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LIMIT CYCLES IN A CLASS OF WHITTAKER-HILL TYPE EQUATIONS

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Abstract. In this work, we establish an explicit upper bound on the maximum number of limit cycles that can bifurcate from the periodic solutions of the linear oscillator $\ddot{x}+x=0$. We consider a class of perturbations governed by a Whittaker–Hill-type differential equation. By reformulating the analysis as an equivalent planar first-order system, we apply the first-order averaging method to investigate the bifurcation of limit cycles. As a result, a precise upper bound on the number of such limit cycles is obtained.

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1. Introduction

The study of planar differential systems plays a central role in the qualitative theory of dynamical systems, where a key object of interest is the limit cycle. A limit cycle is a closed, isolated trajectory representing a stable or unstable periodic solution in the phase space. Unlike general periodic orbits, limit cycles do not form continuous families; this isolation makes them particularly significant in understanding nonlinear dynamics. The concept of limit cycles originated in the foundational work of Henri Poincaré at the end of the 19th century [25]. Alongside this notion, Poincaré introduced several groundbreaking tools that remain essential in modern dynamical systems theory, such as the phase portrait, which provides a global qualitative view of solution behaviors, and the Poincaré map. This return mechanism captures the behavior of trajectories near periodic orbits (see [11,14]). In 1900, David Hilbert formulated a collection of 23 unsolved problems to chart the future course of mathematical research. Among these, the latter half of the famous sixteenth problem posed by Hilbert [8] focuses on establishing a supreme bound for the quantity of limit cycles associated with any given two-dimensional polynomial vector field of a specified order. Despite extensive progress, this problem remains open and is considered one of the most profound unsolved questions in mathematics, alongside the Riemann conjecture. A classical and effective method for constructing limit cycles involves perturbing integrable

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systems, particularly those possessing a center, where all nearby orbits are closed curves. Under small perturbations, some of these closed orbits may undergo bifurcation and evolve into isolated limit cycles, a process extensively analyzed using techniques from bifurcation theory and perturbation analysis, (see for instance [1,3,21,23,24,26]).

George William Hill (1838–1914) introduced the second-order differential equation

$$\ddot{x} + P(t)x = 0, \tag{1}$$

in his work [9], where P is a time-dependent function that repeats periodically. This class of equations, now commonly referred to as Hill-type equations, plays a vital role in various scientific and engineering disciplines (see [6,10,12,16,17]). Such equations arise in contexts ranging from celestial mechanics and electrical circuits to the study of metallic conductivity and particle accelerators like cyclotrons. The Mathieu equation [15], distinguished by a solitary periodic cosine term, alongside the Whittaker–Hill equation [22], whose potential function contains two such terms, represent prominent specific variations of the general Hill equation. A notable generalization of Hill's equation is the Whittaker–Hill equation, which introduces additional harmonic terms into the potential. It takes the form:

$$\ddot{x} + (A_1 + A_2\cos(2\theta) + A_3\cos(4\theta))\,x = 0, \tag{2}$$

where A_1, A_2, A_3 are constants, and θ is a real-valued independent variable. This equation describes systems with doubly periodic coefficients and arises in the analysis of vibrations and stability in mechanical and electrical systems.The Whittaker–Hill equation extends the classical Mathieu equation by incorporating a higher-order cosine term, thus enriching the structure of its stability regions and eigenvalue spectrum. It has been extensively discussed by F.M. Arscott; see [19], [20].

Chen and Llibre, in their work [5], analyzed a class of differential equations that extend the classical Mathieu equation. They considered the perturbed system given by

$$\ddot{x} + \epsilon(\cos^m(\theta) + 1)P(x, y) + x = 0, \tag{3}$$

In this context, ϵ represents a sufficiently small positive constant, while P is defined as an n-th order polynomial. Their study focused on how isolated closed orbits emerge from the unperturbed trajectories of the harmonic oscillator $\ddot{x} + x = 0$ when subjected to the disturbance given in equation 3. Ultimately, the researchers determined a theoretical ceiling for the quantity of emerging limit cycles across various scenarios, which are governed by whether the polynomial degrees n and m share an odd or even status.

