Asia Pac. J. Math. 2026 **13**:71

SIUP-HOMOMORPHISMS AND THEIR CONGRUENCES IN SHEFFER STROKE IUP-ALGEBRAS

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Received Feb. 3, 2026

Abstract. In this paper, we advance the study of Sheffer stroke IUP-algebras by establishing a structural framework for SIUP-homomorphisms and their associated congruences. We define the kernel of an SIUP-homomorphism and explore its correspondence with several specialized subsets of SIUP-algebras. Moreover, we introduce and characterize the notions of congruence, the weak SIUP-property, and idempotency, providing new insights into the interconnections among these fundamental concepts. The results presented here contribute to a deeper understanding of the algebraic structure of SIUP-algebras and lay the groundwork for further extensions to generalized or fuzzy settings.

2020 Mathematics Subject Classification. 03G25; 08A05; 03B50.

Key words and phrases. Sheffer stroke IUP-algebra; SIUP-homomorphism; congruence; weak SIUP-property; idempotency.

1. Introduction

Algebraic structures are among the most fundamental pillars of mathematics, providing a unified framework for describing and analyzing abstract patterns of operations. Within this broad landscape, abstract algebra encompasses a diverse family of systems—such as groups, rings, and lattices—that underpin both theoretical mathematics and numerous applied disciplines. Among these, logical algebras constitute a particularly intriguing class of structures arising from the synthesis of logic and algebra. Rooted in the study of implication and inference, logical algebras capture the algebraic essence of reasoning processes and have therefore received sustained attention from many mathematicians over the past decades. Representative examples include BE-algebras [12], pseudo-BE-algebras [3], eBE-algebras [27], pseudo-eBE-algebras [30], pseudo-CI-algebras [28], eGE-algebras [2], PSRU-algebras [41], KU-algebras [26], UP-algebras [7], UP-bialgebras [14], and several subsequent extensions such as KU/UP-algebras [29] and IUP-algebras [8]. These developments collectively demonstrate the richness

DOI: 10.28924/APJM/13-71

and versatility of logical algebraic frameworks, which continue to inspire new generalizations and structural analyses in modern algebra.

