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FOUR-DIMENSIONAL EXTENSION OF EXTENDED REAL OPERATIONS FOR TRIANGULAR FUZZY NUMBERS

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Abstract. We generalized triangular fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$ and from $\mathbb{R}^2$ to $\mathbb{R}^3$. By defining parametric operations on α-cuts, which are subsets of $\mathbb{R}^3$, we derived parametric operations for triangular fuzzy numbers defined on $\mathbb{R}^3$. Using Zadeh's extension principle, we then defined and computed extended real operations in four-dimensional space. Based on this framework, we introduced a new formulation that extends the two-dimensional case and generalizes it to three- and four-dimensional spaces in a consistent manner. The resulting four-dimensional structures are visualized graphically. To facilitate interpretation, function values are encoded by color intensity, as in the three-dimensional case, while an additional time axis is incorporated to represent the fourth dimension.

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1. Introduction

Among operations on fuzzy sets, one of the most fundamental is the extended real operation based on Zadeh's extension principle ([14, 15]). Among various computational approaches, the theory of performing the four basic arithmetic operations on triangular fuzzy numbers using α-cuts is well established and is presented in most textbooks on introductory fuzzy theory. In this paper, we focus exclusively on triangular fuzzy numbers. For the four arithmetic operations, addition and subtraction are defined uniformly and can be computed straightforwardly, whereas multiplication and division require separate definitions for positive and negative cases, resulting in more complicated calculations. While addition and subtraction preserve the form of fuzzy numbers, multiplication and division generally do not.

We extend this framework to a two-dimensional space. When the two-dimensional extension is restricted to one dimension, it must coincide with the corresponding one-dimensional result. Since

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the one-dimensional case has a triangular form, its two-dimensional extension is represented by a conical structure in three-dimensional space. The intersection of this cone with a vertical plane passing through its vertex yields a one-dimensional triangular fuzzy number.

In the one-dimensional setting, a direct extension to two dimensions is not possible because algebraic operations are performed independently on the endpoints of each α-cut. To address this limitation, a new parametric operation was introduced by reinterpreting the one-dimensional operation ([2]). Under this parametric formulation, multiplication and division can be defined uniformly without distinguishing between positive and negative cases. The resulting computations are consistent with those obtained using the classical extended real operation, and improved results for multiplication and division were reported in ([2]).

Based on this approach, a two-dimensional extended operation preserving the results of the one-dimensional extended real operation was defined and computed ([6]). It was further shown that restricting a two-dimensional triangular fuzzy number to one dimension—obtained as the triangular fuzzy exponent on a plane generated by a vertical cut through the vertex—is consistent with the established one-dimensional results ([5]).

A three-dimensional operation that preserves the two-dimensional results was defined and computed ([12]). In general, a three-dimensional triangular fuzzy number cannot be directly visualized, since its graph would require a four-dimensional space. To address this issue, the three-dimensional domain is represented geometrically, and the value of the membership function at each point in the domain is encoded by color intensity. The three-dimensional domain is modeled as an ellipsoid. When this ellipsoid is intersected by a plane passing through its longest axis, the resulting two-dimensional region—an ellipse including its interior—is obtained on the plane, and the corresponding membership function values are again represented using color intensity. If the color intensity is further interpreted as a function value in three-dimensional space, a two-dimensional cone-shaped graph is produced.

All aspects requiring revision in the two- and three-dimensional studies have been addressed and refined in this paper. An examination of the revised components shows that the associated formulas vary consistently with the dimension, indicating that the extension to four-dimensional space is both natural and mathematically sound. Since a four-dimensional graph cannot be directly visualized in three-dimensional space, it is represented in the same manner as a three-dimensional graph by encoding function values through color intensity, while incorporating an additional time-like axis to represent the fourth dimension. Along this time axis, the graph gradually appears and disappears, and at each time instant a three-dimensional graph analogous to the conventional three-dimensional representation is obtained. The most faithful visualization of the four-dimensional structure is therefore achieved through a video representation, in which the time axis is interpreted as actual time.

In this paper, we defined and computed extended real operations in four-dimensional space using Zadeh's extension principle. We introduced a new formulation for extending the two-dimensional case and generalized it to three- and four-dimensional spaces in a consistent manner. For $n = 2, 3, 4$, we show that restricting the n-dimensional results to $(n-1)$-dimensional space coincides with the corresponding established $(n-1)$-dimensional results. In addition, previously identified inaccuracies in earlier formulations have been corrected. To facilitate the interpretation of four-dimensional structures, function values are visualized using color intensity, as in the three-dimensional case, together with the incorporation of an additional time axis to represent the fourth dimension. The proposed results provide a solid foundation for future studies in finite-dimensional spaces and offer a conceptual basis for extensions to infinite-dimensional settings, with potential applications across a wide range of disciplines.

2. 2-Dimensional Expansion of Extended Real Operation for Triangular Fuzzy Number

Let A be a fuzzy set on $\mathbb{R}$ with the membership function $\mu_A(x)$. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in \mathbb{R} \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 and A^1 are the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ and $\{x \in \mathbb{R} \mid \mu_A(x) = 1\}$, respectively. In the calculations between two fuzzy numbers, the concept of α-cut is very important. Furthermore, some operations between α-cuts are very useful and α-set plays a very important role in a 2-dimensional case.

Definition 2.1. A fuzzy set A is convex if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1),\ \mu_A(x_2)),$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Definition 2.2. A fuzzy number A is a convex fuzzy set on $\mathbb{R}$ such that
(1) there exists unique $x \in \mathbb{R}$ with $\mu_A(x) = 1$,
(2) $\mu_A(x)$ is piecewise continuous.

We call the fuzzy number A is *continuous* if the membership function $\mu_A(x)$ of A is continuous. If A is a continuous fuzzy number, then the α-cut A_α of A is a closed interval in $\mathbb{R}$.

Definition 2.3. A fuzzy number A is called *positive* (*negative*) if its membership function is such that $\mu_A(x) = 0, \forall x < 0$ ($\forall x > 0$).

Zadeh had defined the extension principle([14]). Zimmermann introduced the same basic concepts in [15] as follows:

Definition 2.4. ([15]) Let $X = X_1 \times \cdots \times X_n$ be a cartesian product and μ_i be a fuzzy set in X_i, respectively, and $f : X \to Y$ be a mapping. Then the extension principle allows us to define a fuzzy set

ν in Y by $\nu(y) = \sup\limits_{(x_1,\cdots,x_n)\in f^{-1}(y)} \min\{\mu_1(x_1), \cdots, \mu_n(x_n)\}$ if $f^{-1}(y) \neq \emptyset$ and $\nu(y) = 0$ if $f^{-1}(y) = \emptyset$.

For $n = 1$, the extension principle reduces to a fuzzy set $\nu = f(\mu)$ defined by

$$\nu(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset. \end{cases}$$

Definition 2.5. ([14]) For two fuzzy sets A and B, the extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For $x \in A, y \in B$,

(1) $\mu_{A(+)B}(z) = \sup\limits_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}$,
(2) $\mu_{A(-)B}(z) = \sup\limits_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}$,
(3) $\mu_{A(\cdot)B}(z) = \sup\limits_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}$,
(4) $\mu_{A(/)B}(z) = \sup\limits_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}$,

Remark 2.6. ([14]) Let A and B be fuzzy sets and $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B, A(-)B, A(\cdot)B$ and $A(/)B$ can be calculated as follows.

(1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$
(2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$
(3) $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [\min\{a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\}, \max\{a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\}]$
(4) $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [\min\{a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}\}, \max\{a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}\}]$

If A and B be positive fuzzy sets, then the α-cuts of $A(\cdot)B$ and $A(/)B$ can be calculated as follows.
(3)′ $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}]$
(4)′ $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}]$

When A and B are not positive, the α-cuts of $A(\cdot)B$ and $A(/)B$ defined in a much more complex manner. One of the most famous fuzzy numbers is the triangular fuzzy number, and many research findings on triangular fuzzy numbers have been presented ([1,3,8,9].

Definition 2.7. A triangular fuzzy number on $\mathbb{R}$ is a fuzzy number A, which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_3 \leq x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3, \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3$. It is denoted by $A = (a_1, a_2, a_3)$.

The results obtained using Remark 2.6 for one-dimensional triangular fuzzy numbers are well established. We consider an extension to two dimensions that preserves the results of one-dimensional operations on triangular fuzzy numbers. When the two-dimensional extension is restricted to one dimension, it should coincide with the corresponding one-dimensional results. Since the one-dimensional case has a triangular form, its two-dimensional extension is naturally represented as a conical structure in three-dimensional space. By intersecting this cone with a vertical plane passing through its apex, the resulting cross-section yields a one-dimensional triangular fuzzy number. As noted in Remark 2.6, in the one-dimensional setting algebraic operations on real numbers are performed independently at each endpoint via α-cuts, which makes a direct extension to two dimensions infeasible. To overcome this limitation, the one-dimensional operations were reinterpreted, and new parameterized operations were defined and computed to enable a consistent two-dimensional extension ([2]).