This paper investigates the emergence of limit cycle bifurcations within a perturbed generalized Whittaker–Hill equation, formulated as follows

$$\ddot{x} + \varepsilon\left(1 + \cos^n(\theta) + \cos^n(2\theta)\right)\psi(x, y) + x = 0, \tag{4}$$

here, ε serves as a sufficiently small positive constant, $\psi(x, y)$ is defined as an m-th order polynomial, and n acts as an integer greater than or equal to zero. Furthermore, one can recast the aforementioned second-order model into an equivalent two-dimensional first-order system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon\left(1 + \cos^n(\theta) + \displaystyle\sum_{k=0}^{n}\binom{n}{k}(-1)^k 2^{n-k}\cos^{2(n-k)}(\theta)\right)\psi(x, y), \end{cases} \tag{5}$$

In this expression, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ stands for the classic binomial combinatorial factor, with θ defined as $\arctan\left(\dfrac{y}{x}\right)$. In this study, our goal is to deduce the highest possible count of limit cycles emerging via bifurcation from the natural periodic solutions of the unforced center $\dot{x} = y, \dot{y} = -x$, driven by the perturbing influence outlined in equation (4). By transforming this equation into an equivalent planar system and applying averaging theory, we derive conditions involving the degree m of the polynomial ψ and the parity of the parameter n, yielding sharp estimates for the maximal number of limit cycles.

2. Preliminary tools

This section outlines the principal mathematical tools underpinning the main results of this study. Foremost among these is averaging theory, which constitutes the primary analytical framework adopted herein. Averaging theory focuses on determining whether solutions with a period of T occur in T-periodic systems of differential equations that are perturbed by a small parameter ϵ. The foundational contributions to this theory are presented in [4] and [13]. A formal summary of the theory is provided below.

Theorem 2.1 (First-Order Averaging Theory)**.** *Consider the differential system*

$$\dot{x}(t) = \epsilon F_1(t, x) + \epsilon^2 R(t, x, \epsilon), \tag{6}$$

Here, x belongs to an open region $D \subset \mathbb{R}^n$, and time $t \geq 0$. Suppose that F_1, R, along with their specific partial derivatives $\frac{\partial F_1}{\partial x}$, $\frac{\partial^2 F_1}{\partial x^2}$, and $\frac{\partial R}{\partial x}$, exist, are continuous, and remain uniformly bounded by some constant M. This bound M does not depend on ϵ over the set $[0, \infty) \times D$, provided $0 \leq \epsilon \leq \epsilon_0$. Moreover, $R(t, x, \epsilon)$ and $F_1(t, x)$ share a period T in the variable t, where T is unaffected by ϵ.

Define the averaged function *as*

$$F(x) = \frac{1}{T}\int_0^T F_1(t, x)\, dt. \tag{7}$$

Assume we can find a specific τ inside D for which $F(\tau) = 0$, with the Jacobian matrix satisfying

$$\left|\frac{\partial F}{\partial x}(\tau)\right| \neq 0. \tag{8}$$

Consequently, provided that $|\epsilon|$ is adequately small, system (6) possesses a trajectory of period T, denoted by $x(t,\epsilon)$, which satisfies

$$\lim_{\varepsilon\to 0} x(0,\varepsilon) = \tau.$$

Furthermore, the stability characteristics of the periodic cycle are dictated by the eigenvalues derived from the Jacobian matrix $\frac{\partial F}{\partial x}\big|_{x=\tau}$. The system achieves asymptotic stability provided that the real components of all these eigenvalues are strictly negative. Conversely, the emergence of instability occurs if a positive real part exists in any single eigenvalue.

Consequently, the starting values yielding T-periodic solutions within the perturbed system (6) are directly identified by the simple zeros of the averaged mapping F.

For a more detailed discussion of averaging theory and using it in the context of periodic solution detection, see [18].

In our work, Descartes' Rule of Signs is used to evaluate the simple roots of the averaged function.

Theorem 2.2. [2]. *Consider a polynomial whose coefficients are real numbers*

$$f(r) = a_{j_1} r_1^{j_1} + a_{j_2} r_2^{j_2} + a_{j_3} r_3^{j_3} .... + a_{j_n} r_n^{j_n}.$$

where $0 \le j_1 < j_2 < \cdots < j_n$, and each a_{j_i} represents a non-zero real parameter for $i \in \{1,2,\ldots,n\}$. A sign variation between consecutive coefficients a_{j_i} and $a_{j_{i+1}}$ is defined to occur whenever their product is negative ($a_{j_i} a_{j_{i+1}} < 0$). Assuming m denotes the total count of these sign alterations, the polynomial $f(r)$ will possess no more than m strictly positive real roots. Furthermore, it is always feasible to assign values to the coefficients of $f(r)$ to ensure the existence of precisely $n-1$ positive real zeros.