One of the logical algebraic structures considered in this work is the IUP-algebra. This algebraic structure was first introduced and investigated by Iampan et al. [8], who provided the original definition of an IUP-algebra. In their pioneering work, four distinguished subsets were defined, namely strong IUP-ideals, IUP-subalgebras, IUP-filters, and IUP-ideals. Moreover, the relationships among these subsets were examined, and several additional properties of IUP-algebras were established. Subsequently, IUP-algebras have attracted considerable attention from researchers, and ongoing investigations have continued to broaden the theory in several directions. For example, in 2023, Chanmanee et al. [6] advanced the study of IUP-algebras by examining direct products of infinite collections of IUP-algebras. Their work introduced the concept of weak direct products and established several fundamental results concerning (anti-)IUP-homomorphisms. In a complementary line of research, Chanmanee et al. [5] conducted a parallel investigation into external direct products of dual IUP-algebras, thereby enriching the structural methodology for combining IUP-algebraic systems. Further progress was made in 2024, when Kuntama et al. [13] incorporated principles of fuzzy set theory into the IUP-algebraic framework. This integration led to the formulation of fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals, providing a mathematical approach for representing uncertainty and imprecision within algebraic structures. Extending this direction, Suayngam et al. [37] employed Fermatean fuzzy sets in the context of IUP-algebras, with particular emphasis on characteristic functions and the analysis of t-level and t-strong level subsets. The theoretical expansion continued through the incorporation of additional uncertainty models. Specifically, intuitionistic fuzzy sets were applied to IUP-algebras by Suayngam et al. [39], followed by the introduction of neutrosophic sets in [35]. In these studies, necessary and sufficient conditions for the existence and characterization of the corresponding IUP-subsets were systematically derived. Subsequently, Suayngam et al. [38] proposed the use of Pythagorean fuzzy sets to define analogous IUP-substructures, highlighting both their structural properties and level-wise representations. More recently, Suayngam et al. [34] presented a hybrid extension by integrating Pythagorean neutrosophic sets with IUP-algebras. This fusion introduced Pythagorean neutrosophic IUP-subalgebras, ideals, filters, and strong ideals, and explored their fundamental algebraic properties. In the same period, Surat et al. [40] investigated hesitant fuzzy subalgebraic systems in IUP-algebras, thereby extending the fuzzy-theoretic treatment of IUP-structures to settings in which membership information may be expressed through sets of possible values rather than a single degree. Along a related line, Khonyong et al. [11] examined fuzzy translations in IUP-algebras through the notions of extensions and contractions, further clarifying the behavior of fuzzy IUP-substructures under transformation processes. Another significant development was presented by Phaeyai et al. [25], who studied the structural properties of (l, r)- and (r, l)-derivations in IUP-algebras, thereby broadening the operational and algebraic perspectives of the theory. In 2025, Inthachot et al. [9] further enriched the theory by embedding bipolar fuzzy sets into the IUP-algebraic setting. This approach enables the simultaneous representation of positive and negative degrees of membership, making it particularly relevant to bipolar decision-making models and logical programming frameworks. Finally, Suayngam et al. [33] unified several strands of previous research by introducing intuitionistic neutrosophic IUP-algebras. This comprehensive framework amalgamates multiple forms of uncertainty within a single algebraic structure and provides deeper insights into the interrelationships among various classes of IUP-subsets. Most recently, Suayngam et al. [36] introduced four new concepts of IUP-ideals, namely transmitted, reflected, resonant, and dominant IUP-ideals. This contribution further diversified the ideal-theoretic landscape of IUP-algebras and opened a new avenue for investigating the internal behavior and classification of ideal-related structures. Taken together, these studies illustrate a coherent and progressive evolution of IUP-algebraic research—from its foundational theory to a versatile and increasingly unified framework that integrates fuzzy logic, uncertainty modeling, derivational structures, and refined ideal-theoretic concepts—underscoring its growing significance in both theoretical investigations and potential applications.

From a broader perspective, the Sheffer stroke operation, first introduced by Sheffer [31], stands as one of the most elegant and powerful logical connectives. It possesses the unique ability to express all Boolean functions and axioms through a single binary operation, thereby serving as a complete functional basis for classical logic. This remarkable universality has long fascinated mathematicians and logicians alike, as it allows significant axiomatic reduction and structural simplification within logical and algebraic systems. Over the decades, the Sheffer stroke has inspired numerous generalizations across diverse algebraic frameworks—ranging from ortholattices [4] and orthoimplication algebras [1] to basic algebras [24], MV-algebras [23], Hilbert algebras [21], and BL-algebras [22]. Further extensions have emerged in the study of BG-algebras [17], UP-algebras [15], BE-algebras [10], and Sheffer-based variants of BCK-, BCH-, BM-, and INK-algebras [16,18–20]. Together, these developments highlight the Sheffer stroke as a unifying operation that bridges logic and algebra, revealing its enduring importance in the general theory of non-classical and implicational algebras.

Congruences play a fundamental role in the classification and structural analysis of algebraic systems. They provide a means of identifying when two elements behave equivalently under the algebra's operations, thereby allowing the construction of quotient structures that preserve essential properties. In the context of IUP-algebras and their generalizations, several notions of ideals and subalgebras have been investigated; however, the corresponding congruence relations and their interactions with homomorphisms remain unexplored. This gap motivates the present study, which aims to introduce and characterize congruences in the setting of Sheffer stroke IUP-algebras (SIUP-algebras). By establishing a homomorphism–congruence correspondence, this work contributes to the foundational understanding of structure-preserving mappings and sets the stage for future studies on isomorphisms and decomposition results within the SIUP framework.

