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A VARIATIONAL ITERATION METHOD APPROACH FOR FRACTIONAL FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS: EXISTENCE, ULAM STABILITY AND NUMERICAL ANALYSIS

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Abstract. This paper presents an analytical and numerical study of a mixed fractional Volterra-Fredholm integro-differential equation. First, a theoretical analysis is conducted to establish the existence and uniqueness of the solution under appropriate conditions using Banach's fixed point theorem. Furthermore, the stability of the solution is studied in the sense of Ulam-Hyers, which guarantees the robustness of the model in the face of small perturbations. In a second step, a numerical method based on the Variational Iteration Method (VIM) was used to obtain an approximate solution to the equation under consideration. The convergence of the method is analysed and rigorously justified. Numerical simulations are performed to evaluate the performance of the proposed scheme. The results obtained show rapid convergence and high accuracy of the approximate solutions, in excellent agreement with the theoretical results, thus confirming the effectiveness of the method for studying this type of fractional equations.

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1. Introduction

Integral-differential equations, which combine differential and integral operators, are a fundamental tool for the mathematical modelling of phenomena exhibiting non-local or memory effects [6, 9]. Among these, Volterra and Fredholm formulations play a central and complementary role. The former, with variable integration limits, are suited to memory systems [1], while the latter, with fixed limits, describe extended non-local interactions [12]. When these equations also incorporate nonlinearities and fractional derivatives, their analytical solution often becomes inaccessible, requiring the development of robust numerical and approximate methods. Numerous numerical approaches have been proposed to

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solve these complex equations. These include spectral methods, which offer high accuracy for regular solutions [10], finite difference and finite element methods, which are valued for their geometric flexibility, and collocation and decomposition techniques such as the Adomian method [2,13]. More recently, variational methods have emerged as a promising alternative, combining mathematical rigour and numerical efficiency. Among these, the variational iteration method (VIM) stands out for its power and flexibility [3,6,14]. Based on an iterative variational principle, it allows the construction of a convergent sequence of successive approximations without resorting to linearisation or spatial discretisation, and adapts naturally to non-linearities and complex boundary conditions, including Volterra and Fredholm equations [8]. The main objective of this study is the analysis and numerical solution, via VIM, of a general class of nonlinear fractional partial integro-differential equations of the mixed Fredholm-Volterra type, described by the following system:

(1)
$$\begin{cases} {}^{C}D_t^{\eta}\mathscr{U}(x,t) = S(x,t) + \varepsilon_1 \displaystyle\int_{\alpha}^{x} \mathcal{K}_1(y,t)\Phi_1(\mathscr{U}(y,t))\,dy + \varepsilon_2 \int_{\alpha}^{\beta} \mathcal{K}_2(y,t)\Phi_2(\mathscr{U}(y,t))\,dy, \quad x \in [\alpha,\beta], t \in [0,T] \\ \\ \mathscr{U}(x,0) = \mathscr{U}_0(x), \quad \forall\, x \in [\alpha,\beta]. \end{cases}$$

In this formulation,

- ${}^{C}D_t^{\eta}$ is the Caputo fractional derivative of order η with respect to t,
- $\mathscr{U}(x,t)$ is the unknown function to be determined, defined for $x \in [\alpha,\beta]$ and $t \in [0,T]$,
- $0 < \eta \leq 1$ is the order of the fractional derivative in the Caputo sense,
- $S(x,t)$ is a known source function, defined for $x \in [\alpha,\beta]$ and $t \in [0,T]$,
- $\mathcal{K}_1$ and $\mathcal{K}_2$ are integral kernels,
- Φ_1 and Φ_2 are non-linear functions depending on $\mathscr{U}$,
- $\mathscr{U}(x,0) = \mathscr{U}_0(x)$ is the initial condition, defined for $x \in [\alpha,\beta]$.

In this work, we conduct a comprehensive theoretical analysis aimed at establishing the existence, uniqueness and stability of solutions in the Ulam–Hyers sense. In parallel, we develop and implement a numerical scheme based on the Variational Iteration Method (VIM), specially designed for the class of equations under consideration. This dual analytical and numerical approach allows us to validate the efficiency and robustness of the proposed method, without resorting to restrictive assumptions on the nonlinearities or operators involved. This paper is structured as follows. Section 2 covers the fundamental prerequisites and definitions relating to fractional calculus. Section 3 presents the basic idea behind the VIM method. Section 4 is devoted to studying the conditions for the existence and uniqueness of the solution. Section 5 presents the adaptation of the VIM method to our problem and discusses its convergence. Section 6 addresses stability analysis in the sense of Ulam–Hyers. Section 7 illustrates the method with numerical examples that validate its accuracy and applicability. Finally, Section 8 summarises the main results and outlines prospects for further research.

2. Preliminaries

Definition 2.1. (**Gamma function**) [4,15]

The **Gamma function** denoted by Γ is defined by:

$$\Gamma\colon\ \{z\in\mathbb{C}\mid \Re(z)>0\}\longrightarrow\mathbb{C},\quad z\longmapsto\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt.$$

Definition 2.2. (**Riemann-Liouville integral**) [4,15]

Let $\mathscr{U}:[a,b]\to\mathbb{R}$ be an integrable function, and let $\alpha>0$ be a real number. The fractional Riemann-Liouville integral of order α of $\mathscr{U}$ is defined by:

$$I_{a+}^{\alpha}\mathscr{U}(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\mathscr{U}(\tau)\,d\tau:\ t\geqslant a \tag{2}$$

Definition 2.3. (**Caputo derivative**) [15]

The fractional Caputo derivative of order $\alpha>0$ of a function $\mathscr{U}\in C^n([t_0,+\infty),\mathbb{R})$ is denoted and defined by: $\forall\, t\geqslant t_0$,

$$ {}^{C}D_t^{\alpha}\mathscr{U}(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{t_0}^{t}\frac{\mathscr{U}^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, & if\ 0\leqslant n-1<\alpha<n,\ n=\lceil\alpha\rceil,\quad n\in\mathbb{N},\\[2ex] \dfrac{d^n}{dt^n}\mathscr{U}(t), & if\quad \alpha=n.\end{cases} \tag{3}$$

In particular, for $\alpha\in(0,1)$:

$$ {}^{C}D_t^{\alpha}\mathscr{U}(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{\mathscr{U}'(s)}{(t-s)^{\alpha}}ds. \tag{4}$$

Theorem 2.1.