To facilitate the proof of Theorem (3.1), we first state the following indispensable formulas.

$$\int_0^{2\pi} \cos^i \eta \sin^j \eta d\eta = 0,$$

assuming i or j takes an odd value, and

$$M_{i,j}(2\pi) = \int_0^{2\pi} \cos^i \eta \sin^j \eta d\eta = \frac{(j-1)!!\,(i-1)!!}{2^{\frac{i}{2}}\left(\frac{i}{2}\right)(i+j)(j+i-2)\ldots(i+2)},$$

in the case that i and$j$ are even, see [7].

3. The main result

The principal findings we established in this work are detailed below.

Theorem 3.1. *Assume a small parameter, $\varepsilon > 0$. Consider system (5) as a perturbation applied to the harmonic center defined by $\dot{x} = y$, $\dot{y} = -x$. According to the principles of first-order averaging, identifying the isolated* *roots of the associated averaged equation is equivalent to locating the limit cycles bifurcating from the unperturbed periodic orbits.*

(a) *If both parameters m and n hold odd values, then the number of bifurcating limit cycles does not exceed $m-1$.*
(b) *If m is odd and n is even, then this number is at most $\frac{m-1}{2}$.*
(c) *If n is odd while m remains even, then this number is at most m.*
(d) *If both n and m are even, then this number is at most $\frac{m}{2}-1$.*

Proof. We write

$$\psi(x,y)=\sum_{j+i=0}^{m}\omega_{i,j}x^{i}y^{j},$$

then system (5), under the change of coordinates to polar form $(r,\theta), r>0$ takes the form

$$\begin{cases}\dot{r}=-\varepsilon H_1(r,\theta)\\ \dot{\theta}=-1-\varepsilon H_2(r,\theta),\end{cases} \tag{9}$$

in which

$$\begin{aligned}H_1(r,\theta) &= \sum_{j+i=0}^{m}\omega_{i,j}r^{i+j}\sin^{j+1}(\theta)\cos^{i}(\theta)+\sum_{j+i=0}^{m}\omega_{i,j}r^{j+i}\cos^{i+n}(\theta)\sin^{j+1}(\theta)\\ &\quad+\sum_{j+i=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{i,j}(-1)^{k}2^{n-k}r^{i+j}\sin^{j+1}(\theta)\cos^{i+2(n-k)}(\theta)\\ H_2(r,\theta) &= \sum_{i+j=0}^{m}\omega_{i,j}r^{i+j-1}\sin^{j}(\theta)\cos^{j+1}(\theta)\\ &\quad+\sum_{j+i=0}^{m}\omega_{i,j}r^{i+j-1}\cos^{j+n+1}(\theta)\sin^{j}(\theta)\\ &\quad+\sum_{j+i=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{i,j}(-1)^{k}2^{n-k}r^{i+j-1}\cos^{2(n-k)+j+1}(\theta)\sin^{j}(\theta)\end{aligned}$$

By adopting θ to serve as the primary independent variable, the governing system (9) is systematically recast, yielding the following mathematical expression

$$\frac{dr}{d\theta}=\varepsilon F_1(r,\theta)+O\left(\varepsilon^{2}\right), \tag{10}$$

with

$$F_1(r,\theta)=H_1(r,\theta).$$

To explore the limit cycles governed by equation (10), we employ the first-order averaging method. Consequently, relying on the theoretical framework established in Section 2, the problem simplifies to determining the simple positive zeros of the associated function.

$$F(r)=\frac{1}{2\pi}\int_{0}^{2\pi}F_1\left(r,\theta\right)d\theta, \tag{11}$$

The analysis is divided into two primary cases depending on the parity of m; each of these cases is then further subdivided according to whether n is even or odd.

Case (1). When m takes on odd values, we proceed by studying the averaged function through two distinct subcases.