In this paper, we advance the theory of Sheffer stroke IUP-algebras (abbreviated as SIUP-algebras) by establishing the foundation of SIUP-homomorphisms and their structural properties. We define the kernel of an SIUP-homomorphism and explore its deep connection with congruences, the weak SIUP-property, and idempotency. Through this approach, a homomorphism–congruence correspondence is formulated, enriching the algebraic understanding of structure-preserving mappings within the SIUP framework. The Preliminaries section provides the formal definitions of the Sheffer stroke operation and SIUP-algebras, together with their fundamental properties and four characteristic subsets. The Main Results section presents the new theoretical findings concerning kernels, congruences, and weak properties, highlighting their logical interrelations. Finally, the Conclusion summarizes the principal results and outlines potential avenues for further study, particularly on SIUP-isomorphisms and algebraic decompositions.

2. Preliminaries

Before proceeding to the Main Results, we first present the definition of the Sheffer stroke operation and the formal definition of SIUP-algebras. In addition, several distinctive properties of SIUP-algebras are discussed. We also introduce the four special subsets associated with SIUP-algebras and recall the previously established relationships among them. These preliminary notions and results provide the necessary theoretical foundation for the developments presented in the subsequent sections.

Definition 2.1. [31] *Let $X = (X, |)$ be a groupoid. The operation $|$ is said to be a Sheffer stroke if it satisfies the following conditions:*

(S1) $(\forall x, y \in X)(x \,|\, y = y \,|\, x)$

(S2) $(\forall x, y \in X)((x \,|\, x) \,|\, (x \,|\, y) = x)$

(S3) $(\forall x, y, z \in X)(x \,|\, ((y \,|\, z) \,|\, (y \,|\, z)) = ((x \,|\, y) \,|\, (x \,|\, y)) \,|\, z)$

(S4) $(\forall x, y, \in X)((x \,|\, ((x \,|\, x) \,|\, (y \,|\, y))) \,|\, (x \,|\, ((x \,|\, x) \,|\, (y \,|\, y))) = x)$

Definition 2.2. [32] *A Sheffer stroke IUP-algebra (briefly, SIUP-algebra) is an algebra $X = (X, |, \uparrow, 0)$ of type $(2, 2, 0)$, where X is a nonempty set, $|$ and $\uparrow$ are binary operations on X, and 0 is a constant in X. The binary operation $\uparrow$ is defined by*

$$x \uparrow y = x \,|\, (y \,|\, y)$$

for all $x, y \in X$. The following axioms must also be satisfied:

(SIUP-1) $(\forall x \in X)(x \uparrow x = 0 \,|\, 0)$

$(\forall x, y, z \in X)((((x \uparrow y) \,|\, (y \uparrow x)) \uparrow ((x \uparrow z) \,|\, (z \uparrow x))) \,|\, (((x \uparrow z) \,|\, (z \uparrow x)) \uparrow ((x \uparrow y) \,|\, (y \uparrow x)))$

(SIUP-2) $= (y \uparrow z) \mid (z \uparrow y))$

Lemma 2.1. [32] *Let $X = (X, \mid, \uparrow, 0)$ be an SIUP-algebra. Then the following features hold for all $x, y, z \in X$:*

(1) $x \mid (x \mid (x \mid x)) = x \mid x$, *that is,* $x \mid (x \uparrow x) = x \mid x$

(2) $(x \mid (0 \mid 0)) \mid (x \mid (0 \mid 0)) = x$, *that is,* $(x \uparrow 0) \mid (x \uparrow 0) = x$

(3) $0 \mid x = 0 \mid 0$

(4) $(x \mid (x \mid x)) \mid (x \mid x) = x$, *that is,* $(x \uparrow x) \uparrow x = x$

(5) $(0 \mid 0) \mid (x \mid x) = x$, *that is,* $(0 \mid 0) \uparrow x = x$

(6) $x \mid (((x \mid (y \mid y)) \mid (y \mid y)) \mid ((x \mid (y \mid y)) \mid (y \mid y))) = 0 \mid 0$, *that is,* $x \uparrow ((x \uparrow y) \uparrow y) = 0 \mid 0$

(7) $(x \mid (0 \mid 0))) \mid (0 \mid (x \mid x)) = x$, *that is,* $(x \uparrow 0) \mid (0 \uparrow x) = x$.

