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STABILITY OF A-Q FUNCTIONAL EQUATION IN ULTRAMETRIC BANACH ALGEBRA

SHAYMAA ALSHYBANI

Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq

shaymaa.farhan@qe.edu.iq

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Abstract. This paper deals with a mixed type additive quadratic functional equation

$$\begin{aligned}\aleph(\vartheta+2\iota+3\zeta)+\aleph(\vartheta-2\iota+3\zeta)+\aleph(\vartheta+2\iota-3\zeta)\\ +\aleph(\vartheta-2\iota-3\zeta)=4\aleph(\vartheta)\\ +8[\aleph(\iota)+\aleph(-\iota)]+18[\aleph(\zeta)+\aleph(-\zeta)]\end{aligned}$$

For any $\vartheta,\iota,\zeta\in H$, in non-Archimedean Banach algebra. By the direct and fixed point methods, we studied the previous equation's stability. The results are presented within non-Archimedean Banach algebra.

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1. Introduction

Functional equations are substantial to mathematical analysis, and the notion of stability is one of their major properties, which indicates that the in exact solution is close to the perfect one. This research focuses on the stability of a F-E within N-A B-As, where the image of convergence differs from the classic model.

The paper inspects an equation showing both additive and quadratic behavior, the function nature is specified by the symmetry: in the odd case, it's additive. In the even case, it's quadratic. The purpose of this work is expanding the range of results connected to stability in this unusual algebraic and analytic framework.

Where non-standard measures and ideal convergence phenomena dominate. The findings pant to progress the theory of stability in an analytical scope that remains underrepresented. Applications of

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N-A space include advanced fields in mathematics and physics, most notably number theory (p-adic numbers) and quantum mechanics. These applications rely on a strong trigonometric inequality, which facilitates the study of fixed points, the analysis of functions, and the approximation of solutions in geometric and algebraic contexts. There are a huge number of sources on this subject, from the time this issue began with Ulam [12] until today. We will refer to some of them [1–5,7–10,13].

In this paper, we investigate the generalized Ulam-Hyers stability of a mixed type A-Q F-E:

$$\begin{aligned}\aleph(\vartheta+2\iota+3\zeta)+\aleph(\vartheta-2\iota+3\zeta)+\aleph(\vartheta+2\iota-3\zeta)\\ +\aleph(\vartheta-2\iota-3\zeta)=4\aleph(\vartheta)+8[\aleph(\iota)+\aleph(-\iota)]\\ +18[\aleph(\zeta)+\aleph(-\zeta)]\end{aligned}\tag{1.1}$$

In N-A B-A.

2. Preliminaries

See [6,11] Let ρ indicate a field and a function $\lfloor\cdot\rfloor:\rho\to[0,\infty)$, is a N-A valuation absolute function if it satisfies the inequality $\lfloor\vartheta+\iota\rfloor\leq\max\{\lfloor\vartheta\rfloor,\lfloor\iota\rfloor\}\leq\lfloor\vartheta\rfloor+\lfloor\iota\rfloor,\vartheta,\iota\in\rho$.

The related field ρ is indicated to as a N-A field. Clear that, $\lfloor 1\rfloor=\lfloor -1\rfloor=1$ and $\lfloor\imath\rfloor\leq 1\forall\imath\geq 1$, A petty example of a N-A valuation is the function $\lfloor\cdot\rfloor$ occupation everything unless 0 into 1 and $\lfloor 0\rfloor=0$. We constantly suppose in addition that $\lfloor\cdot\rfloor$ is nontrivial, i.e., there is a $\zeta\in\rho$ such that $\lfloor\zeta\rfloor\neq 0,1$.

Let $\mathbb{S}$ be a linear space on ρ with a N-A nontrivial valuation $\lfloor\cdot\rfloor$. A function $[\![\cdot]\!]:\mathfrak{x}\to[0,\infty)$ is a N-A norm if it is a norm over ρ with ultrametric inequality i.e.,

$[\![\vartheta+\iota]\!]\leq\max\{[\![\vartheta]\!],[\![\iota]\!]\}\forall\vartheta,\iota\in\mathfrak{x}$, Then $(\mathfrak{x},[\![\cdot]\!])$ is called a N-A space.