Theorem 2.8. ([2]) *Let A be a fuzzy number defined on $\mathbb{R}$ and $A_\alpha = \{x \in A \mid \mu_A(x) \geq \alpha\}$ be a α-cut of A. Then for all $\alpha \in [0,1]$, there exists a piecewise continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $A_\alpha = \{f_\alpha(t) | t \in [0,1]\}$.*

Corollary 2.9. ([2]) *Let A be a continuous fuzzy number defined on $\mathbb{R}$. Then the α-cut $A_\alpha = \{x \in A \mid \mu_A(x) \geq \alpha\}$ becomes a closed interval $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ on $\mathbb{R}$. And for all $\alpha \in [0,1]$, there exists a continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $[a_1^{(\alpha)}, a_2^{(\alpha)}] = \{f_\alpha(t) | t \in [0,1]\}$.*

In the case of parametric operations, multiplication and division can also be defined at once without separately defining positive and negative values. The results of the newly defined parametric operations are the same as those of the existing extended real operations, and in the cases of multiplication and division, even more improved results were obtained ([2]).

Definition 2.10. Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and $f_A(t), f_B(t)$ be the parametric α-functions of A and B, respectively. The parametric addition $A(+)_pB$, parametric subtraction $A(-)_pB$, parametric multiplication $A(\cdot)_pB$ and parametric division $A(/)_pB$ are fuzzy numbers that have their α-cuts as follows.

(1) $A(+)_pB$: $(A(+)_pB)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0,1]\}$

(2) $A(-)_pB$: $(A(-)_pB)_\alpha = \{f_A(t) - f_B(1-t) \mid t \in [0,1]\}$

(3) $A(\cdot)_pB$: $(A(\cdot)_pB)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0,1]\}$

(4) $A(/)_pB$: $(A(/)_pB)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}$

Theorem 2.11. ([2]) *Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_pB = A(+)B, A(-)_pB = A(-)B, A(\cdot)_pB = A(\cdot)B$ and $A(/)_pB = A(/)B$.*

Corollary 2.12. ([2]) *Let A and B be two triangular fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_pB = A(+)B, A(-)_pB = A(-)B, A(\cdot)_pB = A(\cdot)B$ and $A(/)_pB = A(/)B$.*

Remark 2.13. Zadeh defined the max-min operators for positive and negative fuzzy numbers in different ways. By defining them using the parametric operators as shown above, they can be defined in a single way without distinguishing between positive and negative values, and the result will be the same as the one obtained by calculating for positive fuzzy numbers in Zadeh's method.

Example 2.14. Let A and B be two triangular fuzzy numbers having membership functions

$$\mu_A(x) = \begin{cases} 1 - \frac{|x-16|}{8}, & 8 \leq x < 24, \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_B(x) = \begin{cases} 1 - \frac{|x-12|}{5}, & 7 \leq x < 17, \\ 0, & \text{otherwise}. \end{cases}$$

Then we have the followings.

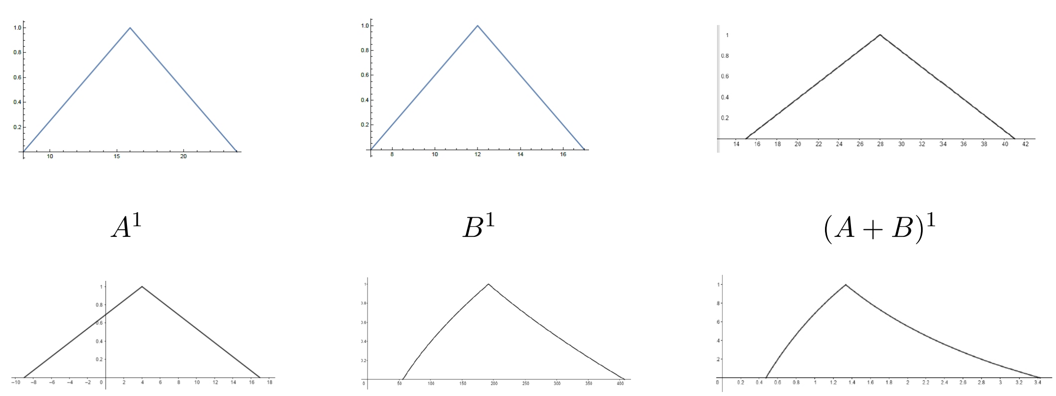
(1) $\mu_{A(+)B}(x) = \begin{cases} 1 - \frac{|x-28|}{13}, & 15 \leq x < 41, \\ 0, & \text{otherwise}, \end{cases}$

(2) $\mu_{A(-)B}(x) = \begin{cases} 1 - \frac{|x-4|}{13}, & -9 \leq x < 17, \\ 0, & \text{otherwise}, \end{cases}$

(3) $\mu_{A(\cdot)B}(x) = \begin{cases} \frac{1}{20}(-24 + \sqrt{2(8+5x)}, & 56 \leq x < 192, \\ \frac{1}{20}(64 - \sqrt{2(8+5x)}, & 192 \leq x < 408, \\ 0, & \text{otherwise}, \end{cases}$

(4) $\mu_{A(/)B}(x) = \begin{cases} \frac{-8+17x}{8+5x}, & \frac{8}{17} \leq x < \frac{16}{12}, \\ \frac{24-7x}{8+5x}, & \frac{16}{12} \leq x < \frac{24}{7}, \\ 0, & \text{otherwise}, \end{cases}$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ need not to be 2-dimensional triangular fuzzy numbers.



A^1 B^1 $(A+B)^1$

$(A-B)^1$ $(AB)^1$ $(A/B)^1$

Figure 1. 1-dimensional space

We defined and computed the 2-dimensional extension operations that preserve the results of 1-dimensional extended real number operations ([6]). When the result of the 2-dimensional triangular fuzzy number is restricted to one dimension (i.e., when it is cut by a vertical plane passing through the apex, generating a 1-dimensional triangular fuzzy number in the plane), it matches the existing 1-dimensional result ([5]).

We define the 2-dimensional triangular fuzzy number on $\mathbb{R}^2$ as a generalization of the triangular fuzzy number on $\mathbb{R}$. Next, we aim to define parametric operations between two 2-dimensional triangular fuzzy numbers. To do this, we need to compute the operations between the α-cuts in $\mathbb{R}^2$. While an α-cut is an interval in $\mathbb{R}$, it is a region in $\mathbb{R}^2$, which means we cannot directly apply the existing methods for computing operations between α-cuts. We reinterpret the existing method from a different perspective and apply this approach to the α-cut of the region in $\mathbb{R}^2$.

Definition 2.15. A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the *2-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1)^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha \ \ (0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \ \ (k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional triangular fuzzy number $A = (a, x_1, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$A_\alpha = \left\{(x,y) \in \mathbb{R}^2 \middle| b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2(1-\alpha)^2\right\}.$$

In Remark 2.6, if $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ is the α-cut of $A = (a_1, a_2, a_3)$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the α-cut of $B = (b_1, b_2, b_3)$, then $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$. However in a 2-dimensional case, $A_\alpha(+)B_\alpha$ can not be calculated by the same way since α-cuts are not intervals but subsets of $\mathbb{R}^2$. For the calculation in a 2-dimensional case, we consider the operations of α-cuts on $\mathbb{R}$ by using a parameter as in Definition 2.10.

Definition 2.16. A 2-dimensional fuzzy number A defined on $\mathbb{R}^2$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) \geq \alpha\}$$

are convex subsets in $\mathbb{R}^2$.

Theorem 2.17. [6] *Let A be a convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist piecewise continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, then the α-set A^α is a closed circular convex subset in $\mathbb{R}^2$.