[15] Let $\alpha>0$ be a real number, $n=\lceil\alpha\rceil$, $\mathscr{U}$ a function n-times continuously derivable on $[0,t]$. Then, the composition of the fractional integral I_t^{α} and the Caputo derivative ${}^{C}D_t^{\alpha}$ verifies:

$$I_t^{\alpha}\left({}^{C}D_t^{\alpha}\mathscr{U}\right)(t)=\mathscr{U}(t)-\sum_{k=0}^{n-1}\frac{\mathscr{U}^{(k)}(0)}{\Gamma(k+1)}t^k$$

Definition 2.4. (**Laplace Transform**) [15]

A function $\mathscr{U}:[0,\infty)\to\mathbb{R}$ *is said to be of* exponential order $\alpha\in\mathbb{R}$ *if there exist constants* $M>0$ *and* $T>0$ *such that*

$$|\mathscr{U}(t)|\leq Me^{\alpha t}\quad \textit{for all } t\geq T.$$

The Laplace transform *of such a function is defined as*

$$F(s) = \mathcal{L}\{\mathscr{U}(t)\}(s) = \int_0^{\infty} e^{-st}\mathscr{U}(t)dt,$$

which converges absolutely for $\Re(s) > \alpha$*.*

Theorem 2.2. [15]

Let $\alpha > 0$ with $n-1 < \alpha < n$, $n \in \mathbb{N}$. Assume $\mathscr{U}$ is continuously differentiable up to order $(n-1)$ on $[t_0, +\infty)$ and its n-th derivative exists with exponential order. If ${}_{t_0}^{C}D_t^{\alpha}\mathscr{U}$ is piecewise continuous on $[t_0, +\infty)$, then, we have:

$$\mathcal{L}\left\{{}^{C}D_t^{\alpha}\mathscr{U}(t)\right\} = s^{\alpha}F(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1}\mathscr{U}^{(j)}(t_0). \tag{5}$$

3. The Variational Iteration Method: Basic Principle

In this section, we review the basic principles of the VIM method. The variational iteration method, its theoretical principles, and its application to various types of differential equations are detailed in the literature. [5–7,9,14]. To illustrate the fundamental idea of the variational iteration method, consider the following general non-linear equation in operational form,

$$\mathcal{L}(\psi(x)) + \mathcal{N}(\psi(x)) = S(x), \quad x \in [a, b] \tag{6}$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, and $f(x)$ is a function called the non-homogeneous term.

The equation (6) can be written in the form

$$\mathcal{L}(\psi(x)) + \mathcal{N}(\psi(x)) - S(x) = 0, \quad x \in [a, b] \tag{7}$$

Let ψ_n be the n^{th} approximate solution of equation (7), it follows that:

$$\mathcal{L}(\psi_n(x)) + \mathcal{N}(\psi_n(x)) - S(x) \neq 0$$

Equation (7) can be solved iteratively using the VIM via the correction functional defined by:

$$\psi_{n+1}(x) = \psi_n(x) + \int_a^x \lambda(x,\xi)\{\mathcal{L}(\psi_n(\xi)) + \mathcal{N}(\tilde{\psi}_n(\xi)) - S(\xi)\}\, d\xi, \quad n = 0, 1, \ldots$$

where λ is the general Lagrange multiplier that can be optimally determined using variational theory. The construction of successive approximations ψ_{n+1} of the solution ψ is based on the use of the previously identified Lagrange multiplier and the choice of a starting function. For more details, see [5–7].

4. Existence and Uniqueness Analysis

In this section, we will establish the conditions for the existence and uniqueness of problem (1) using Banach's fixed point theorem. Let $\Omega_1 \times \Omega_2 = [a,b] \times [0,T]$ and $\mathscr{X} = C(\Omega_1 \times \Omega_2)$ be the space of continuous functions on $\Omega_1 \times \Omega_2$ equipped with the uniform norm:

$$\|\mathscr{U}\|_\infty = \sup_{(x,t)\in\Omega_1\times\Omega_2} |\mathscr{U}(x,t)|.$$

For this purpose, it will be necessary to introduce assumptions regarding the functions Φ_1, Φ_2, $\mathcal{K}_1$ and $\mathcal{K}_2$ defined above.

- $\mathcal{H}_1 : \|\Phi_i(\mathscr{U}_1) - \Phi_i(\mathscr{U}_2)\|_\infty \leq \rho_i \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty, \quad i = 1,2$
- $\mathcal{H}_2 : S \in C([a,b] \times [0,T])$,
- $\mathcal{H}_3 : \|\mathcal{K}_i\|_\infty = \sup_{(x,t)\in\Omega_1\times\Omega_2} |\mathcal{K}_i(x,t)| \leq C_{\mathcal{K}_i}, \quad i = 1,2.$

where:

ρ_1, ρ_2, $\mathcal{K}_1$ and $\mathcal{K}_2$ are four real positive constants.

Now, we consider the constants $\rho > 0$, $\varepsilon > 0$ and $\varpi > 0$ such that:

$$\rho = \max(\rho_1, \rho_2), \quad \varepsilon = \max(\varepsilon_1, \varepsilon_2) \ \text{ and } \ \varpi = \max(C_{\mathcal{K}_1}, C_{\mathcal{K}_2}).$$

Lemma 4.1.

A function $\mathscr{U} \in \mathscr{X}$ is a solution to the fractional problem of order η (1) if and only if $\mathscr{U}$ is a solution to the following fractional integral equation:

$$\mathscr{U}(x,t) = \mathscr{U}_0(x) + I_t^\eta S(x,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \Big[\varepsilon_1 \int_\alpha^x \mathcal{K}_1(y,\tau)\Phi_1(\mathscr{U}(y,\tau)\,dy + \varepsilon_2 \int_\alpha^\beta \mathcal{K}_2(y,\tau)\Phi_2(\mathscr{U}(y,\tau)\,dy \Big] d\tau \tag{8}$$

Proof. By applying the integral operator I_t^η to the differential equation (1) and using the theorem 2 we obtain (4.1) □

Theorem 4.1.

Let us assume that hypotheses $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are satisfied and that $\dfrac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)} < 1$. Then problem (1) has a unique solution.

Proof. First, we define an operator $\mathcal{G} : \ \mathscr{X} \longrightarrow \mathscr{X}$ by:

$$\mathcal{G}[\mathscr{U}](x,t) = \mathscr{U}_0(x) + I_t^\eta S(x,t) + \mathscr{I}[\mathscr{U}](x,t), \tag{9}$$

where:

$$\mathscr{I}[\mathscr{U}](x,t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \left(\sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,\tau)\Phi_i(\mathscr{U}(y,\tau))dy \right) d\tau, \tag{10}$$

with $\Lambda_1 = [\alpha, x]$ et $\Lambda_2 = [\alpha, \beta]$.

