W}_i^n - \frac{\delta t}{\delta x/2} \left(\mathcal{F}(\mathcal{W}_i^n) \right) - \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_i, \mathcal{W}_{i+1})) \right) \\ &= \mathcal{W}_i^n - \frac{\delta t}{\delta x} \left(\mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_i, \mathcal{W}_{i+1})) - \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_{i-1}, \mathcal{W}_i)) \right), \end{aligned} \tag{3.12}$$

which can be reformulated in the conservative form (3.4) with a numerical flow given by

$$\mathcal{F}_{\Delta}(\mathcal{W}_i^n, \mathcal{W}_{i+1}^n) = \mathcal{F}(\mathcal{W}_{\mathcal{R}}(0; \mathcal{W}_i^n, \mathcal{W}_{i+1}^n)).$$

Note that the Godunov scheme is consistent since $\mathcal{W}_{\mathcal{R}} \left(\frac{x}{t}, \mathcal{W}, \mathcal{W} \right) = \mathcal{W}$ so that $\mathcal{F}_{\Delta}(\mathcal{W}, \mathcal{W}) = \mathcal{F}(\mathcal{W})$.

The Godunov scheme provides an accurate and stable simulation of the flow of a homogeneous mixture of incompressible fluids. The CFL condition plays a critical role in ensuring that the numerical evolution respects the physical constraints of propagation and diffusion.

3.2. **Godunov-type Scheme of homogeneous problem.** The works of Harten, Lax, and van Leer (see [22]) have shown that the exact solution of the Riemann problem, $\mathcal{W}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right)$ whose approximation is $\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right)$ must satisfy the following integral consistency condition:

$$\frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \mathcal{W}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right) dx = \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right) dx. \tag{3.13}$$

Under this condition, the average of the exact solution of the Riemann problem can be explicitly written as follows:

$$\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right) = \begin{cases} \mathcal{W}_i & \text{if } x < \eta^L_{i+1/2}\delta t, \\ \overline{\mathcal{W}}\left(\frac{x}{\delta t}\right) & \text{if } \eta^L_{i+1/2}\delta t < x < \eta^R_{i+1/2}\delta t, \\ \mathcal{W}_{i+1} & \text{if } x > \eta^R_{i+1/2}\delta t, \end{cases} \tag{3.14}$$

where $\eta^L_{i+1/2}$ and $\eta^R_{i+1/2}$ are characteristic velocities.

Lemma 3.2. [24] *Let $\mathcal{W}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right)$ be the exact solution of the Riemann problem (3.5), then:*

$$\begin{aligned} \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \mathcal{W}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_i, \mathcal{W}_{i+1}\right) dx = \frac{1}{2}\left(\mathcal{W}_i + \mathcal{W}_{i+1}\right) \\ -\frac{\delta t}{\delta x}\left(\mathcal{F}(\mathcal{W}_{i+1}) - \mathcal{F}(\mathcal{W}_i)\right) \end{aligned} \tag{3.15}$$

under the CFL condition defined by (3.8).

At time t^n, we assume that an approximation to the piecewise constant solution is known:

$$\widetilde{\mathcal{W}}_{\Delta}\left(x, t^n\right) = \mathcal{W}^n_i, \text{ if } x \in \left(x_{i-1/2}, x_{i+1/2}\right).$$

Then, we introduce an approximation to the solution at time $t^n + \delta t$, given by:

$$\widetilde{\mathcal{W}}_{\Delta}\left(x, t^n + \delta t\right) = \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\delta t}, \mathcal{W}^n_i, \mathcal{W}^n_{i+1}\right), \quad \text{if } x \in \left(x_{i-1/2}, x_{i+1/2}\right), \tag{3.16}$$

where $\widetilde{\mathcal{W}}_{\mathcal{R}}$ satisfies the consistency condition (3.13).