Corollary 2.18. ([6]) *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

Definition 2.19. Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$
$$B^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication and parametric division of two fuzzy numbers A and B are fuzzy numbers whose α-sets are given as follows:

(1) parametric addition $A(+)_pB :$

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(2) parametric subtraction $A(-)_pB :$

$$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$
$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

(3) parametric multiplication $A(\cdot)_pB :$

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} \dfrac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)}, & \text{if } 0 \le t \le \pi \\ \dfrac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)}, & \text{if } \pi \le t \le 2\pi, \end{cases} \quad y_\alpha(t) = \begin{cases} \dfrac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)}, & \text{if } 0 \le t \le \pi \\ \dfrac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)}, & \text{if } \pi \le t \le 2\pi \end{cases}$$

$$x_\alpha(t) = \begin{cases} \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)}, & \text{if } 0 \le t \le \pi \\ \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)}, & \text{if } \pi \le t \le 2\pi, \end{cases} \quad y_\alpha(t) = \begin{cases} \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)}, & \text{if } 0 \le t \le \pi \\ \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)}, & \text{if } \pi \le t \le 2\pi \end{cases}$$

For $\alpha = 0$, $(A(*)_pB)^0 = \lim\limits_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and if $\alpha = 1$, $(A(*)_pB)^1 = \lim\limits_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +, -, \cdot, /$.

Theorem 2.20. *([6]) Let $A = (a_1,\ x_1,\ b_1,\ y_1)^2$ and $B = (a_2,\ x_2,\ b_2,\ y_2)^2$ be two 2-dimensional triangular fuzzy numbers. Then we have the following.*

(1) $A(+)_pB = \Big(a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big)^2$

(2) $A(-)_pB = \Big(a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big)^2$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos t + a_1a_2(1-\alpha)^2\cos^2 t,$$

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin t + b_1b_2(1-\alpha)^2\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(1-\alpha)\cos t}, \qquad y_\alpha(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(1-\alpha)\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ need not to be 2-dimensional triangular fuzzy numbers.

Example 2.21. Let $A = (6,\ 3,\ 8,\ 5)^2$ and $B = (4,\ 2,\ 5,\ 3)^2$. Then by Theorem 2.19, we have the following.

(1) $A(+)_pB = (10,\ 5,\ 13,\ 8)^2$

(2) $A(-)_pB = (10,\ 1,\ 13,\ 2)^2$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + 24(1-\alpha)\cos t + 24(1-\alpha)^2\cos^2 t,$$

$$y_\alpha(t) = 15 + 49(1-\alpha)\sin t + 40(1-\alpha)^2\sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{3 + 6(1-\alpha)\cos t}{2 - 4(1-\alpha)\cos t}, \qquad y_\alpha(t) = \frac{5 + 8(1-\alpha)\sin t}{3 - 5(1-\alpha)\sin t}.$$

Thus $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ need not to be 2-dimensional triangular fuzzy numbers.

Example 2.22. Let $A = (6,\ 10,\ 8,\ 16)^2$ and $B = (4,\ 8,\ 5,\ 12)^2$. Then by Theorem 2.19, we have the following.

(1) $A(+)_p B = (10,\ 18,\ 13,\ 28)^2$

(2) $A(-)_p B = (10,\ 2,\ 13,\ 4)^2$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 80 + 88(1-\alpha)\cos t + 24(1-\alpha)^2\cos^2 t,$$

$$y_\alpha(t) = 192 + 176(1-\alpha)\sin t + 40(1-\alpha)^2\sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{10 + 6(1-\alpha)\cos t}{8 - 4(1-\alpha)\cos t}, \qquad y_\alpha(t) = \frac{16 + 8(1-\alpha)\sin t}{12 - 5(1-\alpha)\sin t}.$$

Thus $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ need not to be 2-dimensional triangular fuzzy numbers.

Theorem 2.23. ([5]) *Parametric operations defined on $\mathbb{R}^2$ in Definition 2.18 are the generalization of parametric operations defined on $\mathbb{R}$ in Definition 2.10.*

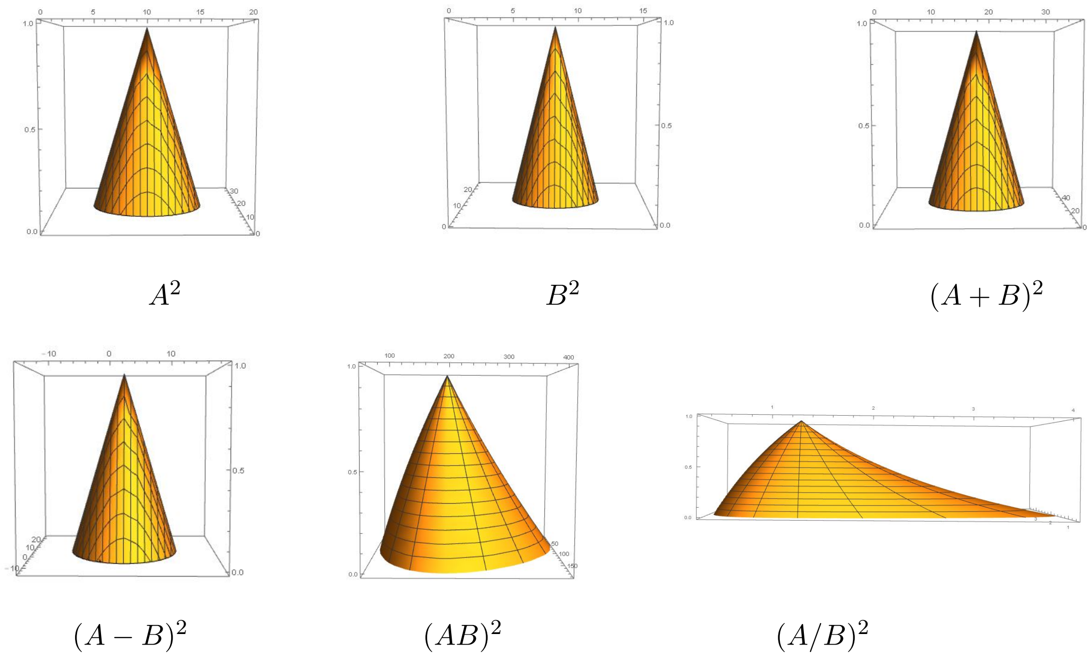


Figure 2. 2-dimensional space

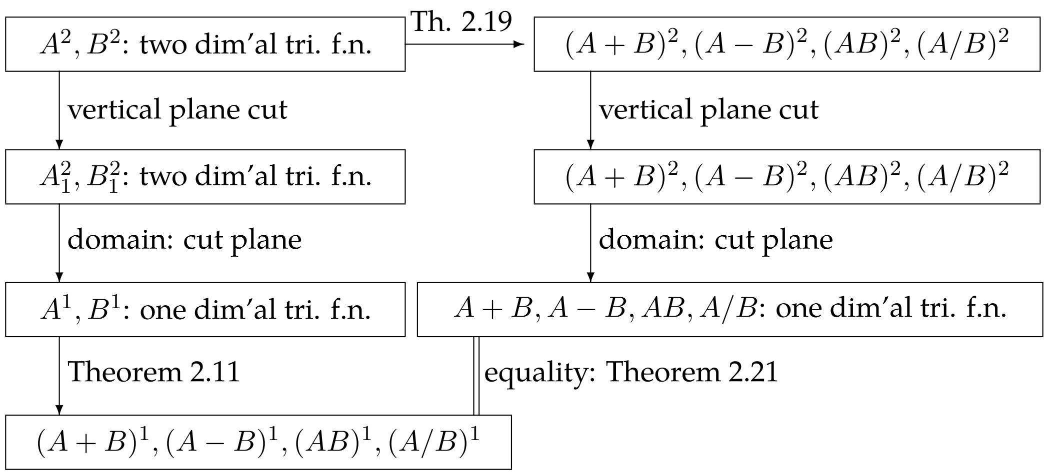
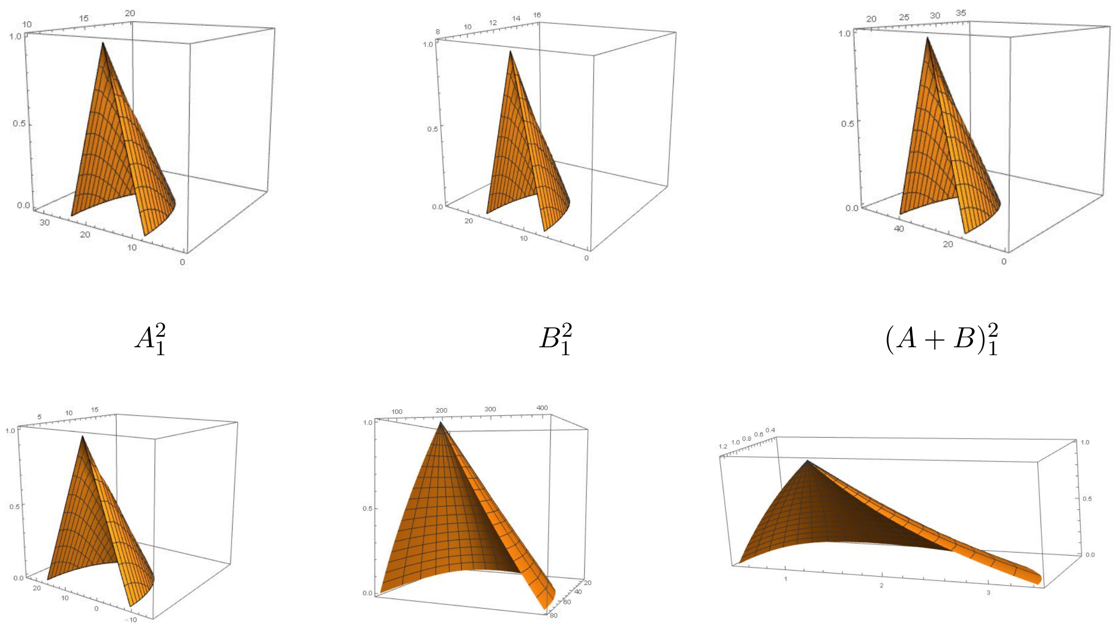