Let $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{X}$, then we have:

$$\begin{aligned}\mathcal{G}[\mathscr{U}_1](x,t) - \mathcal{G}[\mathscr{U}_2](x,t) &= \mathscr{I}[\mathscr{U}_1](x,t) - \mathscr{I}[\mathscr{U}_2](x,t),\\ &= \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left(\mathcal{F}(\mathscr{U}_1)(x,\tau) - \mathcal{F}(\mathscr{U}_1)(x,\tau)\right)d\tau,\end{aligned}$$

with,

$$\mathcal{F}(\mathscr{U})(x,t) = \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,\tau)\Phi_i(\mathscr{U}(y,\tau))dy. \tag{11}$$

Under assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$, we have:

$$\begin{aligned}|\mathcal{F}(\mathscr{U}_1(x,t)) - \mathcal{F}(\mathscr{U}_2(x,t))| &= \left|\sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,\tau)\left[\Phi_i(\mathscr{U}_1(y,t)) - \Phi_i(\mathscr{U}_2(y,t))\right]dy\right|,\\ &\leq \sum_{i=1}^{2} \varepsilon_i \left|\int_{\Lambda_i} \mathcal{K}_i(y,\tau)\left[\Phi_i(\mathscr{U}_1(y,t)) - \Phi_i(\mathscr{U}_2(y,t))\right]dy\right|,\\ &\leq \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \left|\mathcal{K}_i(y,\tau)\left[\Phi_i(\mathscr{U}_1(y,t)) - \Phi_i(\mathscr{U}_2(y,t))\right]\right|dy,\\ &\leq \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \left|\mathcal{K}_i(y,\tau)\left[\Phi_i(\mathscr{U}_1(y,t)) - \Phi_i(\mathscr{U}_2(y,t))\right]\right|dy,\\ &\leq \sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i}\rho_i\mu(\Lambda_i)\|\mathscr{U}_1 - \mathscr{U}_2\|_\infty,\end{aligned}$$

or $\mu(\Lambda_i)$ denotes the **Lebesgue measure** of Λ_i.

$$\mu(\Lambda_1) = x - \alpha \quad \text{for } x \in [\alpha, \beta]$$

$$\mu(\Lambda_2) = \beta - \alpha.$$

We have:

$$\mu(\Lambda_1) \leq \mu(\Lambda_2) = \beta - \alpha, \tag{12}$$

hence

$$|\mathcal{F}(\mathscr{U}_1(x,t)) - \mathcal{F}(\mathscr{U}_2(x,t))| \leq \sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i}\rho_i\mu(\Lambda_i)\|\mathscr{U}_1 - \mathscr{U}_2\|_\infty. \tag{13}$$

Thus, using relation (13), we can write:

$$\begin{aligned}|\mathcal{G}[\mathscr{U}_1](x,t) - \mathcal{G}[\mathscr{U}_2](x,t)| &\leq \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}|\mathcal{F}(\mathscr{U}_1(x,t)) - \mathcal{F}(\mathscr{U}_2(x,t))|d\tau,\\ &\leq \frac{1}{\Gamma(\eta)}\left(\sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i}\rho_i\mu(\Lambda_i)\right)\|\mathscr{U}_1 - \mathscr{U}_2\|_\infty \int_0^t (t-\tau)^{\eta-1}d\tau,\end{aligned}$$

$$= \frac{t^\eta}{\Gamma(\eta+1)} \left(\sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i} \rho_i \mu(\Lambda_i) \right) \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty.$$

Thus we have:

$$|\mathcal{G}[\mathscr{U}_1](x,t) - \mathcal{G}[\mathscr{U}_2](x,t)| \leq \frac{t^\eta}{\Gamma(\eta+1)} \left(\sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i} \rho_i \mu(\Lambda_i) \right) \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty. \tag{14}$$

Taking the uniform norm in inequality (14), we obtain the following estimate:

$$\begin{aligned} \sup_{(x,t)\in\Omega_1\times\Omega_2} |\mathcal{G}[\mathscr{U}_1(x,t)] - \mathcal{G}[\mathscr{U}_2(x,t)]| &\leq \left[\frac{T^\eta}{\Gamma(\eta+1)} \left(\sum_{i=1}^{2} \varepsilon_i C_{\mathcal{K}_i} \rho_i \mu(\Lambda_i) \right) \right] \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty, \\ &\leq \left[\frac{T^\eta(\beta-\alpha)}{\Gamma(\eta+1)} \left(\varepsilon_1 C_{\mathcal{K}_1} \rho_1 + \varepsilon_2 C_{\mathcal{K}_2} \rho_2 \right) \right] \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty, \\ &\leq \frac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)} \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty, \end{aligned}$$

hence:

$$\|\mathcal{G}\mathscr{U}_1 - \mathcal{G}\mathscr{U}_2\|_\infty \leq \frac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)} \|\mathscr{U}_1 - \mathscr{U}_2\|_\infty. \tag{15}$$

Thus, if $\frac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)} < 1$, the operator $\mathcal{G}$ is strictly contracting, which guarantees the uniqueness of the solution by Banach's fixed point theorem. □

5. VIM-Based Algorithm for Solving Problem (1)

In this section, the Variational Iteration Method (VIM) is deployed to obtain the solution to problem (1). The formulation is guided by the foundational theorem that follows.

Theorem 5.1.

Let the nonlinear fractional Volterra-Fredholm partial integro-differential problem given by equation (1). The variational iteration method (VIM) constructs a sequence of successive approximations $\{\mathscr{U}_n\}_{n\geq 0} \in \mathscr{X}$ defined by the following recurrence relation:

$$\begin{cases} \mathscr{U}_0(x,t) = \mathscr{U}_0(x), \\ \mathscr{U}_{n+1}(x,t) = \mathscr{U}_n(x,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \left(\sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,\tau) \Phi_i(\mathscr{U}_n(y,\tau))\, \mathrm{d}y \right) \mathrm{d}\tau \quad n \geq 0. \end{cases} \tag{16}$$

Proof. Using the notation (11) the equation (reffonde) can be written in the form:

$${}^C\mathcal{D}_t^\eta \mathscr{U}(x,t) - S(x,t) - \mathcal{F}(\mathscr{U})(x,t) = 0. \tag{17}$$

By multiplying equation (17) with the general Lagrange multiplier, we obtain:

$$\lambda(\tau) \left[{}^C\mathcal{D}_t^\eta \mathscr{U}(x,t) - S(x,t) - \mathcal{F}(\mathscr{U})(x,t) \right] = 0. \tag{18}$$

Now, applying the Riemann–Liouville operator I_t^η to both sides of relation (18) yields:

$$I_t^\eta\left[\lambda(\tau)\left({}^{C}\mathcal{D}_t^\eta\mathscr{U}(x,t)-S(x,t)-\mathcal{F}(\mathscr{U})(x,t)dy\right)\right]=0 \tag{19}$$

According to the variational iteration method, we can construct the following correction functional:

$$\mathscr{U}_{n+1}(x,t)=\mathscr{U}_n(x,t)+I_t^\eta\left[\lambda(\tau)\left({}^{C}\mathcal{D}_t^\eta\mathscr{U}(x,t)-S(x,t)-\mathcal{F}(\mathscr{U}_n)(x,t)\,dy\right)\right] \tag{20}$$

To estimate the value of λ, we use the following approximation to the correction function:

$$\mathscr{U}_{n+1}(x,t)=\mathscr{U}_n(x,t)+\int_0^t\left[\lambda(\tau)\left({}^{C}\mathcal{D}_\tau^\eta U(x,\tau)-S(x,\tau)-\mathcal{F}(\tilde{\mathscr{U}}_n)(x,\tau)dy\right)\right]d\tau \tag{21}$$

For $0<\eta\le 1$, we get:

$$\delta\mathscr{U}_{n+1}(x,t)=\delta\mathscr{U}_n(x,t)+\delta\left(\int_0^t\lambda(\tau)\mathscr{U}_n'(\tau)d\tau\right) \tag{22}$$