Therefore, using definition (3.16), this Godunov-type scheme can be rewritten in the conservative and consistent form as follows:

$$\mathcal{W}^{n+1}_i = \mathcal{W}^n_i - \frac{\delta t}{\delta x}\left(\mathcal{F}_{\Delta}(\mathcal{W}_i, \mathcal{W}_{i+1}) - \mathcal{F}_{\Delta}(\mathcal{W}_{i-1}, \mathcal{W}_i)\right), \tag{3.17}$$

where the numerical flux function takes the following expression:

$$\mathcal{F}_{\Delta}(\mathcal{W}_i, \mathcal{W}_{i+1}) = \mathcal{F}(\mathcal{W}^n_{i+1}) - \frac{\delta t}{2\delta x}\mathcal{W}^n_{i+1} + \frac{1}{\delta x}\int_{x_{i+1/2}}^{x_{i+1}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\delta t}, \mathcal{W}^n_i, \mathcal{W}^n_{i+1}\right) dx,$$

or alternatively:

$$\mathcal{F}_{\Delta}(\mathcal{W}_L, \mathcal{W}_R) = \mathcal{F}(\mathcal{W}_L) + \frac{\delta t}{2\delta x}\mathcal{W}_L - \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{0} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t}, \mathcal{W}_L, \mathcal{W}_R\right) dx.$$

3.3. **Godunov-type scheme for the pollution model of the Kazhikhov-Smagulov equations.** In this section, we write the Godunov-type scheme adapted for the Kazhikhov-Smagulov pollution model, using the spatial and temporal discretization introduced above. Thus, we are interested in the approximation of the solution at time $t^{n+1} = t^n + \delta t$.

Thanks to the work carried out in [7,16,17], this approximation is written as:

$$\mathcal{W}_i^{n+1} = \frac{1}{\delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{W}_\Delta(x, t^n + \delta t) dx, \tag{3.18}$$

with

$$\mathcal{W}_\Delta(x, t^n + \delta t) = \widetilde{\mathcal{W}}_\Delta\left(\frac{x - x_{i+1/2}}{t}; \mathcal{W}_i^n, \mathcal{W}_i^{n+1}\right), \quad \text{if } (x,t) \in (x_i, x_{i+1}) \times (0, \delta t);$$

where $\widetilde{\mathcal{W}}_\Delta\left(\frac{x}{t}; \mathcal{W}_L, \mathcal{W}_R\right)$ is the approximation of the solution to the Riemann problem defined by

$$\begin{cases} \partial_t \mathcal{W} + \partial_x \mathcal{F}(\mathcal{W}) = \partial_x \left(\mathcal{D}(\mathcal{W}) \partial_x \mathcal{W}\right) + \mathcal{S}(\mathcal{W}), \\ \\ \mathcal{W}(x, t=0) = \begin{cases} \mathcal{W}_L & \text{if } x < 0 \\ \mathcal{W}_R & \text{if } x > 0. \end{cases} \end{cases} \tag{3.19}$$

Using the integral consistency condition (3.13) as in [22,24] and integrating (3.19) over $(-\delta x/2, \delta x/2) \times (0, \delta t)$, we obtain:

$$\begin{aligned} \frac{1}{\delta x} \int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \mathcal{W}_{\mathcal{R}}(x, \delta t, \mathcal{W}_L, \mathcal{W}_R) dx &= \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R) - \frac{1}{\delta x} \int_0^{\delta t} \mathcal{F}(\mathcal{W}_{\mathcal{R}}(\delta x/2, t, \mathcal{W}_L, \mathcal{W}_R)) dt \\ &+ \frac{1}{\delta x} \int_0^{\delta t} \mathcal{F}(\mathcal{W}_{\mathcal{R}}(-\delta x/2, t, \mathcal{W}_L, \mathcal{W}_R)) dt + \frac{1}{\delta x} \int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \int_0^{\delta t} \mathcal{S}(\mathcal{W}_{\mathcal{R}}(x, t, \mathcal{W}_L, \mathcal{W}_R)) dx dt \\ &+ \frac{1}{\delta x} \int_0^{\delta t} \mathcal{D}(\mathcal{W}_{\mathcal{R}}(\delta x/2, t, \mathcal{W}_L, \mathcal{W}_R)) dt - \frac{1}{\delta x} \int_0^{\delta t} \mathcal{D}(\mathcal{W}_{\mathcal{R}}(-\delta x/2, t, \mathcal{W}_L, \mathcal{W}_R)) dt \end{aligned} \tag{3.20}$$

where we use the notation $\mathcal{D}(\mathcal{W}_{\mathcal{R}})$ instead of $\mathcal{D}(\mathcal{W}_{\mathcal{R}}) \partial_x \mathcal{W}_{\mathcal{R}}$ for simplicity.