Figure 3. Summary

Remark 2.24. ([5]) The 2-dimensional case is an extension of the 1-dimensi-onal concept. Similar to the 1-dimensional case, the 2-dimensional fuzzy number is defined on the entire xy-plane without distinguishing between positive and negative regions. When a 2-dimensional fuzzy number is cut by a plane parallel to the xz-plane or yz-plane at the apex, a 1-dimensional fuzzy number is generated in the cutting plane. By slicing the 2-dimensional result into a plane and reducing it to one dimension, it matches the 1-dimensional result. To clarify, we cut a 2-dimensional fuzzy number by parallel planes such as the xz-plane or yz-plane. But cutting with any vertical plane at the apex will yield the same 1-dimensional result. The above result is proven in [5], and for clearer explanation, the corresponding graph is presented below.



A_1^2 B_1^2 $(A+B)_1^2$

$(A-B)_1^2$ $(AB)_1^2$ $(A/B)_1^2$

Figure 4. 2-dimensional space(vertical plane cut)

3. 3-Dimensional Expansion of Extended Real Operation for Triangular Fuzzy Number

We define the 3-dimensional triangular fuzzy number on $\mathbb{R}^3$ as a generalization of the triangular fuzzy number on $\mathbb{R}^2$. Next, we aim to define parametric operations between two 3-dimensional triangular fuzzy numbers. To achieve this, we need to compute the operations between α-cuts in $\mathbb{R}^3$. While an α-cut is a region in $\mathbb{R}^2$, in $\mathbb{R}^3$, the α-cut is a subset of $\mathbb{R}^3$, meaning that the existing methods for computing operations between α-cuts cannot be directly applied. We reinterpret the existing method from a different perspective and apply this approach to the α-cut of a subset in $\mathbb{R}^3$.

Definition 3.1. ([12]) A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}}, & \text{if } b^2c^2(x-x_1)^2 \\ \qquad +c^2a^2(y-y_1)^2 + a^2b^2(z-z_1)^2 \le a^2b^2c^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b, c > 0$ is called the *3-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1,\ c,\ z_1)^3$.

Note that $\mu_A(x,y)$ is a cone in $\mathbb{R}^2$, but it is not possible to know the shape of $\mu_A(x,y,z)$ in $\mathbb{R}^3$. The α-cut A_α of a 3-dimensional triangular fuzzy number $A = (a, x_1, b, y_1, c, z_1)^3$ is the following set

$$A_\alpha = \left\{ (x,y,z) \in \mathbb{R}^3 \middle| \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} \le (1-\alpha)^2 \right\}$$

Definition 3.2. ([12]) A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called a *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) \geq \alpha\}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 3.3. ([12]) *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s)$, $f_2^\alpha(s,t)$, and $f_3^\alpha(s,t)(0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2})$, such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}.$$

Definition 3.4. ([12]) Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^3$, and

$$\begin{aligned} A^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\} \\ &= \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}, \\ B^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_B(x,y,z) = \alpha\} \\ &= \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B are fuzzy numbers whose α-sets are given as follows:

(1) parametric addition $A(+)_pB$:

$$\begin{aligned} (A(+)_pB)^\alpha = \{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(2) parametric subtraction $A(-)_pB$:

$$\begin{aligned} (A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\}, \\ (A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$\begin{aligned} (A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ f_2^\alpha(s,t) \cdot g_2^\alpha(s,t),\ f_3^\alpha(s,t) \cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\} \end{aligned}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | \quad 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \quad \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 3.5. ([12]) *Let* $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1)^3$ *and* $B = (a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2)^3$ *be two 3-dimensional triangular fuzzy numbers. Then, we have the following:*

(1) $A(+)_pB = \Big(a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2,\ c_1 + c_2,\ z_1 + z_2\Big)^3.$

(2) $A(-)_pB = \Big(a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2,\ c_1 + c_2,\ z_1 - z_2\Big)^3.$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2 s,$$

$$y_\alpha(s,t) = y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s\cos t + b_1b_2(1-\alpha)^2\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = z_1z_2 + (z_1c_2 + z_2c_1)(1-\alpha)\sin s\sin t + c_1c_2(1-\alpha)^2\sin^2 s\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(1-\alpha)\cos s}, \qquad y_\alpha(s,t) = \frac{y_1 + b_1(1-\alpha)\sin s\cos t}{y_2 - b_2(1-\alpha)\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{z_1 + c_1(1-\alpha)\sin s\sin t}{z_2 - c_2(1-\alpha)\sin s\sin t}.$$

Therefore, $A(+)_pB$ *and* $A(-)_pB$ *become 3-dimensional triangular fuzzy numbers, but* $A(\cdot)_pB$ *and* $A(/)_pB$ *are not 3-dimensional triangular fuzzy numbers.*

Example 3.6. ([12]) Let $A = (6,\ 3,\ 8,\ 5,\ 4,\ 7)^3$ and $B = (4,\ 2,\ 5,\ 3,\ 6,\ 4)^3$. Then, by Theorem 3.5, we have the following:

(1) $A(+)_pB = (10,\ 5,\ 13,\ 8,\ 10,\ 11)^3$

(2) $A(-)_pB = (10,\ 1,\ 13,\ 2,\ 10,\ 3)^3$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = 6 + 24(1-\alpha)\cos s + 24(1-\alpha)^2 \cos^2 s,$$

$$y_\alpha(s,t) = 15 + 49(1-\alpha)\sin s \cos t + 40(1-\alpha)^2 \sin^2 s \cos^2 t$$

and

$$z_\alpha(s,t) = 28 + 58(1-\alpha)\sin s \sin t + 24(1-\alpha)^2 \sin^2 s \sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = \frac{3 + 6(1-\alpha)\cos s}{2 - 4(1-\alpha)\cos s}, \quad y_\alpha(s,t) = \frac{5 + 8(1-\alpha)\sin s \cos t}{3 - 5(1-\alpha)\sin s \cos t},$$

$$z_\alpha(s,t) = \frac{7 + 4(1-\alpha)\sin s \sin t}{4 - 6(1-\alpha)\sin s \sin t}.$$

Therefore, $A(+)_p B$ and $A(-)_p B$ become 3-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ are not 3-dimensional triangular fuzzy numbers.

Theorem 3.7. ([13]) *Parametric operations on $\mathbb{R}^3$ in Definition 3.4 are the generalization of parametric operations on $\mathbb{R}^2$ in Definition 2.5, which are the generalization of Zadeh's max-min composition operations on $\mathbb{R}$.*

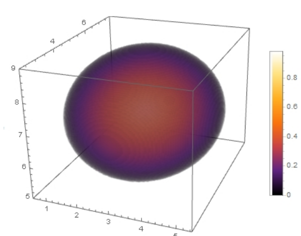


Figure 5.

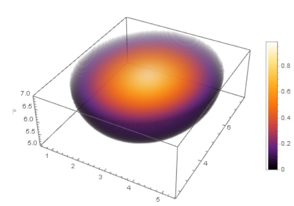


Figure 6.

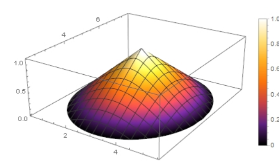


Figure 7.

Figure 5 shows the graph of $A = (6,3,8,5,4,7)^3$, with the values of the membership function represented by color. By slicing the graph of $A = (6,3,8,5,4,7)^3$ at the plane $z = 7$, Figure 6 is obtained. Figure 7 represents the 3-dimensional membership function restricted to the domain of the 2-dimensional cutting plane. Applying Theorem 3.5 to the results in Figure 5 and restricting them to two dimensions yields a result that matches the outcome obtained in two dimensions. Figure 3 Summary states that restricting the 2D result to 1D yields a result that is consistent with the 1D outcome. According to Theorem 3.7, similarly to Figure 3 Summary, the same result can be obtained in both 3D and 2D.