Here a sequence $\{x_\imath\},\imath\in\mathbb{N}$ is Cauchy iff $\{\vartheta_{\imath+1}-\vartheta_\imath\}_{\imath\in\mathbb{N}}\to 0$. By a complete N-A space we mean one in which every Cauchy sequence is convergent. A N-A B.A is a complete N-A algebra A which satisfies

$$[\![\vartheta\iota]\!]\leq[\![\vartheta]\!][\![\iota]\!]\ \forall\ \vartheta,\iota\in A$$

Theorem 2.1. [10] *Let $(\mathcal{E},m)$ be a complete generalized metric space and let $\Omega:\mathcal{E}\longrightarrow\mathcal{E}$ be a strictly contractive mapping with Lipschitz constant $\iota<1$, Then for each gives element $x\in\mathcal{E}$, either $m\left(\mathrm{J}^\imath x,\mathrm{J}^{\imath+1}x\right)=\infty$ for all $n\in\mathrm{N}_0$ (non-negative integers) or there exists $n_0\in\mathrm{N}_0$ such that*

1. *$m\left(\mathrm{J}^\imath\vartheta,\mathrm{J}^{\imath+1}\vartheta\right)<\infty,\imath\geq n_0$,*
2. *The sequence $\{\mathrm{J}^n\vartheta\}$ converges to a F-B λ^* of J,*
3. *y^* is the unique F-B of J in the set $Y=\{y\in\mathcal{E},m(\mathrm{J}^{\imath_0}\vartheta,\lambda)<\infty\}$,*
4. *$m(\iota,\iota^*)\leq\frac{1}{1-\iota}m(\iota,J\iota)\forall\vartheta,\iota\in Y$*

Definition 2.2. [10] let $\mathbb{S},\ \mathbb{D}$ be B- As. A real linear mapping $\mathcal{Q}:\mathbb{S}\to\mathbb{D}$ is said to be a homomorphism if $\mathcal{Q}(\vartheta\iota)=\mathcal{Q}(\vartheta)\mathcal{Q}(\iota)$ for all $\vartheta,\iota\in X$,

Definition 2.3. [10] Let $\mathbb{S}.\mathbb{D}$ be B-As. A real linear mapping $\mathcal{Q}:\mathbb{S}\to\mathbb{D}$ is said to be a derivation if $\mathcal{Q}(\vartheta\iota)=\iota\mathcal{Q}(\vartheta)+\vartheta\mathcal{Q}(\iota)$ for all $\vartheta,\iota\in X$,

3. Direct stability results for (1.1)

Let $\mathbb{S},\mathbb{D}$ are non-Archimedean Banach algebras, and

$$\begin{aligned}D_{\aleph_e}(\vartheta\cdot\iota\cdot\zeta)=&\aleph(\vartheta+2\iota+3\zeta)+\aleph(\vartheta-2\iota+3\zeta)\\&+\aleph(\vartheta+2\iota-3\zeta)+\aleph(\vartheta-2\iota-3\zeta)\\&-4\aleph(\vartheta)-8[\aleph(y\iota)+\aleph(-\iota)]-18[\aleph(\zeta)+\aleph(-\zeta)]\end{aligned}$$

Theorem 3.1. [10] *Let* $i\in\{1,-1\}$ *and* $|3|<1.\Lambda:\mathbb{S}^3\to[0,\infty)$ *be a function such that* $\sum_{i=1}^{\infty}\frac{\Lambda\left(3^{i\imath}\vartheta,3^{i\imath}\iota,3^{in}\zeta\right)}{|3^{i\imath}|}<\infty$ *and* $\lim_{n\to\infty}\frac{\Lambda\left(3^{i\imath}\vartheta,3^{i\imath}\iota,3^{i\imath}\zeta\right)}{|3^{i\imath}|}=0\forall\vartheta,\iota,\zeta\in\mathbb{S}$ *And let* $\aleph_e:\mathbb{S}\to\mathbb{D}$ *be an odd function,* $\aleph_e(0)=0,\aleph_e$ *satisfies the inequality*

$$\|D_{\aleph_e}(\vartheta,\iota,\zeta)\|_Y\leq\Lambda(\vartheta,\iota,\zeta) \tag{3.1}$$

$$\|\aleph_e(\vartheta\iota)-\aleph_e(\vartheta)\ \aleph_e(\iota)\|\leq\Lambda(\vartheta,\iota,0)\ \forall\ \vartheta,\iota,\zeta\in\mathbb{S} \tag{3.2}$$