Using the integration by parts formula, we obtain the following relationship:

$$\delta\mathscr{U}_{n+1}(x,t)=\delta\mathscr{U}_n(x,t)+\lambda(\tau)\delta\mathscr{U}_n(x,\tau)|_{\tau=t}-\int_0^t\lambda'(\tau)\delta\mathscr{U}_n(x,\tau)d\tau,$$

$$\delta\mathscr{U}_{n+1}(x,\tau)=(1+\lambda(\tau)|_{\tau=t})\delta\mathscr{U}_n(x,\tau)-\int_0^t\lambda'(\tau)\delta\mathscr{U}_n(x,\tau)d\tau.$$

The condition of existence of extremum of $\mathscr{U}_{n+1}$ implies $\delta\mathscr{U}_{n+1}=0$.

$$\delta\mathscr{U}_{n+1}=0\Longleftrightarrow 1+\lambda(\tau)|_{\tau=t})\delta\mathscr{U}_n(x,\tau)-\textstyle\int_0^t\lambda'(\tau)\delta\mathscr{U}_n(x,\tau)d\tau=0.$$

This gives us:

$$\lambda'(\tau)=0,\ \text{ et } 1+\lambda(\tau)|_{\tau=t}=0.$$

Thus the general Lagrange multiplier is:

$$\lambda(\tau)=-1 \tag{23}$$

Substituting λ by its value in (20) gives the following relationship:

$$\begin{aligned}\mathscr{U}_{n+1}(x,t)=\mathscr{U}_n(x,t)+I_t^\eta\Bigg(&-{}^{C}\mathcal{D}_t^\eta\mathscr{U}(x,t)+S(x,t)+\varepsilon_1\int_\alpha^x\mathcal{K}_1(y,t)\Phi_1(\mathscr{U}_n(y,t))\,dy\\&+\varepsilon_2\int_\alpha^\beta\mathcal{K}_2(y,t)\Phi_2(\mathscr{U}_n(y,t))\,dy\Bigg).\end{aligned} \tag{24}$$

So the VIM algorithm for the problem (1) is:

$$\begin{cases}\mathscr{U}_0(x,t)=\mathscr{U}_0(x),\\[2ex]\mathscr{U}_{n+1}(x,t)=\mathscr{U}_n(x,t)+\dfrac{1}{\Gamma(\eta)}\displaystyle\int_0^t(t-\tau)^{\eta-1}\left(\sum_{i=1}^{2}\varepsilon_i\int_{\Lambda_i}\mathcal{K}_i(y,\tau)\Phi_i(\mathscr{U}_n(y,\tau))\,\mathrm{d}y\right)\mathrm{d}\tau\quad n\ge 0\end{cases}$$

□

5.1. **Convergence Analysis of the Iterative Scheme.** This section is devoted to the convergence analysis of the sequence of successive approximations generated by the variational iteration method (VIM).

Theorem 5.2.

Let $\mathscr{U} \in C_t([a,b] \times [0,T])$ be the exact solution of equation (1). Consider the sequence of approximations $\{\mathscr{U}_n\}_{n\geq 0} \in C_t([a,b] \times [0,T])$ generated by the VIM scheme. Under the assumptions $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, the sequence $\mathscr{U}_n$ converges uniformly to S on $[a,b] \times [0,T]$, if $\chi < 1$.

Proof. The VIM scheme (16) can be written in operational form:

$$\mathscr{U}_{n+1} = \mathscr{U}_n + I_t^\eta \mathscr{R}(\mathscr{U}_n) \tag{25}$$

where $\mathscr{R}(\mathscr{U}_n)$ is given by:

$$\mathscr{R}(\mathscr{U}_n) = -{}^C\mathcal{D}_t^\eta \mathscr{U}_n + S(x,t) + \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,t)\Phi_i(\mathscr{U}_n(y,t))dy$$

Now let's denote $\mathscr{E}_n(x,t) = \mathscr{U}_n(x,t) - \mathscr{U}(x,t)$ the approximation error. Let $\mathscr{U} \in C([a,b] \times [0,T])$ and $\mathscr{U}_n \in C([a,b] \times [0,T])$ respectively the exact and approximate solution of the probem (1), using the relation (16), we can write

$$\mathscr{U}(x,t) = \mathscr{U}(x,t) + I_t^\eta S(x,t) + \mathscr{I}[\mathscr{U}](x,t), \tag{26}$$

$$\mathscr{U}_{n+1}(x,t) = \mathscr{U}_n(x,t) + I_t^\eta S(x,t) + \mathscr{I}[\mathscr{U}_n](x,t), \tag{27}$$

By subtracting equation (26) from equation (27), we obtain:

$$\begin{aligned}
\mathscr{E}_{n+1}(x,t) =& \mathscr{E}_n + I_t^\eta \left[-{}^C\mathcal{D}_t^\eta \mathscr{E}_n + \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i \left(\Phi_i(\mathscr{U}_n) - \Phi_i(\mathscr{U})\right) dy \right], \\
&= I_t^\eta \left[-{}^C\mathcal{D}_t^\eta \mathscr{E}_n + \sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i \left(\Phi_i(\mathscr{U}_n) - \Phi_i(\mathscr{U})\right) dy \right], \\
&= \mathcal{E}_n(x,t) - \mathcal{E}_n(x,t) + \mathcal{E}_n(x,0) + I_t^\eta \left[\sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i \left(\Phi_i(\mathscr{U}_n) - \Phi_i(\mathscr{U})\right) dy \right], \\
&= \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \left[\sum_{i=1}^{2} \varepsilon_i \int_{\Lambda_i} \mathcal{K}_i(y,\tau) \left(\Phi_i(\mathscr{U}_n(x,\tau) - \Phi_i(\mathscr{U}(x,\tau)\right) dy \right] d\tau, \\
&= \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \left(\mathcal{F}(\mathscr{U}_n)(x,\tau) - \mathcal{F}(\mathscr{U})(x,\tau)\right) d\tau, \\
&= \mathcal{G}[\mathscr{U}_n(x,t)] - \mathcal{G}[\mathscr{U}(x,t)].
\end{aligned}$$

Using the inequalities (14) and (15), we can write:

$$\|\mathscr{E}_{n+1}\|_\infty \leq \frac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)} \|\mathscr{E}_n\|_\infty.$$

By setting $\chi = \dfrac{2T^{\eta}(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)}$, it follows that:

$$\|\mathscr{E}_{n+1}\|_{\infty} \leqslant \chi \|\mathscr{E}_{n}\|_{\infty}.$$

$$\|\mathscr{E}_1\|_\infty \leq \chi\|\mathscr{E}_0\|_\infty, \tag{28}$$

$$\|\mathscr{E}_2\|_\infty \leq \chi^2\|\mathscr{E}_0\|_\infty, \tag{29}$$

$$\|\mathscr{E}_3\|_\infty \leq \chi^3\|\mathscr{E}_0\|_\infty, \tag{30}$$

$$\vdots \tag{31}$$

$$\|\mathscr{E}_n\|_\infty \leq \chi^n\|\mathscr{E}_0\|_\infty, \qquad n \geq 0. \tag{32}$$