4. 4-Dimensional Expansion of Extended Real Operation for Triangular Fuzzy Number

We define the 4-dimensional triangular fuzzy number on $\mathbb{R}^4$ as a generalization of the triangular fuzzy number on $\mathbb{R}^3$. We then define the parametric operations between two 4-dimensional fuzzy numbers. We compute the operations for the parametric operations between two four-dimensional triangular fuzzy numbers and provide examples. From the results in Chapters 2 and 3, it is clear that the results in this chapter represent an extension of the concepts of 3-dimensional space. Here, the results for the four-dimensional case will be presented using graphs that add a time axis.

Definition 4.1. A fuzzy set A with a membership function

$$\mu_A(x,y,z,t) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} + \frac{(t-t_1)^2}{d^2}}, & \text{if } b^2c^2d^2(x-x_1)^2 \\ \qquad +c^2d^2a^2(y-y_1)^2 + d^2a^2b^2(z-z_1)^2 + a^2b^2c^2(t-t_1)^2 \le a^2b^2c^2d^2, & \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b, c, d > 0$, is called the *4-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1,\ c,\ z_1,\ d,\ t_1)^4$.

The α-cut A_α of a 4-dimensional triangular fuzzy number $A = (a, x_1, b, y_1, c, z_1, d, t_1)^4$ is the following set

$$A_\alpha = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} + \frac{(t-t_1)^2}{d^2} \le (1-\alpha)^2\Big\}.$$

Definition 4.2. A 4-dimensional fuzzy number A defined on $\mathbb{R}^4$ is called a *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^4$.

Theorem 4.3. Let A be a continuous convex fuzzy number defined on $\mathbb{R}^4$ and $A^\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)$, and $f_4^\alpha(s,p,q) (0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2})$, such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q),\ f_4^\alpha(s,p,q)) \in \mathbb{R}^4 |\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

Proof. Let $\alpha \in (0,1)$ be fixed. Since A is a convex fuzzy number defined on $\mathbb{R}^4$, the α-cut A_α is a convex subset in $\mathbb{R}^4$. Therefore, the set

$$A^\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) = \alpha\}$$

is a subset in $\mathbb{R}^4$. Let

$$A_3^\alpha = \{(x,y,z) \in \mathbb{R}^3 | (x,y,z,t) \in A_\alpha\}, \quad A_2^\alpha = \{(x,y) \in \mathbb{R}^2 | (x,y,z,t) \in A_\alpha\}$$

and $\bar{A}_3^\alpha, \bar{A}_2^\alpha, \bar{A}^\alpha$ are the boundaries of $A_3^\alpha, A_2^\alpha, A^\alpha$, respectively. The upper surface of A^α is the graph of a continuous concave function $k_2(x,y,z)$, and the lower surface of A^α is also the graph of a continuous convex function $k_1(x,y,z)$ defined on A_3^α. And the upper surface of A_3^α is the graph of a continuous concave function $h_2(x,y)$, and the lower surface of A_3^α is also the graph of a continuous convex function $h_1(x,y)$ defined on A_2^α. Let

$$l = \inf\{x | (x,y) \in \bar{A}_2^\alpha\} \quad \text{and} \quad m = \sup\{x | (x,y) \in \bar{A}_2^\alpha\}.$$

The upper boundary of $\bar{A}_2^\alpha$ is the graph of some continuous concave function $g_2(x)$ defined on $[l,m]$, and the lower boundary of $\bar{A}_2^\alpha$ is also the graph of some continuous convex function $g_1(x)$ defined on $[l,m]$ (see [13]). Define

$$f_1^\alpha(s) = \frac{1}{2}(l-m)(\cos s - 1) + l, \quad \text{if} \quad s \in [0,\pi].$$

Then, $f_1^\alpha(s)$ moves from l to m if $0 \le s \le \pi$. Define

$$f_2^\alpha(s,p) = \frac{1}{2}(g_2(f_1^\alpha(s)) - g_1(f_1^\alpha(s)))(\sin p - 1) + g_2(f_1^\alpha(s)), 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}.$$

Then, $f_2^\alpha(s,p)$ moves from $g_1(f_1^\alpha(s))$ to $g_2(f_1^\alpha(s))$ if $0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}$. We then have

$$A_2^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p)) \in \mathbb{R}^2 | 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}\}.$$

If we define

$$\begin{aligned} f_3^\alpha(s,p,q) &= \frac{1}{2}(h_2(f_1^\alpha(s), f_2^\alpha(s,p)) - h_1(f_1^\alpha(s), f_2^\alpha(s,p)))(\sin q - 1) \\ &\quad + h_2(f_1^\alpha(s), f_2^\alpha(s,p)), 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \end{aligned}$$

then $f_3^\alpha(s,p,q)$ moves from $h_1(f_1^\alpha(s), f_2^\alpha(s,p))$ to $h_2(f_1^\alpha(s), f_2^\alpha(s,p))$ if $0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}$. We then have

$$A_3^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

If we define $f_4^\alpha(s,p,q) =$

$$\begin{cases} k_1(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)), & 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \\ k_2(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)), & \pi \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \end{cases}$$

then we have

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q), f_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

The proof is now complete. □

Remark 4.4. We proved that Theorem 4.3 is satisfied in the case that A is a continuous convex fuzzy number. If A is a piecewise continuous convex fuzzy number, we can prove similarly (see [13]).

Definition 4.5. Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^4$, and

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q), f_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,p), g_3^\alpha(s,p,q), g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B are fuzzy numbers whose α-sets are given as follows:

(1) parametric addition $A(+)_p B$: $(A(+)_p B)^\alpha =$

$$\{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,p) + g_2^\alpha(s,p), f_3^\alpha(s,p,q) + g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) + g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_p B$: $(A(-)_p B)^\alpha =$

$$\{(f_1^\alpha(s) - g_1^\alpha(s),\ f_2^\alpha(s,p) - g_2^\alpha(s,p), f_3^\alpha(s,p,q) - g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) - g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(3) parametric multiplication $A(\cdot)_p B$: $(A(\cdot)_p B)^\alpha =$

$$\{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ f_2^\alpha(s,p) \cdot g_2^\alpha(s,p), f_3^\alpha(s,p,q) \cdot g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) \cdot g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(4) parametric division $A(/)_p B$: $(A(/)_p B)^\alpha =$

$$\left\{\left(\frac{f_1^\alpha(s)}{g_1^\alpha(s)}, \frac{f_2^\alpha(s,p)}{g_2^\alpha(s,p)}, \frac{f_3^\alpha(s,p,q)}{g_3^\alpha(s,p,q)}, \frac{f_4^\alpha(s,p,q)}{g_3^\alpha(s,p,q)}\right) \in \mathbb{R}^4 \middle| 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\right\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \to 0^+}(A(*)_p B)^\alpha$ and $(A(*)_p B)^1 = \lim_{\alpha \to 1^-}(A(*)_p B)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 4.6. *Let* $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1,\ d_1,\ t_1)^4$ *and* $B = (a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2,\ d_2,\ t_2)^4$ *be two 4-dimensional triangular fuzzy numbers. Then, we have the following:*

(1) $A(+)_p B = (a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2,\ c_1 + c_2,\ z_1 + z_2,\ d_1 + d_2,\ t_1 + t_2)^4$

(2) $A(-)_p B = (a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2,\ c_1 + c_2,\ z_1 - z_2,\ d_1 + d_2,\ t_1 - t_2)^4$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = f_1^\alpha(s) \cdot g_1^\alpha(s)$$
$$= x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2 s,$$
$$y_\alpha(s,p) = f_2^\alpha(s,p) \cdot g_2^\alpha(s,p)$$
$$= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s\cos p + b_1b_2(1-\alpha)^2\sin^2 s\cos^2 p,$$
$$z_\alpha(s,p,q) = f_3^\alpha(s,p,q) \cdot g_3^\alpha(s,p,q)$$
$$= z_1z_2 + (z_1c_2 + z_2c_1)(1-\alpha)\sin s\sin p\cos q + c_1c_2(1-\alpha)^2\sin^2 s\sin^2 p\cos^2 q,$$
$$t_\alpha(s,p,q) = f_4^\alpha(s,p,q) \cdot g_4^\alpha(s,p,q)$$
$$= t_1t_2 + (t_1d_2 + t_2d_1)(1-\alpha)\sin s\sin p\sin q + d_1d_2(1-\alpha)^2\sin^2 s\sin^2 p\sin^2 q.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(1-\alpha)\cos s}, \quad y_\alpha(s,p) = \frac{y_1 + b_1(1-\alpha)\sin s\cos p}{y_2 - b_2(1-\alpha)\sin s\cos p},$$
$$z_\alpha(s,p,q) = \frac{z_1 + c_1(1-\alpha)\sin s\sin p\cos q}{z_2 - c_2(1-\alpha)\sin s\sin p\cos q}, \quad t_\alpha(s,p,q) = \frac{t_1 + d_1(1-\alpha)\sin s\sin p\sin q}{t_2 - d_2(1-\alpha)\sin s\sin p\sin q}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 4-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 4-dimensional triangular fuzzy numbers.