So, can find an additive mapping $h:\mathbb{S}\to\mathbb{D}$, h *satisfying the F-E* [1.1] *and the homomorphism property such that*

$$\|\aleph_e(\vartheta)-h(\vartheta)\|\leq\frac{1}{|6^i|}\lim_{\imath\to\infty}\max\left\{\frac{\Lambda\left(3^{im}\vartheta,3^{im}\vartheta,0\right)}{|3^{im}|},0\leq m<\imath,\forall\vartheta\in\mathbb{S}\right\} \tag{3.3}$$

The mapping $h(\vartheta)$ *is defined by* $h(\vartheta)=\lim_{\imath\to\infty}\frac{\aleph_e\left(3^{i\imath}\vartheta\right)}{3^{i\imath}}\ \forall\ \vartheta\in\mathbb{S}$

h *is unique if*

$$\frac{1}{|6^i|}\lim_{m\to\infty}\lim_{\imath\to\infty}\max\left\{\frac{\Lambda\left(3^{ib}\vartheta,3^{ib}\vartheta,0\right)}{|3^{ib}|},m\leq b<\imath+m\right\}=0, \tag{3.4}$$

Proof. Assume i=1. Replacing (ϑ,ι,ζ) by $(\vartheta,\vartheta,0)$ in (3.1) and using oddness of $\aleph_e$, we get

$$\left\|\aleph_e(\vartheta)-\frac{\aleph_e(3\vartheta)}{3}\right\|\leq\frac{1}{|6|}\Lambda(\vartheta,\vartheta,0)\,\forall\ \vartheta\in\mathbb{S} \tag{3.5}$$

Here we replace x by $3^{in}x$ and dividing by 3^{in} in (3.5). We get

$$\left\|\frac{\aleph_e\left(3^{i\imath}\vartheta\right)}{3^{i\imath}}-\frac{\aleph_e\left(3^{i\imath+1}\vartheta\right)}{3^{i\imath+1}}\right\|\leq\frac{1}{|6|\,|3^{i\imath}|}\Lambda\left(3^{i\imath}\vartheta,3^{i\imath}\vartheta,0\right)\to0\text{ as }\imath\to\infty$$

$\forall\ \vartheta\in\mathbb{S}$, Hence the sequence $\left\{\frac{\aleph_e\left(3^{i\imath}\vartheta\right)}{3^{i\imath}}\right\}$ is Cauchy sequence. Since $\mathbb{D}$ is complete, there exists a mapping $h:\mathbb{S}\to\mathbb{D}$ So that $h(\vartheta)=\lim_{n\to\infty}\frac{\aleph_e\left(3^{i\imath}\vartheta\right)}{3^{i\imath}}\forall\vartheta\in\mathbb{S}$. we can prove that

$$\left\|\frac{\aleph_e\left(3^{i\imath}\vartheta\right)}{3^{i\imath}}-\aleph_e(\vartheta)\right\|\leq\frac{1}{|6^i|}\ \max\left\{\frac{\Lambda\left(3^{i\imath}\vartheta,3^{i\imath}\vartheta,0\right)}{|3^{im}|},0\leq m<\imath\right\} \tag{3.6}$$

Making $n \to \infty$ in (3.6).

This proves that (3.3) is true for every $\vartheta \in \mathbb{S}$.