Since $0 < \chi < 1$, then $\chi^n \longrightarrow 0$ when $n \longrightarrow +\infty$ and it follows that $\|\mathcal{E}_n(x,t)| \longrightarrow 0$ when $n \longrightarrow +\infty$. Therefore, the sequence of approximate solutions $\{\mathscr{U}_n\}_{n\geqslant 0}$ converges to the exact solution $\mathscr{U}$. □

6. Ulam–hyers Stability Analysis of the Problem (1)

In this section we study the Ulam-Hyers stability of the problem (1), a concept introduced by Ulam [11] and developed for fractional derivatives. Its importance for approximate solutions justifies the use of non-linear functional analysis methods. To perform the stability analysis, let us rewrite the problem (1) in the form:

$$\begin{cases} {}^C D_t^{\eta}\mathscr{U}(x,t) = S(x,t) + \mathcal{F}(\mathscr{U}(x,t)) \;\; x \in [\alpha,\beta], \;\; t \in [0,T] \\ \\ \mathscr{U}(x,0) = \mathscr{U}_0(x), \;\; \forall\, x \in [\alpha,\beta]. \end{cases} \tag{33}$$

The following definitions are necessary. Let $\Delta > 0$ and consider the following , inequality:

$$\left|{}^C D_t^{\eta}\mathscr{U}(x,t) - S(x,t) - \mathcal{F}\left(\mathscr{U}(x,t)\right)\right| \leq \Delta, \quad (x,t) \in [\alpha,\beta]\times[0,T]), \tag{34}$$

Definition 6.1.

The problem (1) is stable in the Ulam-Hyers sense if there exists a constant $\mathscr{C} > 0$ such that for any $\Delta > 0$ and for any solution $\mathscr{V} \in \mathscr{X}$ satisfying the inequality (34), there exists a solution $\mathscr{U} \in \mathscr{X}$ of the problem (1) satisfying:

$$|\mathscr{V}(x,t) - \mathscr{U}(x,t)| \leq \mathscr{C}\Delta, \;\; (x,t) \in [\alpha,\beta]\times[0,T]. \tag{35}$$

Remark 6.1.

A function $\mathscr{V} \in \mathscr{X}$ is a solution of the inequality (34) if and only if there exists a function $\varphi \in \mathscr{X}$ verifying the following properties:

(i) $|\varphi(x,t)| \leq \Delta$

(ii) ${}^{C}D_t^{\eta}\mathscr{V}(x,t) = S(x,t) + \mathcal{F}(\mathscr{V}(x,t)) + \varphi(x,t), \quad (x,t) \in [\alpha,\beta]\times[0,T]$

Lemma 6.1.

Suppose that $\mathscr{V} \in \mathscr{X}$ satisfies the inequality (34). Then $\mathscr{V}$ satisfies the integral inequality:

$$\left|\mathscr{V}(x,t) - \mathscr{V}(x,0) - \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left[S(x,\tau) + \mathcal{F}(\mathscr{V}(x,\tau))\right] d\tau\right| \leq \Omega\Delta. \tag{36}$$

Proof. From (ii) of Remark 6.1, we have:

$${}^{C}D_t^{\eta}\mathscr{V}(x,t) = S(x,t) + \mathcal{F}(\mathscr{V}(x,t)) + \varphi(x,t)$$

Using the lemma 4.1, we can write:

$$\mathscr{V}(x,t) = \mathscr{V}(x,0) + \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left[S(x,\tau) + \mathcal{F}(\mathscr{V}(x,\tau))\right] d\tau + \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\varphi(\tau) d\tau. \tag{37}$$

Using (i) of Remark 6.1 and the relation (37), we obtain:

$$\begin{aligned}
&\left|\mathscr{V}(x,t) - \mathscr{V}(x,0) - \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left[S(x,\tau) + \mathcal{F}(\mathscr{V}(x,\tau))\right] d\tau\right|, \\
&= \left|\frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\varphi(x,\tau) d\tau\right|, \\
&\leq \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}|\varphi(x,\tau)| d\tau, \\
&\leq \frac{\Delta}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1} d\tau, \\
&= \frac{\Delta}{\Gamma(\eta)}\cdot\frac{t^{\eta}}{\eta} \\
&\leq \frac{\Delta}{\Gamma(\eta)}\cdot\frac{T^{\eta}}{\eta} \leq \Omega\Delta,
\end{aligned}$$

where $\Omega = \dfrac{T^{\eta}}{\eta\Gamma(\eta)}$. This completes the proof. □

Theorem 6.1. *Let us assume that hypotheses $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are satisfied and that $1 - \Omega\mathfrak{L} > 0$. Then, problem (1) is Ulam-Hyers stable.*

Proof. Let $\mathscr{V} \in \mathscr{X} = C_t([\alpha,\beta]\times[0,T])$ a solution of inequality (34) and $\mathscr{U} \in \mathscr{X}$ the unique solution of problem (1). For every $\Delta > 0$ and $(x,t) \in [\alpha,\beta]\times[0,T]$, we have:

$$\begin{aligned}
\|\mathscr{V} - \mathscr{U}\|_{\infty} &= \sup_{(x,t)\in J}\left|\mathscr{V}(x,t) - \mathscr{U}_0 - \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left[S(x,\tau) + \mathcal{F}(\mathscr{U}(x,\tau))\right] d\tau\right| \\
&= \sup_{(x,t)\in J}\left|\mathscr{V}(x,t) - \mathscr{V}(x,0) - \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left[S(x,\tau) + \mathcal{F}(\mathscr{U}(x,\tau))\right] d\tau\right| \\
&\quad + \sup_{(x,t)\in J}\frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\left|\mathcal{F}(\mathscr{U}(x,\tau)) - \mathcal{F}(\mathscr{V}(x,\tau))\right| d\tau.
\end{aligned}$$

Using Lemma 6.1 and the estimate (13), we can write:

$$\|\mathscr{V}-\mathscr{U}\|_\infty \leq \Omega\Delta + \frac{2T^\eta(\beta-\alpha)\rho\varepsilon}{\Gamma(\eta+1)}\int_0^t (t-\tau)^{\eta-1}\|\mathscr{U}(x,\tau)-\mathscr{V}(x,\tau)\|d\tau,$$
$$\leq \Omega\Delta + \mathfrak{L}\Omega\|\mathscr{U}-\mathscr{V}\|_\infty$$

where $\mathfrak{L} = 2(\beta-\alpha)\rho\varepsilon$ So we get:

$$\|\mathscr{V}-\mathscr{U}\|_\infty(1-\Omega\mathfrak{L}) \leq \Omega\Delta$$

Hence:

$$\|\mathscr{V}-U\|_\infty \leq \mathscr{C}\Delta \quad \text{with} \quad \mathscr{C} = \frac{\Omega}{1-\Omega\mathfrak{L}}.$$