Proof. Since A and B are continuous convex fuzzy numbers defined on $\mathbb{R}^4$, by Theorem 4.3, there exist some functions f_i^α, g_i^α $(i = 1,2,3,4)$, such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q), f_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$
$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,p), g_3^\alpha(s,p,q), g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

Since $A = (a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1,\ d_1,\ t_1)^4$ and $B = (a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2, d_2,\ t_2)^4$, we have

$$f_1^\alpha(s) = x_1 + a_1(1-\alpha)\cos s,$$
$$f_2^\alpha(s,p) = y_1 + b_1(1-\alpha)\sin s\cos p,$$
$$f_3^\alpha(s,p,q) = z_1 + c_1(1-\alpha)\sin s\sin p\cos q,$$
$$f_4^\alpha(s,p,q) = t_1 + d_1(1-\alpha)\sin s\sin p\sin q,$$

and

$$g_1^\alpha(s) = x_2 + a_2(1-\alpha)\cos s,$$
$$g_2^\alpha(s,p) = y_2 + b_2(1-\alpha)\sin s\cos p,$$
$$g_3^\alpha(s,p,q) = z_2 + c_2(1-\alpha)\sin s\sin p\cos q$$
$$g_3^\alpha(s,p,q) = t_2 + d_2(1-\alpha)\sin s\sin p\sin q.$$

(1) Since

$$f_1^\alpha(s) + g_1^\alpha(s) = x_1 + x_2 + (a_1 + a_2)(1-\alpha)\cos s,$$

$$f_2^\alpha(s,p) + g_2^\alpha(s,p) = y_1 + y_2 + (b_1 + b_2)(1-\alpha)\sin s \cos p,$$

$$f_3^\alpha(s,p,q) + g_3^\alpha(s,p,q) = z_1 + z_2 + (c_1 + c_2)(1-\alpha)\sin s \sin p \cos q,$$

and

$$f_4^\alpha(s,p,q) + g_4^\alpha(s,p,q) = t_1 + t_2 + (d_1 + d_2)(1-\alpha)\sin s \sin p \sin q,$$

we have

$$\begin{aligned}(A(+)_pB)^\alpha = \Big\{(x,y,z,t)\in\mathbb{R}^4 \Big| &\Big(\frac{x-x_1-x_2}{(a_1+a_2)(1-\alpha)}\Big)^2 + \Big(\frac{y-y_1-y_2}{(b_1+b_2)(1-\alpha)}\Big)^2 \\ &+ \Big(\frac{z-z_1-z_2}{(c_1+c_2)(1-\alpha)}\Big)^2 + \Big(\frac{t-t_1-t_2}{(d_1+d_2)(1-\alpha)}\Big)^2 = 1\Big\}.\end{aligned}$$

Thus $A(+)_pB = \Big(a_1+a_2,\ x_1+x_2,\ b_1+b_2,\ y_1+y_2,\ c_1+c_2,\ z_1+z_2,\ d_1+d_2,\ t_1+t_2\Big)^4$.

(2) Since

$$f_1^\alpha(s) - g_1^\alpha(s) = x_1 - x_2 + (a_1 + a_2)(1-\alpha)\cos s$$

$$f_2^\alpha(s,p) - g_2^\alpha(s,p) = y_1 - y_2 + (b_1 + b_2)(1-\alpha)\sin s \cos p$$

$$f_3^\alpha(s,p,q) - g_3^\alpha(s,p,q) = z_1 - z_2 + (c_1 + c_2)(1-\alpha)\sin s \sin p \cos q.$$

and

$$f_4^\alpha(s,p,q) - g_4^\alpha(s,p,q) = t_1 - t_2 + (d_1 + d_2)(1-\alpha)\sin s \sin p \sin q,$$

we have

$$\begin{aligned}(A(-)_pB)^\alpha = \Big\{(x,y,z,t)\in\mathbb{R}^4 \Big| &\Big(\frac{x-x_1+x_2}{(a_1+a_2)(1-\alpha)}\Big)^2 + \Big(\frac{y-y_1+y_2}{(b_1+b_2)(1-\alpha)}\Big)^2 \\ &+ \Big(\frac{z-z_1+z_2}{(c_1+c_2)(1-\alpha)}\Big)^2 + \Big(\frac{t-t_1+t_2}{(d_1+d_2)(1-\alpha)}\Big)^2 = 1\Big\},\end{aligned}$$

i.e., $A(-)_pB = \Big(a_1+a_2,\ x_1-x_2,\ b_1+b_2,\ y_1-y_2,\ c_1+c_2,\ z_1-z_2,\ d_1+d_2,\ t_1-t_2\Big)^4$.

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$. Then

$$\begin{aligned}x_\alpha(s) &= f_1^\alpha(s)\cdot g_1^\alpha(s) \\ &= x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2 s, \\ y_\alpha(s,p) &= f_2^\alpha(s,p)\cdot g_2^\alpha(s,p) \\ &= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s\cos p + b_1b_2(1-\alpha)^2\sin^2 s\cos^2 p,\end{aligned}$$

$$z_\alpha(s,p,q) = f_3^\alpha(s,p,q)\cdot g_3^\alpha(s,p,q)$$
$$= z_1z_2 + (z_1c_2 + z_2c_1)(1-\alpha)\sin s\sin p\cos q + c_1c_2(1-\alpha)^2\sin^2 s\sin^2 p\cos^2 q,$$
$$t_\alpha(s,p,q) = f_4^\alpha(s,p,q)\cdot g_4^\alpha(s,p,q)$$
$$= t_1t_2 + (t_1d_2 + t_2d_1)(1-\alpha)\sin s\sin p\sin q + d_1d_2(1-\alpha)^2\sin^2 s\sin^2 p\sin^2 q.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$. Then

$$x_\alpha(s) = \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(1-\alpha)\cos s}, \qquad y_\alpha(s,p) = \frac{y_1 + b_1(1-\alpha)\sin s\cos p}{y_2 - b_2(1-\alpha)\sin s\cos p},$$

$$z_\alpha(s,p,q) = \frac{z_1 + c_1(1-\alpha)\sin s\sin p\cos q}{z_2 - c_2(1-\alpha)\sin s\sin p\cos q}, \qquad t_\alpha(s,p,q) = \frac{t_1 + d_1(1-\alpha)\sin s\sin p\sin q}{t_2 - d_2(1-\alpha)\sin s\sin p\sin q}.$$

The proof is now complete. □

Example 4.7. Let $A = (6,\ 3,\ 8,\ 5,\ 4,\ 7,\ 2,\ 5)^4$ and $B = (4,\ 2,\ 5,\ 3,\ 6,\ 4, 3,\ 2)^4$. Then, by Theorem 4.6, we have the following:

(1) $A(+)_pB = (10,\ 5,\ 13,\ 8,\ 10,\ 11,\ 5,\ 7)^4$

(2) $A(-)_pB = (10,\ 1,\ 13,\ 2,\ 10,\ 3,\ 5,\ 3)^4$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = 6 + 24(1-\alpha)\cos s + 24(1-\alpha)^2\cos^2 s,$$
$$y_\alpha(s,p) = 15 + 49(1-\alpha)\sin s\cos p + 40(1-\alpha)^2\sin^2 s\cos^2 p,$$
$$z_\alpha(s,p,q) = 28 + 58(1-\alpha)\sin s\sin p\cos q + 24(1-\alpha)^2\sin^2 s\sin^2 p\cos^2 q,$$
$$t_\alpha(s,p,q) = 10 + 19(1-\alpha)\sin s\sin p\sin q + 6(1-\alpha)^2\sin^2 s\sin^2 p\sin^2 q.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = \frac{3 + 6(1-\alpha)\cos s}{2 - 4(1-\alpha)\cos s}, \qquad y_\alpha(s,p) = \frac{5 + 8(1-\alpha)\sin s\cos p}{3 - 5(1-\alpha)\sin s\cos p},$$

$$z_\alpha(s,p,q) = \frac{7 + 4(1-\alpha)\sin s\sin p\cos q}{4 - 6(1-\alpha)\sin s\sin p\cos q}, \qquad t_\alpha(s,p,q) = \frac{5 + 2(1-\alpha)\sin s\sin p\sin q}{2 - 3(1-\alpha)\sin s\sin p\sin q}.$$

Therefore, $A(+)_pB$ and $A(-)_pB$ become 4-dimensional triangular fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 4-dimensional triangular fuzzy numbers.