We set

$$\mathcal{I}_{i+1/2}^{-} = \frac{1}{\delta x} \int_{-\frac{\delta x}{2}}^{0} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t}, \mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) dx, \quad \mathcal{I}_{i+1/2}^{+} = \frac{1}{\delta x} \int_0^{\frac{\delta x}{2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t}, \mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) dx,$$

we obtain:

$$\begin{aligned} \mathcal{W}_i^{n+1} &= \mathcal{W}_i^n - \frac{\delta t}{\delta x} \left(\mathcal{F}_\Delta(\mathcal{W}_i^n, \mathcal{W}_{i+1}^n) - \mathcal{F}_\Delta(\mathcal{W}_{i-1}^n, \mathcal{W}_i^n)\right) \\ &+ \frac{\delta t}{\delta x}(\overline{\mathcal{D}}_{i+1/2} - \overline{\mathcal{D}}_{i-1/2}) + \frac{\delta t}{2}(\overline{\mathcal{S}}_{i+1/2} + \overline{\mathcal{S}}_{i-1/2}) \end{aligned} \tag{3.21}$$

with the numerical flow function defined by:

$$\mathcal{F}_\Delta(\mathcal{W}_L, \mathcal{W}_R) = \frac{1}{2}(\mathcal{F}(\mathcal{W}_L) + \mathcal{F}(\mathcal{W}_R)) - \frac{\delta x}{4\delta t}(\mathcal{W}_R - \mathcal{W}_L) + \frac{\delta x}{2\delta t}(\mathcal{I}_{LR}^{+} - \mathcal{I}_{LR}^{-}), \tag{3.22}$$

where,

$$\mathcal{I}^{-}_{LR} = \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{0} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t};\mathcal{W}_R,\mathcal{W}_L\right)\,dx \quad \mathcal{I}^{+}_{LR} = \frac{1}{\delta x}\int_{0}^{\frac{\delta x}{2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t};\mathcal{W}_R,\mathcal{W}_L\right)\,dx$$

and $\overline{\mathcal{D}}_{i-1/2}$ and $\overline{\mathcal{D}}_{i+1/2}$ are constants given use to approach the integral on $(0,\delta t)$ of $\mathcal{D}\left(\mathcal{W}_{\mathcal{R}}\left(\frac{-\delta x/2}{\delta t};\mathcal{W}_L,\mathcal{W}_R\right)\right)$ and $\mathcal{D}\left(\mathcal{W}_{\mathcal{R}}\left(\frac{\delta x/2}{\delta t};\mathcal{W}_L,\mathcal{W}_R\right)\right)$.

3.3.1. ***Discretization of source and diffusion terms.*** Constructing the local approximation of the terms $\overline{\mathcal{D}}_{LR}(\mathcal{W}_L,\mathcal{W}_R)$ and $\overline{\mathcal{S}}_{LR}(\mathcal{W}_L,\mathcal{W}_R)$ is crucial for the scheme (3.21) to preserve stationary solutions. These functions, as local approximations, must be defined to satisfy the stationarity condition, in accordance with the definition given in (2.4). So, we introduce the definition of local stationarity as follows.