We conclude that the problem (1) is stable in the Ulam-Hyers sense. □

7. Numerical examples

In this section, we present a numerical example to illustrate the efficiency and performance of the variational iteration method (VIM) for solving the fractional integro-differential equation studied. This example highlights the practical implementation of the numerical scheme proposed in section 5 as well as its numerical behaviour. The approximate solution is obtained from successive iterations of the VIM, then compared to the exact solution in order to evaluate the accuracy and convergence of the numerical scheme. Consider the following problem:

$$(38)\quad \begin{cases} {}^C D_t^\eta \mathscr{U}(x,t) = g(x,t) + \varepsilon_1 \int_0^x 2y\,\sin(\mathscr{U}(y,t))\,dy + \varepsilon_2 \int_0^3 2y\,\tanh(\mathscr{U}(y,t))\,dy, \quad x\in[0,3], \quad t\in[0,1] \\ \\ \mathscr{U}(x,0) = \mathscr{U}_0(x), \quad \forall\, x\in[0,3] \end{cases},$$

with:

$$\mathscr{U}_0(x) = x^2, \quad x\in[0,3],$$
$$g(x,t) = \frac{2}{\Gamma(3-\eta)}t^{2-\eta} - \varepsilon_1(\cos(t^2)-\cos(x^2+t^2)) - \varepsilon_2(\ln(\cosh(9+t^2)) - \ln(\cosh(t^2))),$$
$$\varepsilon_1 = 0.1, \quad \varepsilon_2 = 0.05,$$
$$\Phi_1(\mathscr{U}) = \sin(\mathscr{U}) \text{ et } \Phi_1(\mathscr{U}) = \tanh(\mathscr{U}),$$
$$\|\Phi_1(\mathscr{U}_1)-\Phi_1(\mathscr{U}_2)\|_\infty \leq \|\mathscr{U}_1-\mathscr{U}_2\|_\infty,$$
$$\|\Phi_2(\mathscr{U}_1)-\Phi_2(\mathscr{U}_2)\|_\infty \leq \|\mathscr{U}_1-\mathscr{U}_2\|_\infty$$

Thus $\rho_1 = \rho_2 = 1$. In this example, we consider two cases: $\eta = 0.8$ and $\eta = 0.2$, We have:

$$\chi = \frac{2T^\eta(\beta-\alpha)\rho\varepsilon\varpi}{\Gamma(\eta+1)},$$
$$= \frac{2T^\eta(\beta-\alpha)\max\{\rho_1,\rho_2\}.\max\{\varepsilon_1,\varepsilon_2\}.\max\{C_{\mathcal{K}_1},C_{\mathcal{K}_2}\}}{\Gamma(\eta+1)},$$

thus we have:

$$\chi = \begin{cases} \dfrac{0.2}{\Gamma(1.2)} \approx 0.217825 < 1, & \text{if } \eta = 0.2 \\ \dfrac{0.2}{\Gamma(1.8)} \approx 0.21473 < 1, & \text{if } \eta = 0.8. \end{cases}$$

Thus, since all the assumptions of Banach's fixed point theorem are satisfied, we can conclude that the problem under consideration has a unique solution.

Now, we present the numerical approximation of the solution obtained using the variational iteration method (VIM). The behaviour of the approximate solution is illustrated graphically by the curves in Figure 1 for $\eta = 0.8$ and Figure 4 for $\eta = 0.2$. In order to evaluate the accuracy of the approximation, the corresponding absolute error curves are also shown in Figure 3 and Figure 5. Furthermore, Tables 1 and 2 present a detailed comparison between the exact solution and the approximate solution, as well as the absolute differences observed.

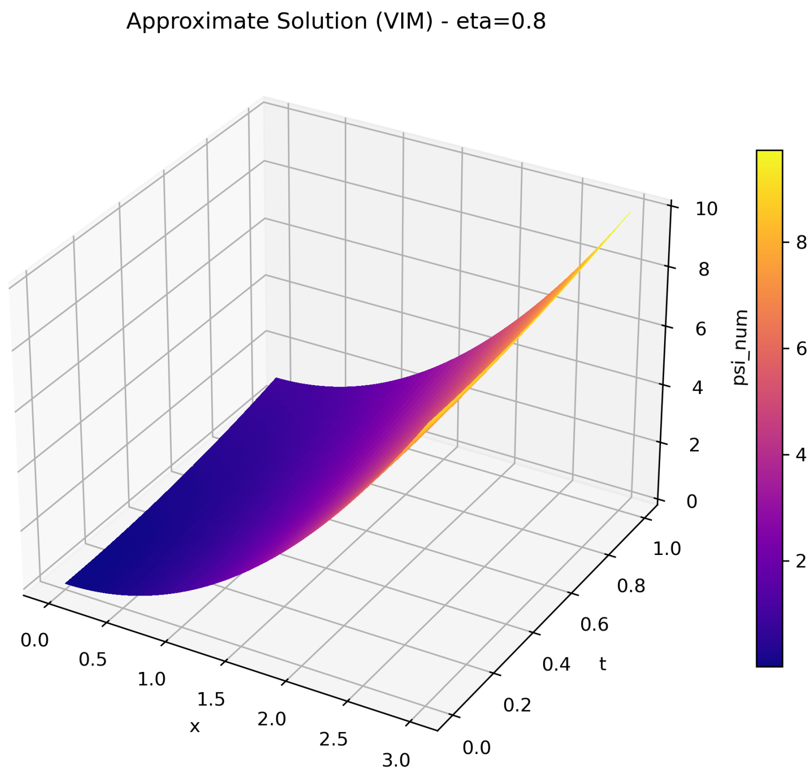

Figure 1. Approximate solution obtained by the VIM method for $\eta = 0.8$.