To prove h satisfies F-E (1.1), we change $(\vartheta, \iota, \zeta)$ by $\left(3^{i\imath}\vartheta, 3^{i\imath}\iota, 3^{i\imath}\zeta\right)$ and dividing by 3^{in} in **(1.3)**, we obtain

$$\frac{1}{|3^{i\imath}|}\left\|D_{\aleph_e}\left(3^{i\imath}\vartheta, 3^{i\imath}\iota, 3^{i\imath}\zeta\right)\right\| \leq \frac{1}{|3^{i\imath}|}\Lambda\left(3^{i\imath}\vartheta, 3^{i\imath}\iota, 3^{i\imath}\zeta\right)$$

$\forall \vartheta, \iota, z \in \mathbb{S}$. making $n \to \infty$ in the a bove inequality and utilizing the definition of $h(\vartheta)$, we notice that

$$D_h(\vartheta, \iota, \zeta) = 0$$

Hence h satisfies [1.1] $\forall \vartheta, \iota, \zeta, \in \mathbb{S}$, and h is an additive mapping.

$$\begin{aligned}\|h(\vartheta\iota) - h(\vartheta)h(\iota)\| &= \lim_{\imath\to\infty}\left\|\frac{\aleph_e\left(3^{in}\vartheta 3^{in}\iota\right)}{3^{i\imath}3^{i\imath}} - \frac{\aleph_{e(3^{i\imath}\vartheta)}}{3^{i\imath}}\frac{\aleph_{e(3^{i\imath}\iota)}}{3^{i\imath}}\right\| \\ &\lim_{\imath\to\infty}\frac{1}{|3^{2i\imath}|}\left\|\aleph_e\left(3^{i\imath}\ \vartheta\ 3^{i\imath}\iota\right) - \aleph_e\left(3^{i\imath}\vartheta\right)\aleph_e\left(3^{i\imath}\iota\right)\right\| \\ &\leq \lim_{n\to\infty}\frac{\Lambda\left(3^{i\imath}\vartheta, 3^{i\imath}\iota, o\right)}{|3^{2i\imath}|} = 0\forall\ \vartheta \in X\end{aligned}$$

$\therefore h(\vartheta\iota) = h(\vartheta)h(\iota) \cdot \forall \vartheta, \iota \in \mathbb{S}$

To show h is unique, Let $\acute{h}$ be another additive mapping satisfying (1.1) and (3.3). then

$$\begin{aligned}\left\|h(\vartheta) - \acute{h}(\vartheta)\right\| &\leq \max\left\{\|h(\vartheta) - \aleph_e(\vartheta)\|, \left\|\aleph_e(\vartheta) - \acute{h}(\vartheta)\right\|\right\} \\ &= \frac{1}{|6^i|}\lim_{\imath\to\infty}\max_{0\leq m<\imath}\left\{\frac{\Lambda\left(3^{im}\vartheta, 3^{im}\vartheta, e\right)}{|3^{im}|}\right\} \\ &\leq \frac{1}{|6^i|}\lim_{m\to\infty}\lim_{\imath\to\infty}\max_{m\leq b<\imath+m}\left\{\frac{\Lambda\left(3^{ib}\vartheta, \vartheta, e\right)}{|3^{ib}|}\right\} = o\forall\vartheta \in \mathbb{S}\end{aligned}$$

Hence h is unique.

For $i = -1$, we can prove a similar stability result. This completes the proof of the theorem.

Corollary 3.2 is an instant outcome of Theorem 3.1 □

Corollary 3.2. *Let r be nonnegative real number.*

$\aleph_e : \mathbb{S} \to \mathbb{D}$ *an odd function satisfies the inequalities:*

$$\begin{aligned}\|D_{\aleph_e}(\vartheta, \iota, \zeta)\| &\leq \\ \|\aleph_e(\vartheta\iota) - \aleph_e(\vartheta)\aleph_e(\iota)\| &\leq \begin{cases}\frac{\|\vartheta\|\cdot\|\iota\|}{\|\vartheta\|^r + \|\iota\|^r + \|\zeta\|^r}, & r < 3 \\ \frac{1}{\|\vartheta\|^r + \|\iota\|^r + \|\zeta\|^r}, & r < 1 \\ \frac{\|\vartheta\|\ \|\iota\|}{\|\vartheta\|^r}, & r < 3 \\ \|\vartheta\|^r \cdot \|\iota\|^r, & r \geq 1 \\ \|\vartheta\|^r + \|\iota\|^r + \|\zeta\|^r, & r > 1\end{cases}\end{aligned}$$

$\forall \vartheta, \iota, \zeta \in X$. *So, we can find a unique additive mapping* $h: \mathbb{S} \to \mathbb{D}$ *and confirms the homomorphism property such that*

$$\|\aleph_e(\vartheta) - h(\vartheta)\| \leq \frac{1}{|6|} \lim_{m\to\infty} \max_{0\leq m<\iota} \begin{cases} \frac{|3^n|^{3-r}\|\vartheta\|^2}{2\|\vartheta\|^r} \\ \frac{|3|^{1-r}}{2\|\vartheta\|^r} \\ |3^n|^{3-r}\|\vartheta\|^{2-r} \\ |3^n|^{2r-1}\|\vartheta\|^{2r} \\ 2|3^n|^{r-1}\|\vartheta\|^r \end{cases}$$

for all $\vartheta \in \mathbb{S}$.