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with two binary operations $\mid$ and $\uparrow$ defined by the following Cayley tables:*

$\mid$	0	1	2	3	4	5	6	7
0	2	2	2	2	2	2	2	2
1	2	3	3	2	2	3	3	2
2	2	3	0	1	5	4	7	6
3	2	2	1	1	5	6	5	6
4	2	2	5	5	5	2	5	2
5	2	3	4	6	2	4	3	6
6	2	3	7	5	5	3	7	2
7	2	2	6	6	2	6	2	6

$\uparrow$	0	1	2	3	4	5	6	7
0	2	2	2	2	2	2	2	2
1	3	2	2	3	3	2	2	3
2	0	1	2	3	4	5	6	7
3	1	1	2	2	6	5	6	5
4	5	5	2	2	2	5	2	5
5	4	6	2	3	4	2	6	3
6	7	5	2	3	3	5	2	7
7	6	6	2	2	6	2	6	2

Then $X = (X, \mid, \uparrow, 0)$ is an SIUP-algebra.

The structure of SIUP-algebras, similar to that of IUP-algebras, is characterized by four distinguished subsets, namely transformed Sheffer stroke IUP-subalgebras, transformed Sheffer stroke IUP-filters, derived Sheffer stroke IUP-subalgebras, and transformed Sheffer stroke strong IUP-ideals. These specialized subsets play a crucial role in enhancing our understanding of the internal structure of SIUP-algebras and in enabling their effective application across various mathematical contexts.

Definition 2.3. [32] *A nonempty subset S of an SIUP-algebra $X = (X, \mid, \uparrow, 0)$ is called*

(i) *a transformed Sheffer stroke IUP-subalgebra (briefly, tSIUP-subalgebra) of X if it satisfies the following condition:*

(2.1) $$(\forall x, y \in S)(x \mid y \in S)$$

(ii) *a transformed Sheffer stroke IUP-filter (briefly, tSIUP-filter) of X if it satisfies the following conditions:*

(2.2) *the constant 0 of X is in S*

(2.3) $$(\forall x, y \in X)(x \,|\, y \in S, x \in S \Rightarrow y \in S)$$

(iii) *a derived Sheffer stroke IUP-subalgebra (briefly, dSIUP-subalgebra) of X if it satisfies the following condition:*

(2.4) $$(\forall x, y \in S)((x \uparrow y) \,|\, (y \uparrow x) \in S)$$

(iv) *a transformed Sheffer stroke strong IUP-ideal (briefly, tSsIUP-ideal) of X if it satisfies the following condition:*

(2.5) $$(\forall x, y \in X)(y \in S \Rightarrow x \,|\, y \in S)$$

The following diagram illustrates the structural relationships among several distinguished subsets—namely, the dSIUP-subalgebra, tSIUP-subalgebra, tSsIUP-ideal, and tSIUP-filter—within an SIUP-algebra X.

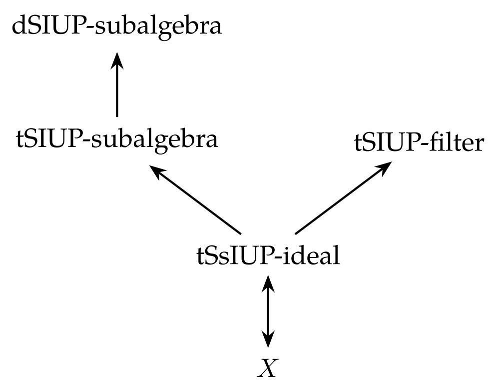

FIGURE 1. Simplified structure of the four concepts of subsets in SIUP-algebras

3. Structural Properties of SIUP-Homomorphisms and Congruences

For this section, we shall present the results obtained from the study of SIUP-algebras, with particular emphasis on the concept of SIUP-homomorphism. Specifically, we will define the notions of SIUP-homomorphism, the weak SIUP-property, the kernel of a function, congruence, and idempotency. Furthermore, we will investigate the relationships between the four special subsets of SIUP-algebras and the aforementioned concepts, and derive additional significant properties, which will be discussed in the subsequent sections.