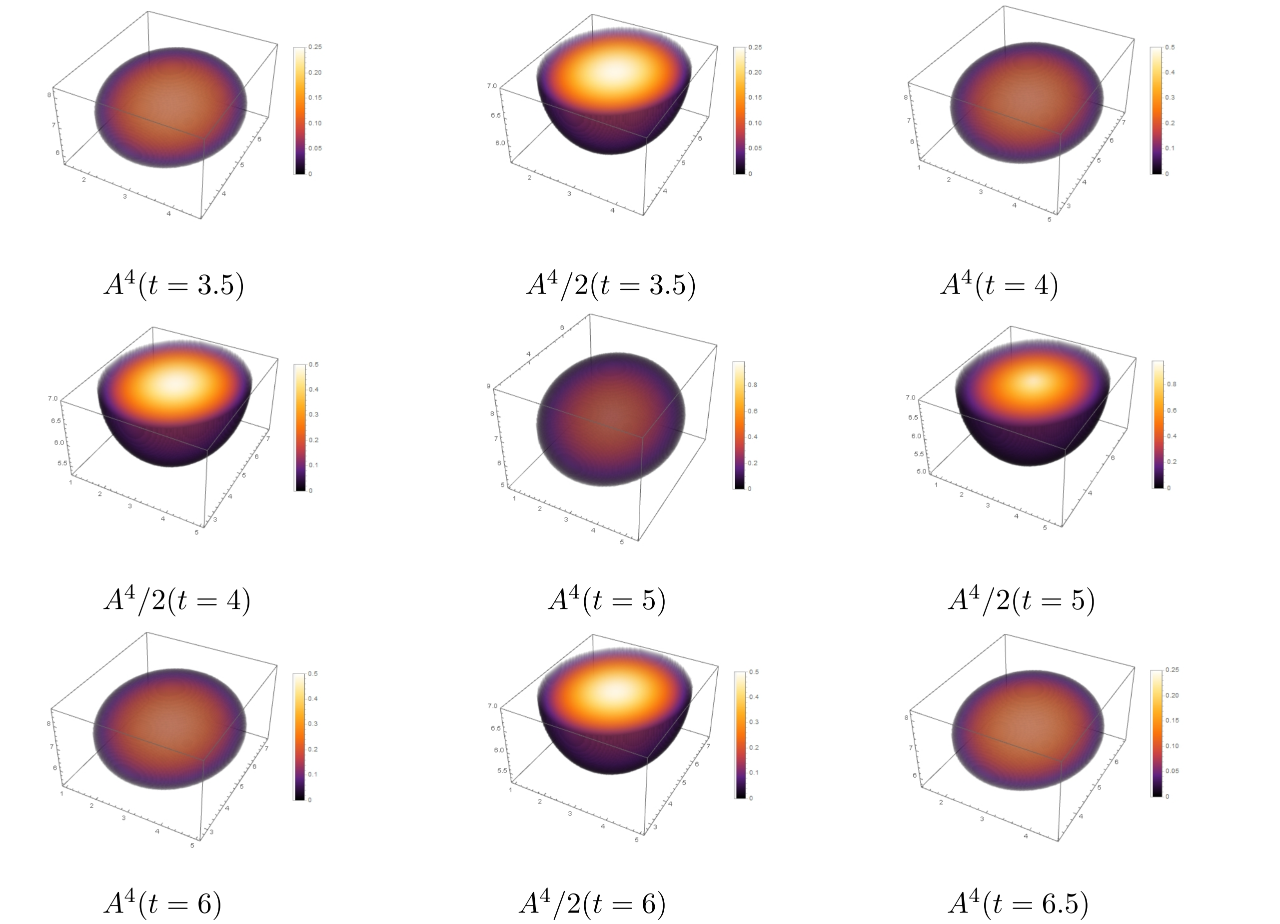
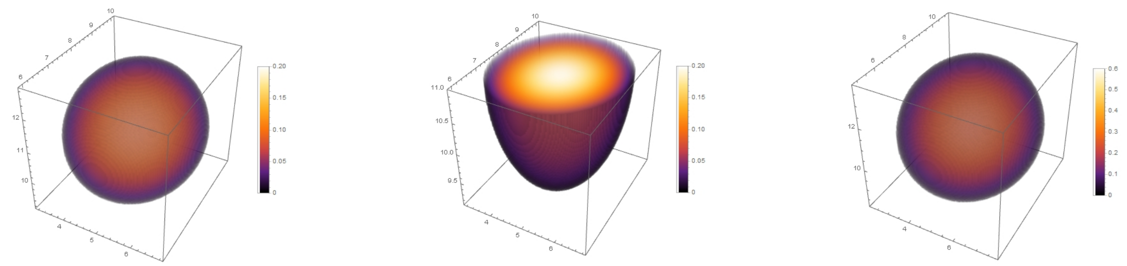
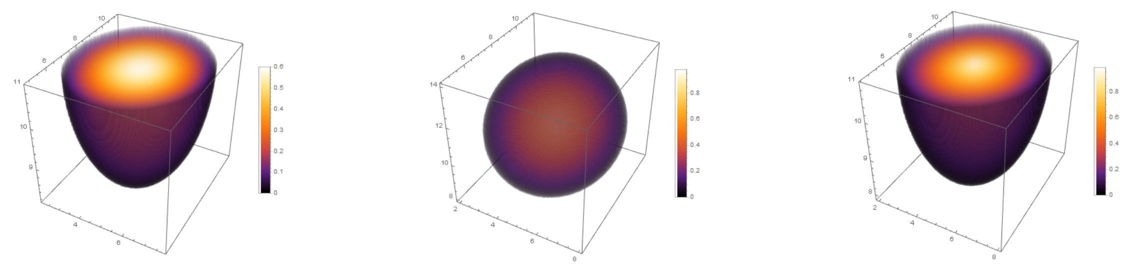


Figure 8. A^4 in 4-dimensional space

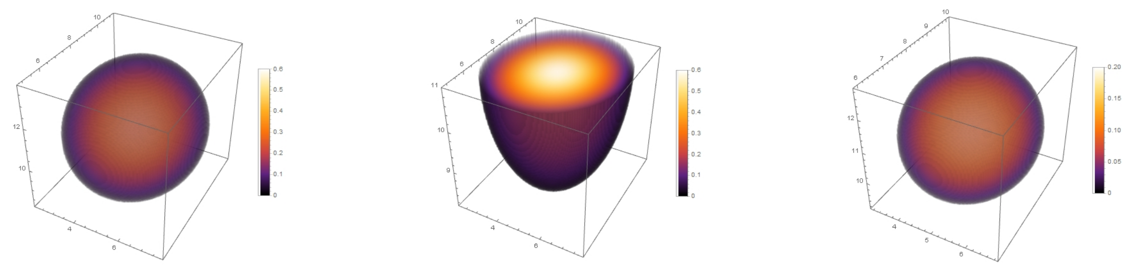
We expressed the 4-dimensional triangle fuzzy number defined in Chapter 4 graphically. Then, the results from Chapter 4 are represented graphically for several fixed time periods and compared with the typical four dimensional graph representation. Since the graphs of A and B, as well as $A + B$ and $A - B$, are similar, only the graphs of A and $A + B$ are presented. Although it is defined in four dimensions, a five-dimensional space is required to draw the graph. However, in practice, since the membership function values are represented by the intensity of color, only four dimensional space is needed.