Theorem 3.3. *Let* $i \in \{1, -1\}$ *and* $\delta : \mathbb{S}^3 \to [0, \infty)$ *be a function such that*

$$\sum_{i=1}^{\infty} \frac{\delta\left(3^{i\iota}\vartheta, 3^{i\iota}y, 3^{i\iota}\zeta\right)}{|3^{i\iota}|} < \infty \text{ and } \lim_{\iota\to\infty} \frac{\delta\left(3^{i\iota}\vartheta, 3^{i\iota}y, 3^{i\iota}\zeta\right)}{|3^{i\iota}|} = 0 \ \forall \vartheta, \iota, \zeta \in X$$

and $\aleph_e : \mathbb{S} \to \mathbb{D}$ *be an odd function satisfies the inequality*

$$\|D\aleph_e(\vartheta, \iota, \zeta)\|_Y \leq \delta(\vartheta, \iota, \zeta)$$

and

$$\|\aleph_e(\vartheta\iota) - x\aleph_e(\iota) - \iota\aleph_e(\vartheta)\|_Y \leq \delta(\vartheta, \iota, o) \quad \forall \vartheta, \iota, \zeta \in \mathbb{S}.$$

Then we can find an additive mapping $h: \mathbb{S} \to \mathbb{D}$, h *satisfy the F- E* [1.1] *and the derivation property such that*

$$\|\aleph_e(\vartheta) - h(\vartheta)\| \leq \frac{1}{|6^i|} \lim_{\iota\to\infty} \max_{e\leq m<\iota} \left\{ \frac{\delta\left(3^{im}\vartheta, 3^{im}\vartheta, 0\right)}{|3^{im}|} \right\}$$

The mapping $h(\vartheta)$ *is defined by*

$$h(\vartheta) = \lim_{\iota\to\infty} \frac{\aleph_e\left(3^{i\iota}\vartheta\right)}{3^{i\iota}}, \quad \forall \vartheta \in \mathbb{S}$$

h *is unique if*

$$\frac{1}{|6^i|} \lim_{m\to\infty} \lim_{\iota\to\infty} \max_{m\leq b<m+\iota} \left\{ \frac{\delta\left(3^{ib}\vartheta, 3^{ib}\vartheta, e\right)}{|3^{ib}|} \right\} = e$$

Proof. The proof is as in the previous theorem, and we only establish the condition of the derivation.

$$\begin{aligned} \|h(\vartheta\iota) - \vartheta h(\iota) - \iota h(\vartheta)\|_Y &= \lim_{n\to\infty} \left\| \frac{\aleph_e(3^{i\iota}\vartheta 3^{i\iota}\iota)}{3^{i\iota}3^{i\iota}} - \frac{3^{i\iota}\iota\aleph_e\left(3^{i\iota}\vartheta\right)}{3^{i\iota}3^{i\iota}} - \frac{3^{i\iota}\vartheta\ \aleph_e\left(3^{i\iota}\iota\right)}{3^{i\iota}3^{i\iota}} \right\| \\ &= \lim_{\iota\to\infty} \frac{1}{|3^{2i\iota}|} \left\| \aleph_e\left(3^{i\iota}\vartheta\iota\right) - 3^{i\iota}\iota\aleph_e\left(3^{i\iota}\vartheta\right) - 3^{i\iota}\iota\aleph_e\left(3^{i\iota}\vartheta\right) \right\| \\ &\leq \lim_{\iota\to\infty} \frac{1}{|3^{2i\iota}|} \delta\left(3^{i\iota}\vartheta, 3^{i\iota}\iota, 0\right) = 0 \end{aligned}$$