Definition 3.1. *Let $(X, |_X, \uparrow_X, 0_X)$ and $(Y, |_Y, \uparrow_Y, 0_Y)$ be SIUP-algebras. A mapping $f : X \to Y$ is called an SIUP-homomorphism if*

$$(\forall x, y \in X)(f(x \,|_X\, y) = f(x) \,|_Y\, f(y)).$$

Definition 3.2. *An SIUP-algebra X is said to have the weak SIUP-property if*

$$(\forall x, y \in X)((x \uparrow y) \,|\, (y \uparrow x) = 0 \Rightarrow x = y).$$

Definition 3.3. *Let* $f : X \to Y$ *be an SIUP-homomorphism. The set* $\ker_f = \{x \in X : f(x) = 0_Y\}$ *is called the kernel of* f*.*

Theorem 3.1. *Let* $f : X \to Y$ *be an SIUP-homomorphism. Then* $\ker_f$ *is a dSIUP-subalgebra of* X*.*

Proof. Assume that $f : X \to Y$ be an SIUP-homomorphism. Let $x, y \in \ker_f$, that is, $f(x) = 0_Y$ and $f(y) = 0_Y$. Then

$$\begin{aligned}
f((x \uparrow_X y) \,|_X\, (y \uparrow_X x)) &= f(x \uparrow_X y) \,|_Y\, f(y \uparrow_X x) \\
&= f(x \,|_X\, (y \,|_X\, y)) \,|_Y\, f(y \,|_X\, (x \,|_X\, x)) \\
&= (f(x) \,|_Y\, (f(y) \,|_Y\, f(y))) \,|_Y\, (f(y) \,|_Y\, (f(x) \,|_Y\, f(x))) \\
&= (0_Y \,|_Y\, (0_Y \,|_Y\, 0_Y)) \,|_Y\, (0_Y \,|_Y\, (0_Y \,|_Y\, 0_Y)) \\
\text{(by (SIUP-1))} \quad &= (0_Y \,|_Y\, 0_Y) \,|_Y\, (0_Y \,|_Y\, 0_Y) \\
\text{(by (S2))} \quad &= 0_Y.
\end{aligned}$$

Thus $(x \uparrow_X y) \,|_X\, (y \uparrow_X x) \in \ker_f$. Hence, $\ker_f$ is a dSIUP-algebra of X. □

Proposition 3.1. *Let* $f : X \to Y$ *be an SIUP-homomorphism. Then for all* $x, y \in X$*,*

(1) $f(0_X) = 0_Y$,
(2) $x \,|_X\, y = 0_X \Rightarrow f(x) \,|_Y\, f(y) = 0_Y$.

Proof. Assume that $f : X \to Y$ be an SIUP-homomorphism.
(1)

$$\begin{aligned}
f(0_X) &= f((0_X \,|_X\, (0_X \,|_X\, 0_X)) \,|_X\, (0_X \,|_X\, (0_X \,|_X\, 0_X))) \\
&= (f(0_X) \,|_Y\, (f(0_X) \,|_Y\, f(0_X))) \,|_Y\, (f(0_X) \,|_Y\, (f(0_X) \,|_Y\, f(0_X))) \\
\text{(by (SIUP-1))} \quad &= (0_Y \,|_Y\, 0_Y) \,|_Y\, (0_Y \,|_Y\, 0_Y) \\
\text{(by (S2))} \quad &= 0_Y.
\end{aligned}$$

(2) Let $x, y \in X$ be such that $x \,|_X\, y = 0_X$. Then

$$\begin{aligned}
f(x) \,|_Y\, f(y) &= f(x \,|_X\, y) \\
&= f(0_X) \\
\text{(by (1))} \quad &= 0_Y.
\end{aligned}$$

Hence, $f(x) \,|_Y\, f(y) = 0_Y$ for all $x, y \in X$. □

Theorem 3.2. *Let $f : X \to Y$ be an SIUP-homomorphism. The relation $\sim$ on X defined by*

$$(\forall x, y \in X)(x \sim y \Leftrightarrow f(x) = f(y))$$

is an equivalence relation.