Subcase (1-1). For n odd, we find:

$$\begin{aligned}
2\pi F(r) &= \int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j}r^{i+j}\cos^{i+n}(\theta)\sin^{j+1}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{j+i=0}^{m}\sum_{k=0}^{n}\binom{n}{k}(-1)^k2^{n-k}\omega_{i,j}r^{i+j}\sin^{j+1}(\theta)\cos^{i+2(n-k)}(\theta)\,d\theta\\
&=\int_0^{2\pi}\sum_{j+2p=2}^{m+1}\omega_{2p-1,j}r^{2p-1+j}\sin^{j+1}(\theta)\sin^{j+1}cos^{2p-1+n}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{j+2p=0}^{m}\omega_{2p,j}r^{2p+j}\cos^{2p}(\theta)\sin^{j+1}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{j+2p=0}^{m}\sum_{k=0}^{n}\binom{n}{k}(-1)^k2^{n-k}\omega_{2p,j}r^{j+2p}\sin^{j+1}(\theta)\cos^{2(n-k+p)}(\theta)\,d\theta\\
&=\int_0^{2\pi}\sum_{1+2p+2q=3}^{m}\omega_{2p-1,2q+1}r^{2p+2q}\sin^{2q+2}(\theta)\cos^{2p-1+n}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{1+2p+2q=1}^{m}\omega_{2p,2q+1}r^{2p+2q+1}\sin^{2q+2}(\theta)\cos^{2p}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{1+2p+2q=1}^{m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}\omega_{2p,2q+1}r^{1+2p+2q}\sin^{2q+2}(\theta)\cos^{2(n-k+p)}(\theta)\,d\theta\\
&=\int_0^{2\pi}\sum_{2p+2q=2}^{m-1}\omega_{2p-1,2q+1}r^{2p+2q}\sin^{2q+2}(\theta)\cos^{2p-1+n}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{2(p+q)=2}^{m+1}\omega_{2p,2q+1}r^{2p+2q+1}\sin^{2q+2}(\theta)\cos^{2p}(\theta)\,d\theta\\
&+\int_0^{2\pi}\sum_{2(p+q)=2}^{m+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k2^{n-k}\omega_{2p,2q+1}r^{2p+2q+1}\sin^{2q+2}(\theta)\cos^{2(n-k+p)}(\theta)\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p+q=1}^{\frac{m-1}{2}} \omega_{2p-1,2q+1} r^{2p+2q} M_{2p-1+n,2q+2}(2\pi) \\
&\quad + \sum_{p+q=1}^{\frac{m+1}{2}} \omega_{2p,2q+1} r^{2p+2q+1} M_{2p,2q+2}(2\pi) \\
&\quad + \sum_{p+q=1}^{\frac{m+1}{2}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} \omega_{2p,2q+1} r^{2p+2q+1} M_{2p+2,2q+2(n-k)}(2\pi) \\
&= \sum_{s=1}^{m} K_s r^s.
\end{aligned}
$$

Therefore, relying upon the fundamental theorems of first-order averaging presented in Section 2, it is deduced that the highest possible quantity of limit cycles produced by this differential system cannot exceed $m-1$ whenever $\varepsilon > 0$ acts as a tiny perturbation parameter.

Subcase (1.2). When n is even, it follows that

$$
\begin{aligned}
2\pi F(r) &= \int_0^{2\pi} \sum_{j+i=0}^{m} \omega_{i,j} r^{j+i} \sin^{1+j}(\theta) \cos^{i}(\theta)\, d\theta + \int_0^{2\pi} \sum_{j+i=0}^{m} \omega_{i,j} r^{j+i} \cos^{i+n}(\theta) \sin^{j+1}(\theta)\, d\theta \\
&\quad + \int_0^{2\pi} \sum_{j+i=0}^{m} \sum_{k=0}^{n} \binom{n}{k} \omega_{i,j} (-1)^k 2^{n-k} r^{i+j} \sin^{j+1}(\theta) \cos^{i+2(n-k)}(\theta)\, d\theta \\
&= \int_0^{2\pi} \sum_{i+2q-1=0}^{m} \omega_{i,2q-1} r^{2q-1+i} \sin^{2q}(\theta) \cos^{i}(\theta)\, d\theta \\
&\quad + \int_0^{2\pi} \sum_{i+2q-1=0}^{m} \omega_{i,2q-1} r^{2q-1+i} \sin^{2q}(\theta) \cos^{i+n}(\theta)\, d\theta \\
&\quad + \int_0^{2\pi} \sum_{i+2q-1=0}^{m} \sum_{k=0}^{n} \binom{n}{k} \omega_{i,2q-1} (-1)^k 2^{n-k} r^{i+2q-1} \sin^{2q}(\theta) \cos^{i+2(n-k)}(\theta)\, d\theta \\
&= \int_0^{2\pi} \sum_{2(p+q)=2}^{m+1} \omega_{2p,2q-1} r^{2p+2q-1} \sin^{2q}(\theta) \cos^{2p}(\theta)\, d\theta \\
&\quad + \int_0^{2\pi} \sum_{2(p+q)=2}^{m+1} \omega_{2p,2q-1} r^{2p+2q-1} \sin^{2q}(\theta) \cos^{2p+n}(\theta)\, d\theta \\
&\quad + \int_0^{2\pi} \sum_{2(p+q)=2}^{m+1} \sum_{k=0}^{n} \binom{n}{k} \omega_{2p,2q-1} (-1)^k 2^{n-k} r^{2p+2q-1} \sin^{2q}(\theta) \cos^{2p+2(n-k)}(\theta)\, d\theta \\
&= \sum_{p+q=1}^{\frac{m+1}{2}} \omega_{2p,2q-1} r^{2p+2q-1} M_{2p,2q}(2\pi) + \sum_{q+p=1}^{\frac{m+1}{2}} \omega_{2p,2q-1} r^{2p+2q-1} M_{2p+n,2q}(2\pi)
\end{aligned}
$$