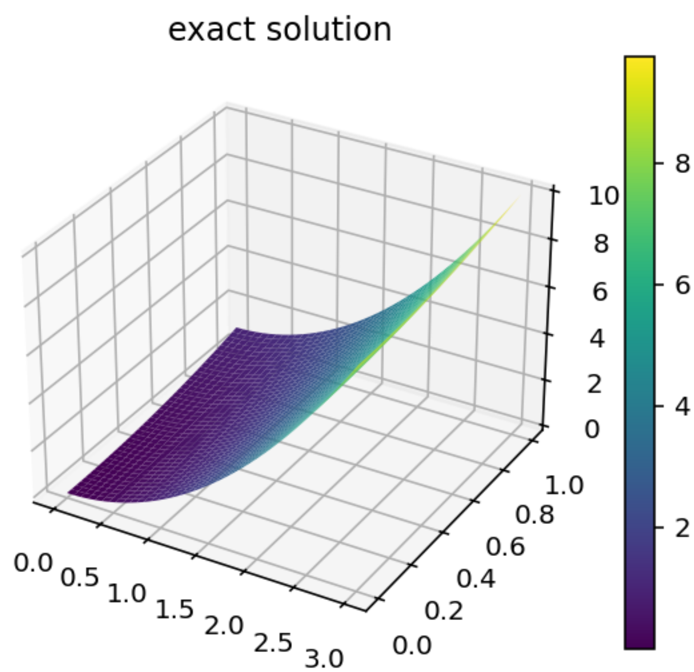

FIGURE 2. Exact solution

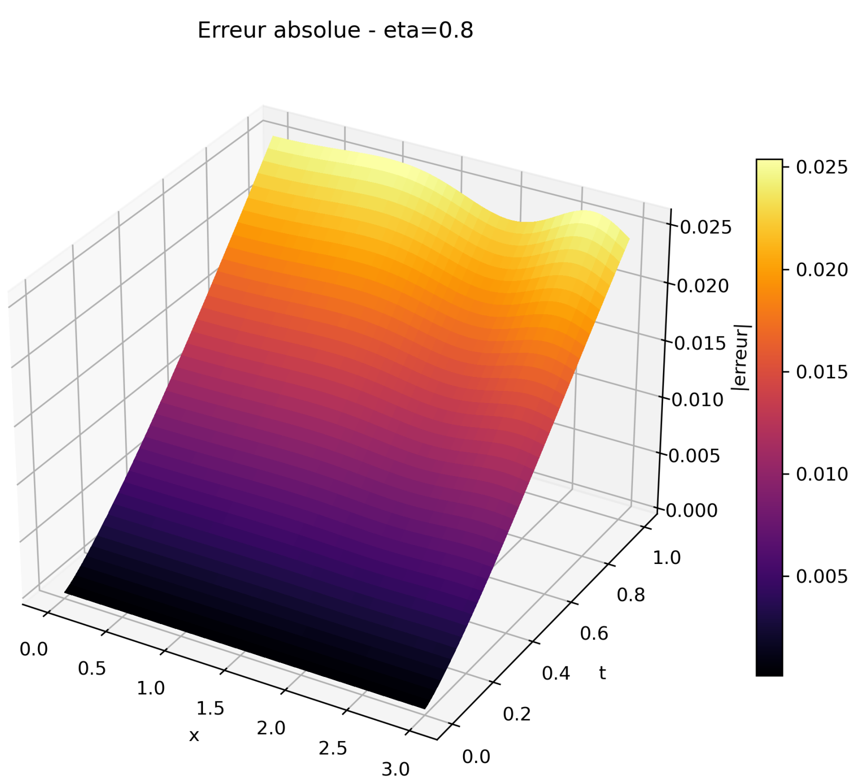

FIGURE 3. Error curve for $\eta = 0.8$

Measurement points		Values of $u(x,t)$		Deviation
Abscissa x	Time t	Exact solution	Approximate solution	absolute
0.000	0.000000	0.000000	0.000000	0.000000
0.325	0.108333	0.117361	0.115652	0.001710
0.675	0.225000	0.506250	0.502074	0.004176
1.000	0.333333	1.111111	1.104316	0.006796
1.325	0.441667	1.950694	1.941083	0.009611
1.675	0.558333	3.117361	3.104887	0.012474
2.000	0.666667	4.444444	4.429642	0.014803
2.325	0.775000	6.006250	5.988385	0.017865
2.675	0.891667	7.950694	7.928235	0.022459
3.000	1.000000	10.000000	9.975779	0.024221

TABLE 1. Comparison between exact solution and approximate solution for $\eta = 0.8$

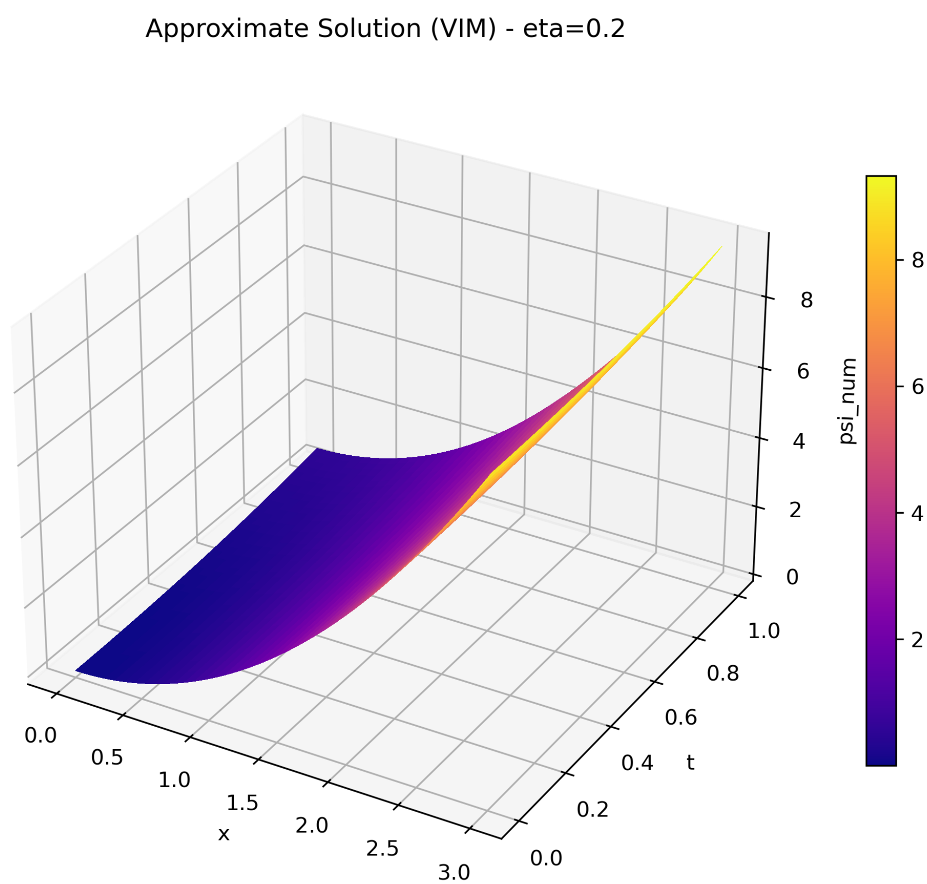

FIGURE 4. Approximate solution obtained by the VIM method for $\eta = 0.2$.