Proof. Assume that $f : X \to Y$ is an SIUP-homomorphism. Let $x, y, z \in X$.

(reflexive): Since $f(x) = f(x)$, we have $x \sim x$.

(symmetric): Let $x \sim y$. Then $f(x) = f(y)$, that is, $f(y) = f(x)$. Thus, $y \sim x$.

(transitive): Let $x \sim y$ and $y \sim z$. Then $f(x) = f(y)$ and $f(y) = f(z)$, that is, $f(x) = f(y) = f(z)$. Thus, $x \sim z$.

Hence, $\sim$ is an equivalence relation on X. □

Definition 3.4. *An equivalence relation $\sim$ on an SIUP-algebra X is called a congruence if*

$$(\forall x, y, u, v \in X)(x \sim y \text{ and } u \sim v \Rightarrow x \,|\, u \sim y \,|\, v).$$

Lemma 3.1. *Let $f : X \to Y$ be an SIUP-homomorphism. The equivalence relation $\sim$ in Theorem 3.2 is a congruence.*

Proof. Let $x, y, u, v \in X$ be such that $x \sim y$ and $u \sim v$. Then $f(x) = f(y)$ and $f(u) = f(v)$. Since f is an SIUP-homomorphism, we have

$$f(x \,|_X\, u) = f(x) \,|_Y\, f(u) = f(y) \,|_Y\, f(v) = f(y \,|_X\, v).$$

Hence, $x \mid u \sim y \mid v$. Therefore, $\sim$ is a congruence relation on the SIUP-algebra X. □

Let $[x]$ be the equivalence class of $x \in X$ and let $X/\sim$ denote the collection of equivalence classes in the equivalence relation $\sim$. Define two binary operations $\circ$ and $\diamond$ on X by

$$[x] \circ [y] = [x \,|\, y]$$

and

$$[x] \diamond [y] = [x \uparrow y]$$

for all $[x], [y] \in X/\sim$.

Theorem 3.3. *Let $f : X \to Y$ be an SIUP-homomorphism. Then $(X/\sim, \circ, \diamond, [0])$ is an SIUP-algebra.*

Proof. Let x, y, z be any elements in X.

(S1):

$$[x] \circ [y] = [x \,|\, y]$$

(by (S1)) $$= [y \,|\, x]$$

$$= [y] \circ [x].$$

(S2):

$$([x] \circ [x]) \circ ([x] \circ [y]) = [(x \mid x) \mid (x \mid y)]$$

(by (S2)) $= [x].$

(S3):

$$[x] \circ (([y] \circ [z]) \circ ([y] \circ [z])) = [x \mid ((y \mid z) \mid (y \mid z))]$$

(by (S3)) $= [((x \mid y) \mid (x \mid y)) \mid z]$

$$= (([x] \circ [y]) \circ ([x] \circ [y])) \circ [z].$$

(S4):

$$([x] \circ (([x] \circ [x]) \circ ([y] \circ [y]))) \circ ([x] \circ (([x] \circ [x]) \circ ([y] \circ [y])))$$

$$= [(x \mid ((x \mid x) \mid (y \mid y))) \mid (x \mid ((x \mid x) \mid (y \mid y)))]$$

(by (S4)) $= [x].$

(SIUP-1):

$$[x] \diamond [x] = [x \uparrow x]$$

(by (SIUP-1)) $= [0 \mid 0]$

$$= [0] \circ [0].$$

(SIUP-2):

$$(((([x] \diamond [y]) \circ ([y] \diamond [x])) \diamond (([x] \diamond [z]) \circ ([z] \diamond [x])))\circ$$

$$((([x] \diamond [z]) \circ ([z] \diamond [x])) \diamond (([x] \diamond [y]) \circ ([y] \diamond [x])))$$

$$= [(((x \uparrow y) \mid (y \uparrow x)) \uparrow ((x \uparrow z) \mid (z \uparrow x))) \mid$$

$$(((x \uparrow z) \mid (z \uparrow x)) \uparrow ((x \uparrow y) \mid (y \uparrow x)))]$$

(by (SIUP-2)) $= [(y \uparrow z) \mid (z \uparrow y)]$

$$= ([y] \diamond [z]) \circ ([z] \diamond [y]).$$

Hence, $(X/\sim, \circ, \diamond, [0])$ is an SIUP-algebra. □

Theorem 3.4. *Let $f : X \to X$ be an SIUP-homomorphism, where X has the weak SIUP-property. Then f is one-to-one if and only if* $\ker_f = \{0\}$.