$$+\sum_{p+q=1}^{\frac{m+1}{2}}\sum_{k=0}^{n}\binom{n}{k}\omega_{2p,2q-1}(-1)^k 2^{n-k} r^{2p+2q-1} M_{2p+2(n-k),2q}(2\pi)$$
$$=\sum_{s=1}^{\frac{m+1}{2}} T_s r^{2s-1}.$$

In this case, the averaged expression $F(r)$ arises from a linear span of the odd-degree terms $\{r, r^3, \ldots, r^{\frac{m+1}{2}}\}$. By invoking Descartes' Rule of Signs, provided the value of m is odd whereas n is even, the maximum number of strictly positive real roots for $F(r)$ is bounded by $\frac{m+1}{2}$. As a result, utilizing the first-order averaging framework detailed in Section 2, it follows that the governed equations possess no more than $\frac{m-1}{2}$ limit cycles for sufficiently small values of the positive perturbation ε.

Case (2). If m is an even, the analysis of the averaged function's structure is divided into two distinct subcases.

Subcase (2-1). Assuming n is odd, we deduce

$$\begin{aligned}
2\pi F(r) &= \int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j} r^{i+j}\sin^{j+1}(\theta)\cos^{i}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j} r^{i+j}\sin^{j+1}(\theta)\cos^{i+n}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{j+i=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{i,j}(-1)^k 2^{n-k} r^{i+j}\sin^{j+1}(\theta)\cos^{i+2(n-k)}(\theta)\,d\theta \\
&= \int_0^{2\pi}\sum_{j+2p=0}^{m}\omega_{2p,j} r^{j+2p}\sin^{j+1}(\theta)\cos^{2p}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{j+2p=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{2p,j}(-1)^k 2^{n-k} r^{2p+j}\sin^{j+1}(\theta)\cos^{2p+2(n-k)}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{2p+1+j=0}^{m}\omega_{2p+1,j} r^{2p+1+j}\cos^{2p+1+n}(\theta)\sin^{j+1}(\theta)\,d\theta \\
&= \int_0^{2\pi}\sum_{2(p+q)=2}^{m}\omega_{2p,2q-1} r^{2p+2q-1}\cos^{2p}(\theta)\sin^{2q}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{2(p+q)=2}^{m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\omega_{2p,2q-1}2^{n-k} r^{2p+2q-1}\sin^{2q}(\theta)\cos^{2p+2(n-k)}(\theta)\,d\theta \\
&+\int_0^{2\pi}\sum_{2(q+p)=2}^{m+2}\omega_{2p+1,j} r^{2p+2q}\cos^{2p+1+n}(\theta)\sin^{2q}(\theta)\,d\theta \\
&= \sum_{q+p=1}^{\frac{m}{2}}\omega_{2p,2q-1} r^{2p+2q-1} M_{2p,2q}(2\pi)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q+p=1}^{\frac{m}{2}}\sum_{k=0}^{n}\binom{n}{k}\omega_{2p,2q-1}(-1)^{k}2^{n-k}r^{2p+2q-1}M_{2p+2(n-k),2q}(2\pi)\\
&+\sum_{p+q=1}^{\frac{m+2}{2}}\omega_{2p+1,j}r^{2p+2q}M_{2p+1+n,2q}(2\pi)\\
&=\sum_{s=1}^{m+1}R_s r^s.
\end{aligned}
$$

In this case, the polynomial form $F(r)$ is constructed as a linear superposition of the basis monomials $\{r, r^2, \ldots, r^{m+1}\}$. Based on Descartes' Rule of Signs, it can be established that under the conditions where m holds an even value whereas n is odd, $F(r)$ possesses a maximum of $m+1$ positive real roots. Consequently, by employing the averaging method of the first order, we deduce that the considered differential system admits at most m limit cycles for sufficiently small values of $\varepsilon > 0$.