Definition 3.1. *A solution is considered locally stationary if for all states $\mathcal{W}_L = (\rho_L, q_L)$ and $\mathcal{W}_R = (\rho_R, q_R)$, we have:*

$$\begin{cases} \rho_L V_L - \rho_R V_R = \lambda\left(\partial_x\rho_R - \partial_x\rho_L\right), \\ \mathcal{B}\left(\mathcal{W}_L,\mathcal{D}_L\right) = \mathcal{B}\left(\mathcal{W}_R,\mathcal{D}_R\right) - \mathcal{A}\left(\mathcal{S}_L,\mathcal{S}_R\right), \\ \partial_x V_{LR} = 0, \end{cases} \tag{3.23}$$

with

$$\mathcal{B}\left(\mathcal{W},\mathcal{D}\right) = \rho V^2 + P(\rho) - \lambda\,\partial_x(\rho V) \text{ and } \mathcal{A}\left(\mathcal{S}_L,\mathcal{S}_R\right) = \mathbf{g}\int_{x_L}^{x_R}\rho(x)\,dx,$$

where $\mathbf{g}$ *is the gravitational field and* λ *is the mass diffusion constant.*

Assume now that the sextuplet $(\mathcal{W}_R,\mathcal{W}_L,\mathcal{D}_L,\mathcal{D}_R,\mathcal{S}_R,\mathcal{S}_L)$ satisfies condition (3.23). In this context, the approximate terms $\overline{\mathcal{D}}_{LR}(\mathcal{W}_R,\mathcal{W}_L)$ and $\overline{\mathcal{S}}_{LR}(\mathcal{W}_R,\mathcal{W}_L)$ must be defined such that the stationary state $\widetilde{\mathcal{W}}_{\mathcal{R}}(\frac{x}{\delta t},\mathcal{W}_R,\mathcal{W}_L)$ is preserved, i.e.,

$$\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{\delta t},\mathcal{W}_R,\mathcal{W}_L\right) = \begin{cases} \mathcal{W}_L & \text{if } x<0, \\ \mathcal{W}_R & \text{if } x>0. \end{cases}$$

Accordingly, when the conditions of (3.22) are met, the approximate source term and viscosity term; $\overline{\mathcal{S}}_{LR}(\mathcal{W}_R,\mathcal{W}_L) = (\delta x\overline{\mathcal{S}}^{\rho}_{LR},\delta x\overline{\mathcal{S}}^{q}_{LR})$ and $\overline{\mathcal{D}}_{LR}(\mathcal{W}_R,\mathcal{W}_L) = (\delta x\overline{\mathcal{D}}^{\rho}_{LR},\delta x\overline{\mathcal{D}}^{q}_{LR})$ respectively must satisfy the following relations to ensure the integral consistency of the stationary solution:

$$\begin{cases} \delta x\overline{\mathcal{S}}^{\rho}_{LR} + \delta x\overline{\mathcal{D}}^{\rho}_{LR} = \lambda(\partial_x\rho)_{LR}, \\ \\ \delta x\overline{\mathcal{D}}^{q}_{LR} + \delta x\overline{\mathcal{S}}^{q}_{LR} = \left(\rho V^2 + P(\rho)\right)_R - \left(\rho V^2 + P(\rho)\right)_L + \lambda\partial_x(\rho V)_{LR} + \mathcal{A}(\mathcal{S}_R,\mathcal{S}_L). \end{cases} \tag{3.24}$$

Since the continuity equation is non-homogeneous, we seek to solve system (3.23) to determine the intermediate states (ρ^*, V^*), or equivalently $q^* = \rho^* V^*$, as functions of the states $\mathcal{W}_L$ and $\mathcal{W}_R$.

3.3.2. ***Approximate Riemann solver.*** In order to complete the construction of the numerical scheme (3.17), it is essential to rigorously define the approximate Riemann solver $\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_R, \mathcal{W}_L\right)$, such that the resulting flux satisfies the integral consistency condition (3.13), which is crucial for ensuring the stability and convergence of the scheme. We consider a self-similar structure for the Riemann problem solution, approximated by one piecewise constant state separated by wave fronts. The numerical profile of the solution is thus given by:

$$\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_R, \mathcal{W}_L\right) = \begin{cases} \mathcal{W}_L & \text{if } \dfrac{x}{t} < \eta_L, \\ \mathcal{W}^* & \text{if } \eta_L < \dfrac{x}{t} < \eta_R, \\ \mathcal{W}_R & \text{if } \dfrac{x}{t} > \eta_R, \end{cases} \tag{3.25}$$

where:

- $\mathcal{W}_L$ and $\mathcal{W}_R$ are the left and right conservative states, respectively;
- $\mathcal{W}^*$ denote the intermediate state introduced to capture the internal structure of the solution;
- η_L and η_R are the characteristic wave speeds associated with the left and right wave fronts.