Proof. Assume that f is one-to-one and let $x \in \ker_f$. Then $f(x) = 0 = f(0)$. Thus $x = 0$. Hence, $\ker_f = \{0\}$.

Conversely, assume that $\ker_f = \{0\}$. Let $u, v \in X$ be such that $f(u) = f(v)$. Then

$$\begin{aligned} f((u \uparrow v) \mid (v \uparrow u)) &= (f(u) \uparrow f(v)) \mid (f(v) \uparrow f(u)) \\ &= (f(u) \uparrow f(u)) \mid (f(u) \uparrow f(u)) \\ \text{(by (SIUP-1))} \quad &= (0 \mid 0) \mid (0 \mid 0) \\ \text{(by (S2))} \quad &= 0. \end{aligned}$$

Thus, $(u \uparrow v) \mid (v \uparrow u) \in \ker_f = \{0\}$. Hence, $(u \uparrow v) \mid (v \uparrow u) = 0$, so $u = v$. Therefore, f is one-to-one. □

Definition 3.5. *An SIUP-homomorphism $f : X \to X$ is called idempotent if $f \circ f = f$.*

Theorem 3.5. *Let $f : X \to X$ be an idempotent SIUP-homomorphism with weak SIUP-property. Then f is one-to-one if and only if f is the identity map.*

Proof. Assume that f is one-to-one. Let $x \in X$. Then

$$\begin{aligned} f((x \uparrow f(x)) \mid (f(x) \uparrow x)) &= (f(x) \uparrow f(f(x))) \mid (f(f(x)) \uparrow f(x)) \\ &= (f(x) \uparrow f(x)) \mid (f(x) \uparrow f(x)) \\ \text{(by (SIUP-1))} \quad &= (0 \mid 0) \mid (0 \mid 0) \\ \text{(by (S2))} \quad &= 0. \end{aligned}$$

By Theorem 3.4, we get $f(x) = x$. Hence, f is the identity map.

The converse is obvious. □

4. Conclusion

In this study, we have extended the framework of SIUP-algebras by placing particular emphasis on SIUP-homomorphisms, which constitute a fundamental and unifying concept within the algebraic structure. Specifically, the kernel of a function has been formally defined, and its connections with distinguished subsets of SIUP-algebras have been thoroughly investigated. These relationships provide deeper insight into how homomorphic mappings preserve or reflect intrinsic structural properties.

In addition, this work introduces and examines several core algebraic notions in the context of SIUP-algebras, including equivalence relations, congruences, the weak SIUP-property, and the concept of idempotency of SIUP-homomorphisms. The interdependencies among these concepts are systematically analyzed, revealing their significance in characterizing algebraic stability and structural compatibility within SIUP-algebras. The results obtained in this research establish a solid theoretical foundation for further investigations. In particular, future work may explore SIUP-isomorphisms in greater depth, focusing on structure-preserving bijections and classification results. Such studies are expected to enhance the understanding of structural equivalence among SIUP-algebras and may facilitate the development of decomposition theorems and representation results.

Moreover, continued research along these lines has the potential to address not only abstract mathematical problems in algebraic logic but also practical issues arising in real-world applications. By formalizing structural transformations and equivalence under homomorphisms, SIUP-algebras may provide useful frameworks for reasoning systems, information processing, and uncertainty modeling. Consequently, this study contributes both theoretical advancement and practical relevance to the ongoing development of SIUP-algebraic theory.

Acknowledgments. This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2252/2568).

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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