$\therefore h(\vartheta\iota) = \vartheta h(\iota) + \iota h(\vartheta)$. □

4. Fixed Point Stability Results for (1.1)

Theorem 4.1. *Let $\mathcal{L} < 1, |3| < 1, \Lambda : \mathbb{S} \to [e,\ \infty)$ is a function with the conditions*

$$\lim_{\iota \to \infty} \frac{\Lambda\left(3^{\iota}\vartheta, 3^{\iota}\iota, 3^{\iota}\zeta\right)}{|3^{\iota}|} = e$$

and

$$\Lambda\left(3\vartheta, 3\iota, 3z\right) \leq |3|\, \mathcal{L}\Lambda\left(\vartheta, \iota, \zeta\right),$$

$\aleph_e : \mathbb{S} \to \mathbb{D}$ *is an odd function,* $\aleph_e\left(0\right) = 0$.

$\aleph_e$ *satisfying*

$$\left\|D_{\aleph_e}\left(\vartheta, \iota, \zeta\right)\right\| \leq \Lambda\left(\vartheta, \iota, \zeta\right)$$

$$\left\|\aleph_e\left(\vartheta\iota\right) - \aleph_e\left(\vartheta\right)\aleph_e\left(\iota\right)\right\| \leq \Lambda\left(\vartheta, \iota, e\right)\ \forall\, \vartheta, \iota, \zeta \in \mathbb{S}$$

Then there exists unique additive mapping $h : \mathbb{S} \to \mathbb{D}$, h satisfies homomorphism property and

$$\left\|\aleph_e\left(\vartheta\right) - h\left(\vartheta\right)\right\| \leq \frac{1}{|6|\left(1 - L\right)}\, \Lambda\left(\vartheta, \vartheta, e\right).$$

Proof. $\left\|\aleph\left(\vartheta\right) - \frac{f(3\vartheta)}{3}\right\| \leq \frac{1}{|6|}\, \Lambda\left(\vartheta, \vartheta, e\right)$ consider the set

$$B = \left\{g : \mathbb{S} \to \mathbb{D}, g\left(e\right) = e\right\}$$

and mention the generalized metric on B.

$$d\left(\aleph, g\right) = \inf\left\{C \in \mathrm{R}^{+} :\ \left\|\aleph\left(\vartheta\right) - g\left(\vartheta\right)\right\| \leq \frac{C}{|6|}\, \Delta\left(\vartheta, \vartheta, e\right)\right\} \forall\, \vartheta \in \mathbb{S}.$$

Easily, we find that (X, d) is complete. Define $J : B \to B$ by

$$J\left(p\left(\vartheta\right)\right) = \frac{p\left(3\vartheta\right)}{3} \forall\, x \in \mathbb{S}$$

Now, let $p, q \in B,\ d\left(p, q\right) = C$

$$\begin{aligned}
\left\|p\left(\vartheta\right) - q\left(\vartheta\right)\right\| &\leq \frac{C}{|6|}\Lambda\left(\vartheta, \vartheta, e\right) \\
\left\|Jp\left(\vartheta\right) - Jq\left(\vartheta\right)\right\| &= \left\|\frac{p\left(3\vartheta\right)}{3} - \frac{q\left(3\vartheta\right)}{3}\right\| \\
&= \frac{1}{|3|}\left\|p\left(3\vartheta\right) - q\left(3\vartheta\right)\right\| \\
&\leq \frac{1}{|3|}\frac{C}{|6|}\Lambda\left(3\vartheta, 3\vartheta, e\right) \\
&\leq \frac{1}{|3|}\frac{C}{|6|}\left|3\right|\ L\Lambda\left(3\vartheta, 3\vartheta, e\right) \\
&= \frac{C\ L\ \Lambda\left(\vartheta, \vartheta, e\right)}{|6|} \\
&\therefore d\ \left(Jp, Jq\right) \leq \mathcal{L}\ C \\
&\Longrightarrow d\ \left(Jp, Jq\right) \leq \mathcal{L}\ d\ \left(p, q\right)
\end{aligned}$$

i.e. J is a strictly contractive mapping on X with Lipschitz constant $\mathcal{L}$. Since