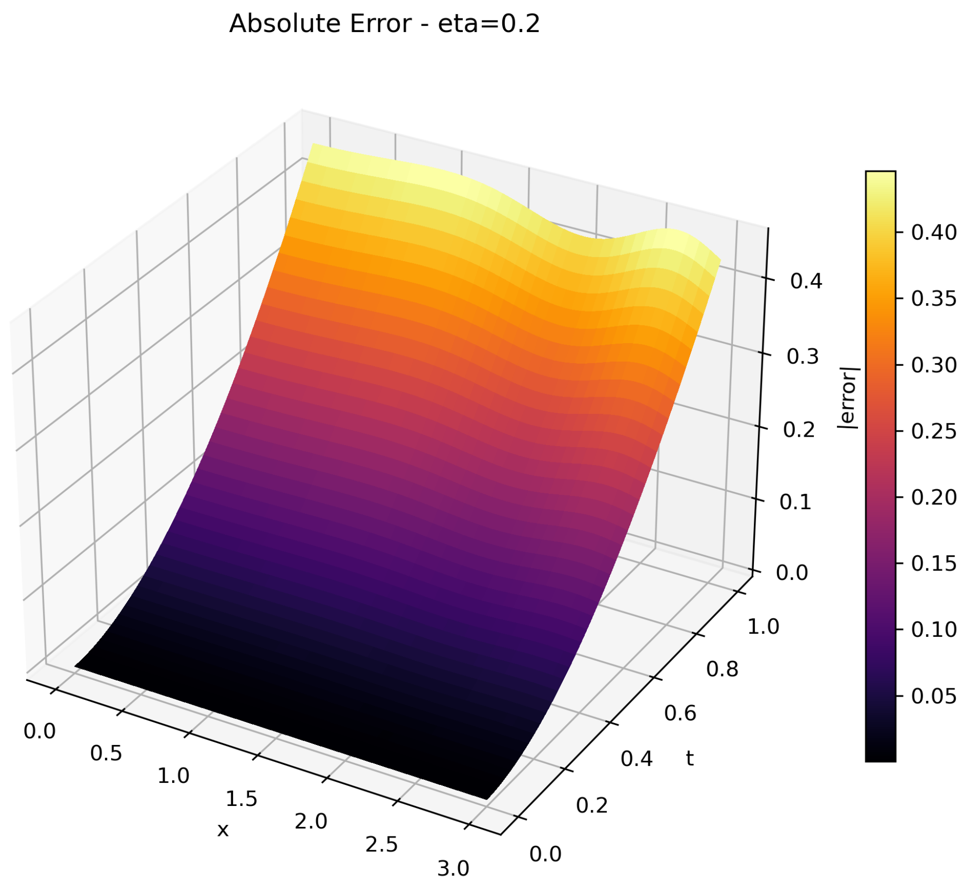

FIGURE 5. Error curve for $\eta = 0.2$

Measurement points		Values of $u(x,t)$		Deviation
Abscissa x	Time t	Exact solution	Approximate solution	absolute
0.000	0.000000	0.000000	0.000000	0.000000
0.325	0.108333	0.117361	0.109266	0.008095
0.675	0.225000	0.506250	0.475500	0.030750
1.000	0.333333	1.111111	1.047634	0.063477
1.325	0.441667	1.950694	1.844684	0.106010
1.675	0.558333	3.117361	2.960315	0.157046
2.000	0.666667	4.444444	4.238018	0.206426
2.325	0.775000	6.006250	5.732801	0.273449
2.675	0.891667	7.950694	7.577967	0.372728
3.000	1.000000	10.000000	9.568366	0.431634

TABLE 2. Comparison between the exact solution and the approximate solution for $\eta = 0.2$

In order to evaluate the numerical stability of the iterative method, we will analyse the variations between two successive approximate solutions. Let $\mathscr{U}^{(j)}(x,t)$ be the solution obtained at iteration j. The error measures we use for our analysis are as follows:

$$\Delta_\infty^{(j)} = \max_{(x,t)\in[a,b]\times[0,T]} \left| \mathscr{U}^{(j)}(x,t) - \mathscr{U}^{(j-1)}(x,t) \right|,$$

$$\Delta_2^{(j)} = \left(\int_0^T \int_\alpha^\beta \left(\mathscr{U}^{(j)}(x,t) - \mathscr{U}^{(j-1)}(x,t) \right)^2 dx\, dt \right)^{1/2},$$

$$\Delta_{\text{moy}}^{(j)} = \frac{1}{(b-a)T} \int_0^T \int_a^b \left| \mathscr{U}^{(j)}(x,t) - \mathscr{U}^{(j-1)}(x,t) \right| dx\, dt.$$

The numerical results obtained for $\eta = 0.2$ and $\eta = 0.8$ show very rapid convergence behaviour of the variational iteration method (VIM). In both cases, the values of the errors L_∞, L_2 and the mean error decrease monotonically and exponentially over the iterations. For $\eta = 0.8$, Table 3, the maximum error decreases from approximately 10^0 in the first iteration to values of the order of 10^{-15} in the ninth iteration, indicating extremely rapid convergence towards the exact solution. Similarly, for $\eta = 0.2$, Table 4, the maximum error decreases by a factor of 10^{-1} at the first iteration to approximately 10^{-15} at the tenth iteration. The L_2 norms and average errors follow the same decreasing trend, reaching orders of magnitude close to 10^{-15} and 10^{-16} respectively after only a few iterations. This steady and significant decrease in errors confirms the numerical stability and efficiency of the proposed method for solving the fractional integro-differential problem under consideration. Furthermore, the similarity of convergence behaviours for two different values of the fractional order η shows that the VIM method remains robust and effective regardless of the value of the fractional parameter. These results demonstrate the method's strong ability to produce highly accurate approximations.

Iteration	Δ_∞ error	Δ_2 error	Mean error
1	1.013432×10^{0}	7.504295×10^{-1}	3.181786×10^{-1}
2	4.235154×10^{-2}	2.040630×10^{-2}	7.367308×10^{-3}
3	2.593759×10^{-3}	4.965370×10^{-4}	1.410336×10^{-4}
4	3.665961×10^{-5}	6.947809×10^{-6}	1.817844×10^{-6}
5	3.401206×10^{-7}	6.777210×10^{-8}	1.969596×10^{-8}
6	2.809287×10^{-9}	7.215074×10^{-10}	2.053938×10^{-10}
7	3.085177×10^{-11}	8.334257×10^{-12}	2.329228×10^{-12}
8	3.295142×10^{-13}	8.775441×10^{-14}	2.342807×10^{-14}
9	3.552714×10^{-15}	9.224375×10^{-16}	2.231252×10^{-16}
10	1.776357×10^{-15}	1.159245×10^{-16}	5.463532×10^{-18}

TABLE 3. Error evolution for $\eta = 0.8$

Iteration	Δ_∞ Error	Δ_2 Error	Mean error
1	6.044576×10^{-1}	4.123715×10^{-1}	1.704951×10^{-1}
2	2.837752×10^{-2}	1.398139×10^{-2}	5.136155×10^{-3}
3	1.095923×10^{-3}	3.210250×10^{-4}	1.112481×10^{-4}
4	1.735241×10^{-5}	8.406002×10^{-6}	2.995379×10^{-6}
5	4.568740×10^{-7}	2.136694×10^{-7}	7.261125×10^{-8}
6	1.212485×10^{-8}	5.623461×10^{-9}	1.875328×10^{-9}
7	3.196701×10^{-10}	1.470888×10^{-10}	4.756745×10^{-11}
8	8.294698×10^{-12}	3.742868×10^{-12}	1.183074×10^{-12}
9	2.109424×10^{-13}	9.251634×10^{-14}	2.859446×10^{-14}
10	5.329071×10^{-15}	2.252065×10^{-15}	6.679215×10^{-16}

Table 4. Error evolution for $\eta = 0.2$

8. Conclusion

This work establishes an analytical and numerical framework for the study of fractional-order mixed Fredholm-Volterra integro-differential equations. The existence and uniqueness of the solution are guaranteed by Banach's fixed point theorem, while stability in the sense of Ulam-Hyers ensures the well-posedness of the model. In digital terms, the variational iteration method (VIM) shows rapid convergence and high accuracy. These results confirm the effectiveness and robustness of the VIM method for analysing this class of equations and open up prospects for its application to more complex fractional models.

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