Subcase (2-2). Let n be even. Then

$$
\begin{aligned}
2\pi F(r) &= \int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j}r^{i+j}\sin^{j+1}(\theta)\cos^{i}(\theta)d\theta+\int_0^{2\pi}\sum_{j+i=0}^{m}\omega_{i,j}r^{i+j}\cos^{i+n}(\theta)\sin^{j+1}(\theta)d\theta\\
&+\int_0^{2\pi}\sum_{j+i=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{i,j}\,(-1)^{k}\,2^{n-k}r^{i+j}\sin^{j+1}(\theta)\cos^{i+2(n-k)}(\theta)d\theta\\
&= \int_0^{2\pi}\sum_{i+2q-1=0}^{m}\omega_{i,2q-1}r^{2q-1+i}\sin^{2q}(\theta)\cos^{i}(\theta)d\theta\\
&+\int_0^{2\pi}\sum_{i+2q-1=0}^{m}\omega_{i,2q-1}r^{2q-1+i}\sin^{2q}(\theta)\cos^{i+n}(\theta)d\theta\\
&+\int_0^{2\pi}\sum_{i+2q-1=0}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{i,j}\,(-1)^{k}\,2^{n-k}r^{i+2q-1}\sin^{2q}(\theta)\cos^{i+2(n-k)}(\theta)d\theta\\
&= \int_0^{2\pi}\sum_{2(p+q)=2}^{m}\omega_{2p,2q-1}r^{2p+2q-1}\sin^{2q}(\theta)\cos^{2p}(\theta)d\theta\\
&+\int_0^{2\pi}\sum_{2(p+q)=2}^{m}\omega_{2p,2q-1}r^{2p+2q-1}\sin^{2q}(\theta)\cos^{2p+n}(\theta)d\theta\\
&+\int_0^{2\pi}\sum_{2(p+q)=2}^{m}\sum_{k=0}^{n}\binom{n}{k}\omega_{2p,2q-1}(-1)^{k}2^{n-k}r^{2p+2q-1}\sin^{2q}(\theta)\cos^{2p+2(n-k)}(\theta)d\theta\\
&= \sum_{q+p=1}^{\frac{m}{2}}\omega_{2p,2q-1}r^{2p+2q-1}M_{2p,2q}(2\pi)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{q+p=1}^{\frac{m}{2}} \omega_{2p,2q-1} r^{2p+2q-1} M_{2p+n,2q}(2\pi) \\
&+\sum_{p+q=1}^{\frac{m}{2}} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \omega_{2p,2q-1} \left(-1\right)^{k} 2^{n-k} r^{2p+2q-1} M_{2p+2(n-k),2q}(2\pi) \\
= &\sum_{s=1}^{\frac{m}{2}} L_s r^{2s-1}.
\end{aligned}
$$

From (2.2), the resulting averaged form $F(r)$ is constructed as a linear expansion of the monomials in the set $\Gamma = \{r, r^3, \ldots, r^{\frac{m}{2}}\}$. By Descartes' Rule of Signs, when both m and n are even, the maximum number of positive real zeros belonging to $F(r)$ is at most $\frac{m}{2}$. Applying the principles of first-order averaging presented in Section 2, we conclude that whenever $\varepsilon > 0$ is adequately small, the differential setup supports no more than $\frac{m}{2} - 1$ limit cycles bifurcating from the origin. Consequently, the theorem (3.1) is proved. □

4. Conclusions

This study is devoted to the investigation of limit cycles associated with a generalized Whittaker–Hill equation. By converting the second-order equation into an equivalent planar first-order system and employing the first-order averaging technique, we derive an explicit upper bound for the number of limit cycles that can bifurcate from the periodic orbits of the underlying linear center system $\dot{x} = y,\ \dot{y} = -x$. The obtained results provide further insight into the bifurcation behavior of Hill-type differential equations and their qualitative dynamics.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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