In this framework, we assume that the waves are successive and symmetric, as suggested in [24]. This assumption allows the scheme to reproduce wave interactions and reflections with reduced computational complexity, while preserving numerical accuracy.

In the following, we derive explicit expressions for the intermediate state $\mathcal{W}^*$ by enforcing the integral consistency condition in the case of a single intermediate state, relevant for problems dominated by a single wave (shock or rarefaction).

We have:

$$\begin{aligned} \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_R, \mathcal{W}_L\right) dx &= \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R) - \frac{\delta t}{\delta x}\left(\mathcal{F}(\mathcal{W}_R) - \mathcal{F}(\mathcal{W}_L)\right) \\ &\quad + \frac{\delta t}{\delta x}(\overline{\mathcal{D}}_R - \overline{\mathcal{D}}_L) + \frac{\delta t}{2}(\overline{\mathcal{S}}_L + \overline{\mathcal{S}}_R) \end{aligned} \tag{3.26}$$

and furthermore

$$\begin{aligned} \frac{1}{\delta x}\int_{-\frac{\delta x}{2}}^{\frac{\delta x}{2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t}, \mathcal{W}_R, \mathcal{W}_L\right) dx &= \mathcal{I}_{LR}^{+} + \mathcal{I}_{LR}^{-} \\ &= 2\eta\frac{\delta t}{\delta x}\mathcal{W}^* + \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R) - \eta\frac{\delta t}{\delta x}(\mathcal{W}_L + \mathcal{W}_R). \end{aligned} \tag{3.27}$$

Using (3.25) and (3.27) we get:

$$\mathcal{W}^* = \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R) - \frac{1}{2\eta}\left(\mathcal{F}(\mathcal{W}_R) - \mathcal{F}(\mathcal{W}_L)\right)$$

$$+\frac{1}{2\eta}(\overline{\mathcal{D}}_R - \overline{\mathcal{D}}_L) + \frac{1}{4\eta\delta x}(\overline{\mathcal{S}}_L + \overline{\mathcal{S}}_R). \tag{3.28}$$

Now, using (3.28) we have the following relations.

$$\begin{cases} \rho^* = \dfrac{\rho_R + \rho_L}{2} - \dfrac{(\rho V)_R - (\rho V)_L}{2\eta} + \dfrac{1}{2\eta}\overline{\mathcal{D}}^{\rho}_{LR} \\ \quad = \rho^{HLL} + \dfrac{1}{2\eta}\overline{\mathcal{D}}^{\rho}_{LR} \\ \\ q^* = \dfrac{(\rho V)_R + (\rho V)_L}{2} - \dfrac{\rho_R V_R^2 + P_R - \rho_L V_L^2 - P_L}{2\eta} \\ \qquad\qquad + \dfrac{1}{2\eta}\overline{\mathcal{D}}^{q}_{LR} + \dfrac{1}{4\eta\delta x}\overline{\mathcal{S}}^{q}_{LR} \\ \quad = q^{HLL} + \dfrac{1}{2\eta}\overline{\mathcal{D}}^{q}_{LR} + \dfrac{1}{4\eta\delta x}\overline{\mathcal{S}}^{q}_{LR} \end{cases} \tag{3.29}$$

In this case of one intermediate state $\mathcal{W}^*$, we have

$$\mathcal{I}^-_{LR} = \frac{\eta\delta t}{\delta x}\mathcal{W}^* + \left(\frac{1}{2} - \frac{\eta\delta t}{\delta x}\right)\mathcal{W}_L \text{ and } \mathcal{I}^+_{LR} = \frac{\eta\delta t}{\delta x}\mathcal{W}^* + \left(\frac{1}{2} - \frac{\eta\delta t}{\delta x}\right)\mathcal{W}_R.$$

So, the numerical flux (3.22) is written as:

$$\mathcal{F}_\Delta(\mathcal{W}_L, \mathcal{W}_R) = \frac{1}{2}(\mathcal{F}(\mathcal{W}_L) + \mathcal{F}(\mathcal{W}_R)) - \frac{\eta}{2}(\mathcal{W}_R - \mathcal{W}_L). \tag{3.30}$$

This symmetric construction provides a consistent and structured way to recover the left and right constant states based on the known approximation $\mathcal{W}^*$ and the local dissipation and source terms.