$(A+B)^4(t=3)$ $(A+B)^4/2(t=3)$ $(A+B)^4(t=5)$



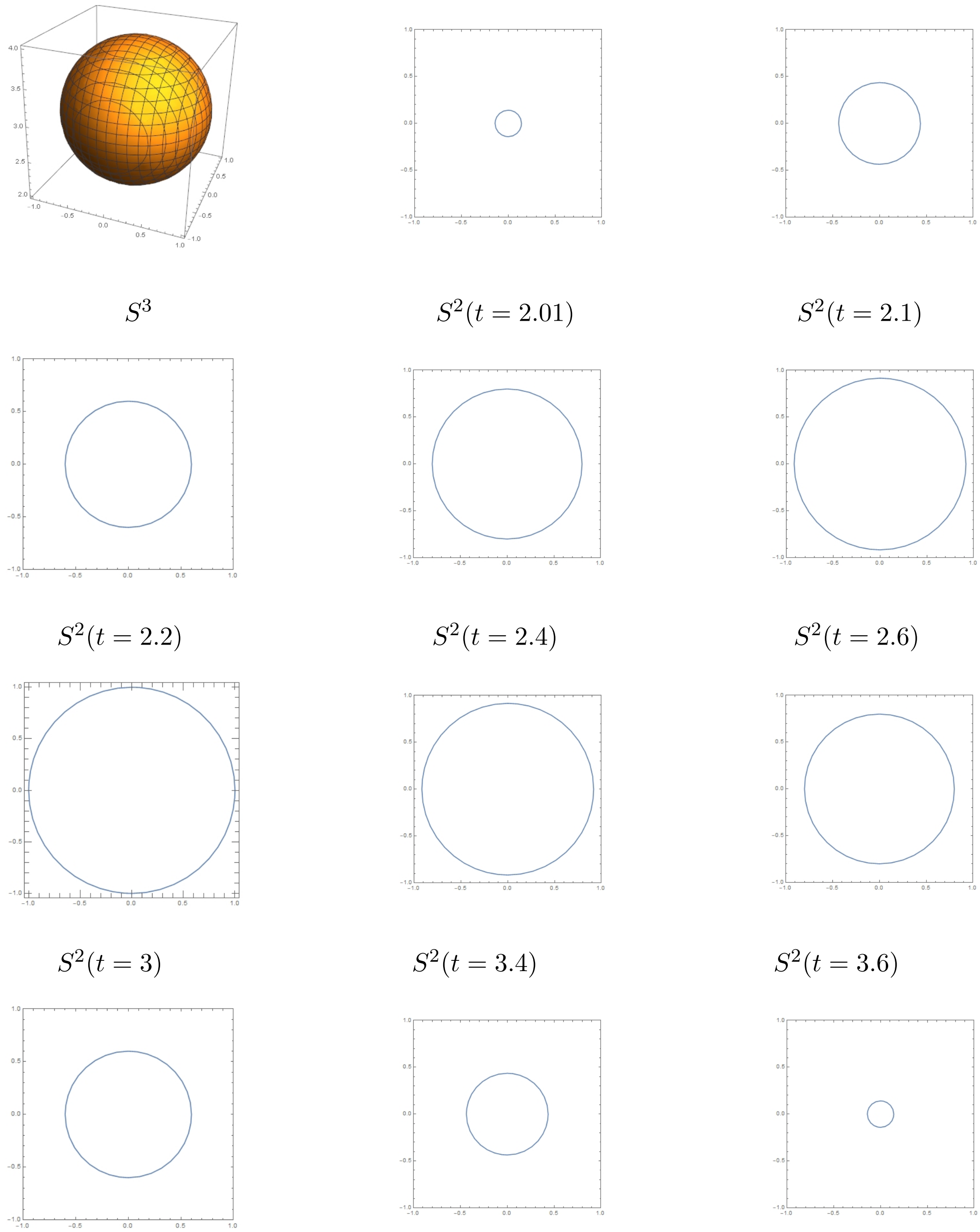
$(A+B)^4/2(t=5)$ $(A+B)^4(t=7)$ $(A+B)^4/2(t=7)$



$(A+B)^4(t=9)$ $(A+B)^4/2(t=9)$ $(A+B)^4(t=11)$

Figure 9. A^4+B^4 in 4-dimensional space

The shapes of all the graphs appear similar. The difference is that, as time progresses (i.e., as the value of t increases), the function values first increase from a small value and then decrease again. When compared with the graph obtained by changing the z-axis of the 3-dimensional sphere to the time axis, it becomes clear that the graph is accurate.



S^3 $S^2(t = 2.01)$ $S^2(t = 2.1)$

$S^2(t = 2.2)$ $S^2(t = 2.4)$ $S^2(t = 2.6)$

$S^2(t = 3)$ $S^2(t = 3.4)$ $S^2(t = 3.6)$

$S^2(t = 3.8)$ $S^2(t = 3.9)$ $S^2(t = 3.99)$

Figure 10. 2-dimensional space w.r.t. time axis

Based on the time axis, we can observe that the 2-dimensional graph appears and disappears within the existing time range, and the size of the circle gradually increases from a small size and then decreases. The gradual increase and decrease of the circle indicate that the function values increase and then decrease, which matches the behavior of the 4-dimensional graph we obtained. This shows that the 4-dimensional graph is consistent with this pattern.

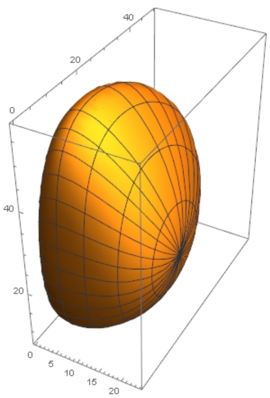


Fig. 11. $(AB)^4(q = -\pi/6, \alpha = 1/2)$

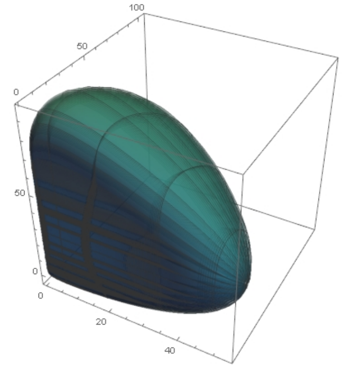


Fig. 12. $(AB)^4(q = -\pi/6, 0 \leq \alpha \leq 1)$

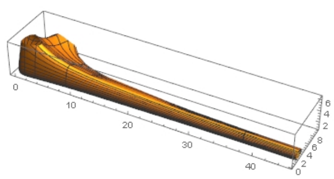


Fig. 13. $(A/B)^4(q = 0, \alpha = 1/2)$

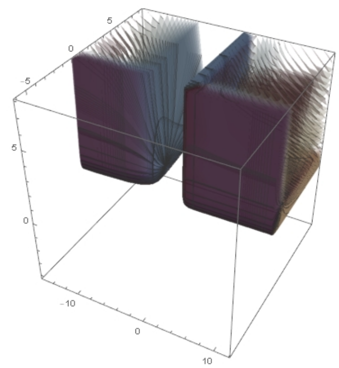


Fig. 14. $(A/B)^4(q = 0, 0 \leq \alpha \leq 1)$

To get a three-dimensional graph for one t value, you can get a three-dimensional graph for one q value. The graph for $q = -\pi/6$ and $\alpha = 1/2$ is given in Figure 11, and the graph for the entire $\alpha(0 \leq \alpha \leq 1)$ value is given in Figure 12. It can be seen that Figure 12 is a graph containing the inner region, one of which is Figure 11. In the case of A/B, only the case of $q = 0$ is presented because it forms too complex. Nevertheless, since it is a graph of an incomprehensible degree, it is considered difficult to handle in terms of application.

5. Conclusion

In Section 2, we newly defined an extension operation that expands 1D to 2D. We introduced the results obtained in 2D and discussed that when the results in 2D are restricted to 1D, they coincide with the results in 1D.

In Section 3, we defined the extended real operation on 3D. We corrected errors in the existing 3D results and discussed that when the 3D results are restricted to 2D, they match the 2D results. We also presented the 3D result graphs.

In Section 4, we defined and calculated the extended real operation on 4D. Using the results from Chapter 3, which were corrected and proven, we naturally proved the extension. Without correcting the 3D results, predicting the 4D results would be difficult. We represented the 4D fuzzy triangular numbers and operation results graphically. Since fuzzy numbers are defined in 4D, five-dimensional space would be required to actually represent them graphically. However, the membership function values were expressed using the intensity of color. The graph was represented not over the entire time axis but for specific parts, making it feasible in 3D space as well. Detailed explanations for each result were also included. The most accurate graph is a video using the time axis.

In the 2D and 3D cases, the results of the parametric equations did not maintain consistency. However, in 4D, the existing results were slightly modified and extended to maintain consistency. As a result, the extension to finite dimensions became predictable. Our research ([5,7,12]) has been cited in studies such as fuzzy decisions, fuzzy linear systems, Fuzzy Control Strategy ([4,8,11]), and the 4D research is expected to be applied in many fields. In particular, representing 4D results graphically by adding a time axis will be of significant help in the field of fuzzy graphics.

In [Kaufmann], addition and other operations for 2D fuzzy numbers were introduced in the discrete case, but no further research was conducted afterward. Our results define continuous fuzzy numbers and prove that they are an extended concept of lower-dimensional fuzzy numbers. In this regard, many applications are expected to be made in fields such as Fuzzy Logic, decision making problems, Fuzzy classification, and fuzzy mathematical model in the future.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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