$$\left\| \aleph_e(\vartheta) - \frac{\aleph_e(3\vartheta)}{3} \right\| \leq \frac{1}{|6|} \Lambda(\vartheta, \vartheta, e)$$

$$\Longrightarrow \|\aleph_e(\vartheta) - J(\aleph_e(\vartheta))\| \leq \tfrac{1}{|6|}\Lambda(\vartheta, \vartheta, e)$$

$$\Longrightarrow d(\aleph_e, J\aleph_e) \leq 1$$

h is a F-B of J, That is $J(h(\vartheta)) = h(\vartheta)$ i.e. $h(3\vartheta) = 3h(\vartheta)\ \forall\ \vartheta \in \mathbb{S}$, h is unique F-B of J in the set $\{g \in B : d(\aleph, g) < \infty\}$, therefore, there exists $k \in (e, \infty)\ s.t. d(\aleph_e, h) < k$.
Hence

$$\|\aleph_e(\vartheta) - h(\vartheta)\| \leq \frac{k}{|6|}\ \Lambda\ (\vartheta, \vartheta, e).$$

Also, $d(J^\iota \aleph_e, h) \to e$ as $n \to \infty$ i.e.

$$\lim_{n\to\infty} J^\iota\ \aleph_e(\vartheta) = h(\vartheta).$$

$$\lim_{\iota\to\infty} \frac{\aleph_e(3^\iota\vartheta)}{3^\iota} = h(\vartheta)\,\forall\ \vartheta \in \mathbb{S}$$

$$d(\aleph_e.h) \leq \frac{1}{1-\mathcal{L}}\, d(\aleph, J\aleph)$$

$$\|\aleph_e(\vartheta) - h(\vartheta)\| \leq \frac{1}{|1-\mathcal{L}|\,|6|}\ \Lambda\ (\vartheta, \vartheta, e)$$

As in the Theorem 3.1, it can be proven that h satisfies the F-E [1.1], also confirms homomorphism property.

We note that if we put the derivation condition instead of the homomorphism condition in the previous theorem, then the theorem is also correct.

Corollary 4.2 is an instant outcome of Theorem 4.1 □

Corollary 4.2. *Let r be a real number.*
$\aleph_e : \mathbb{S} \to \mathbb{D}, \aleph_e$ *is an odd mapping,* $\aleph_e(0) = 0, \mathcal{L} < 1$. *And*

$$\begin{array}{rcl} \|D_{\aleph_e}(\vartheta, \iota, \zeta)\| & \leq & \\ \|\aleph_e(\vartheta\iota) - \aleph_e(\vartheta)\,\aleph_e(\iota)\| & \leq & \end{cases} \begin{cases} \frac{\|\vartheta\|\cdot\|\iota\|}{\|\vartheta\|^r+\|\iota\|^r+\|\zeta\|^r}, & r < 1, \mathcal{L} \geq |3|^{1-r} \\ \frac{1}{\|\vartheta\|^r+\|\iota\|^r+\|\zeta\|^r}, & r \leq -2\,, \mathcal{L} \geq \frac{1}{|3|^{r+1}} \\ \frac{\|\vartheta\|\ \|\iota\|}{\|\vartheta\|^r}, & r < 1\ ,\ L \geq |3|^{1-\mathcal{L}} \\ \|\vartheta\|^r \cdot \|\iota\|^r, & r \geq 1, \mathcal{L} \geq |3|^{2r-1} \\ \|\vartheta\|^r + \|\iota\|^r + \|\zeta\|^r, & r > 1, \mathcal{L} \geq |3|^{r-1} \end{cases}$$

$\forall \vartheta, \iota, \zeta \in \mathbb{S}$, then there exists a unique additive mapping $h : \mathbb{S} \to \mathbb{D}$, h confirms the homomorphism property

$$\|\aleph_e(\vartheta) - h(\vartheta)\| \leq \frac{1}{|6|} \lim_{m\to\infty} \max_{0\leq m<\iota} \begin{cases} \frac{\|\vartheta\|^2}{2\|\vartheta\|^r} \\ \frac{1}{2\|\vartheta\|^r} \\ \|\vartheta\|^{2-r} \\ \|\vartheta\|^{2r} \\ 2\|\vartheta\|^r \end{cases}$$

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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