In Godunov-type scheme enriched with dissipative and source terms, the intermediate state $\mathcal{W}^*$ provide a refined approximation of the local Riemann problem. We establish a relation involving fluxes, dissipation, and source terms, which is instrumental for the design and analysis of the numerical scheme.

Lemma 3.3. *Let $\mathcal{W}_L$ and $\mathcal{W}_R$ be the left and right states at an interface, and let $\mathcal{W}^*$ be the intermediate state such that their average defines (at equilibrium):*

$$\mathcal{W}^* = \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R). \tag{3.31}$$

Assume the relation (3.28), then the following flux difference relation holds:

$$\mathcal{F}(\mathcal{W}_R) - \mathcal{F}(\mathcal{W}_L) = (\overline{\mathcal{D}}_R - \overline{\mathcal{D}}_L) + \frac{1}{2\delta x}(\overline{\mathcal{S}}_L + \overline{\mathcal{S}}_R). \tag{3.32}$$

Proof. Start from the given expression for $\mathcal{W}^*$:

$$\mathcal{W}^* = \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R) - \frac{1}{2\eta}(\mathcal{F}(\mathcal{W}_R) - \mathcal{F}(\mathcal{W}_L)) + \frac{1}{2\eta}(\overline{\mathcal{D}}_R - \overline{\mathcal{D}}_L) + \frac{1}{4\eta\delta x}(\overline{\mathcal{S}}_L + \overline{\mathcal{S}}_R).$$

Moreover, we have $\mathcal{W}^* = \frac{1}{2}(\mathcal{W}_L + \mathcal{W}_R)$; then we conclude this proof by multiplying both sides by 2η. □

Theorem 3.1. *Consider the following CFL-type condition:*

$$\max_{i\in\mathbb{Z}} \left|V_{i+1/2}\right| \delta t \leq \frac{\delta x}{2}. \tag{3.33}$$

The numerical scheme (3.21)-(3.28)-(3.30) and (3.32) is consistent with (1.6). Let $\mathcal{W}_i^n \in \Omega$ for all $i \in \mathbb{Z}$. The updated state $\mathcal{W}_i^{n+1}$, given by (3.21), satisfies the following properties:

1. *Preservation of the positivity of the density: $\rho_i^n > 0$ for all $i \in \mathbb{Z}$.*
2. *Preservation of all equilibrium states: $\mathcal{W}_i^{n+1} = \mathcal{W}_i^n$, $\forall i \in \mathbb{Z}$, provided that $(\mathcal{W}_i^n)_{i\in\mathbb{Z}}$ satisfies:*

$$V_i^{n+1} = V_i^n; \quad \rho_i^{n+1} = \rho_i^n,$$

where $\rho_i^n = \dfrac{\beta_i^n}{\gamma} e^{\mu x_i}$ *with* $\beta_i^n = \dfrac{\nu V_i^n \mathbf{g}}{a V_i^n - \lambda \mathbf{g}}$.

Proof. For (1), regarding the preservation of the positivity of the density, we start from (3.18) and write:

$$\rho_i^{n+1} = \frac{1}{\delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho_\Delta(x, t^n + \delta t) dx, \tag{3.34}$$

where ρ_Δ denotes the exact solution (density) of the Riemann problem associated with the Kazhikhov-Smagulov equations (1D). Since this solution is strictly positive, it follows that $\rho_i^{n+1} > 0$. Therefore, property (1) is satisfied for all $(i, n) \in \mathbb{Z} \times \mathbb{N}$.

For (2), given the sequence $(\mathcal{W}_i^n)_{i\in\mathbb{Z}}$ and under the CFL condition specified by (3.33), as in [4], we have:

$$\widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\Delta t}; \mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) = \begin{cases} \mathcal{W}_i^n, & \text{if } x_{i+1/2} < 0, \\ \mathcal{W}_{i+1}^n, & \text{if } x_{i+1/2} > 0, \end{cases} \tag{3.35}$$

Then we have the following sequence of equalities using (3.35):

$$\begin{aligned} \mathcal{W}_i^{n+1} &= \frac{1}{\delta x} \int_{x_{i-1/2}}^{x_i} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x - x_{i-1/2}}{\delta t}; \mathcal{W}_{i-1}^n, \mathcal{W}_i^n\right) dx \\ &\quad + \frac{1}{\delta x} \int_{x_i}^{x_{i+1/2}} \widetilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{\delta t}; \mathcal{W}_i^n, \mathcal{W}_{i+1}^n\right) dx \\ &= \frac{1}{\delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{W}_i^n \, dx \\ &= \mathcal{W}_i^n. \end{aligned} \tag{3.36}$$

So (2) is satisfied in equilibrium and using Lemma 2.1, we deduce that $\rho_i^n = \dfrac{\beta_i^n}{\gamma} e^{\mu x_i}$ with $\beta_i^n = \dfrac{\nu V_i^n \mathbf{g}}{a V_i^n - \lambda \mathbf{g}}$. □

4. NUMERICAL RESULTS

In this section dedicated to numerical simulation, we study the mixing of two incompressible and viscous fluids, such as water and glycerin or, as an example, a vaccine and blood. The considered interval is $[0, 10]$, or $[0, 30]$ and there are $N = 500$ cells. We model the pressure as an affine function of the density, expressed by the equation $P = a\rho + k$, where $a = 4.25$ and $k = 2$. The initial conditions are defined as follows: $u_L = 0$, $u_R = 1$, $\rho_L = 0.5$, $\rho_R = 0.25$, under the influence of gravity $g = 9.81$. It should be mentioned that the diffusion term is discretized using Simpson's method, while the source term is treated through the arithmetic mean. We highlight and decipher here two fundamental aspects of the results obtained.

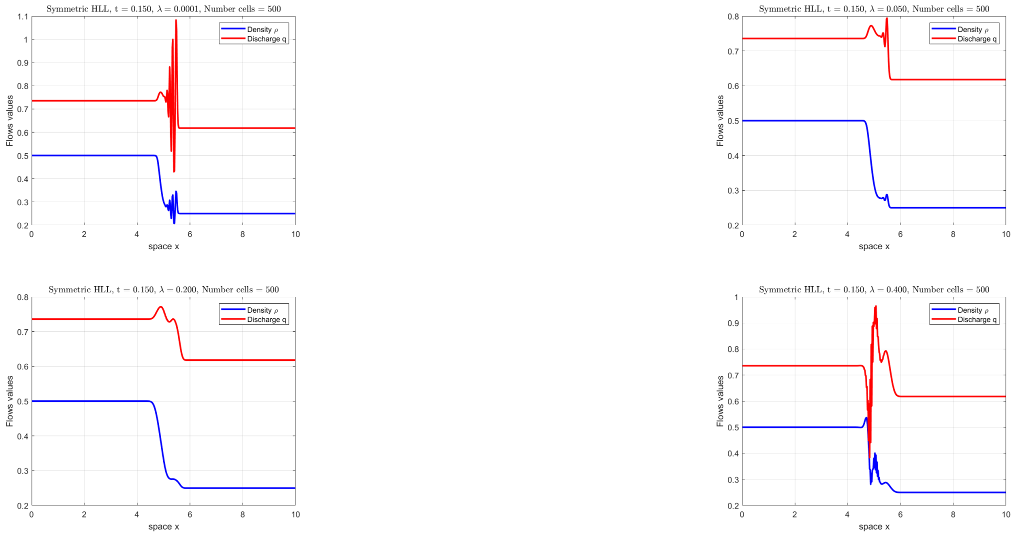

FIGURE 2. Numerical results of the mixture dynamics for different diffusion coefficients.

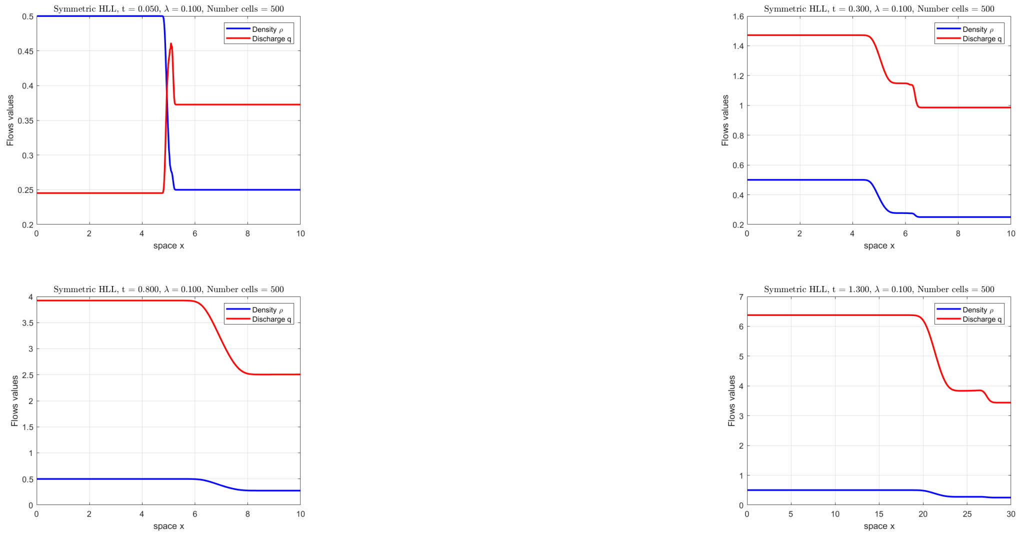

FIGURE 3. Temporal evolution of the mixture dynamics for a fixed diffusion coefficient.

Interpretations of numerical results:

(1) **Impact of the diffusion constant.**
Numerical observations (Figure 2) indicate that for very small values $\lambda \leq 10^{-4}$, the solution (density and momentum) displays numerous discontinuities and oscillations. These effects gradually diminish as λ increases, leading to a more stable profile. Beyond values of about 0.2, another type of discontinuity emerges, this time more diffuse. This behavior reflects the balance between convection and diffusion: when diffusion is weak, sharp fronts remain close to the initial state, while stronger diffusion progressively smooths out the mixture, sometimes creating artificially uniform regions. The results also show that the Godunov-type scheme may generate local oscillations when steep gradients are present (low λ), and may become overly sensitive at high diffusion, producing an artificial flattening of the solution profiles.

(2) **Impact of time for a defined λ.**
For a given value of λ (see Figure 3), the numerical results show that, at short time ($t = 0.05$), the density exhibits a sharp and pronounced discontinuity, while the momentum displays two distinct discontinuities. These irregularities gradually decrease over time and seem to vanish for $t \geq 1.2$. This phenomenon illustrates the dissipative effect of diffusion, which progressively homogenizes the mixture and reduces differences in density and velocity. The results also highlight the ability of the Godunov-type scheme to accurately capture the transient evolution of fronts, while showing that diffusion acts as a mechanism of progressive regularization of the solution, making it more easily resolved by discretization.

5. Conclusion

This digital research highlights the importance of the diffusion constant and time in the behavior of a mixture of incompressible viscous fluids. On a physical level, diffusion represents the process of gradual mixing and homogenization: for low values, distinct interfaces persist, while intense diffusion accelerates flattening and quickly disperses contrasts. As for time, it regulates the transition between the transient phases and the equilibrium state. The Godunov scheme proves reliable for representing these dynamics in intermediate regimes at the numerical level. It correctly captures the interaction between advection and diffusion. However, some sensitivities arise: low diffusion (oscillations or instabilities); high diffusion (excessive smoothing and loss of physical information).

These results highlight the importance of a balanced selection of parameters and an appropriate mesh. They also propose the idea of developing more precise or adaptive models. As a result, this study highlights the dual challenge: ensuring an accurate representation of physics while maintaining digital robustness.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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