

# ON $r$ -FUZZY SEMI-DELTA-OPEN SETS IN ŠOSTAK'S FUZZY TOPOLOGICAL SPACES

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**ABSTRACT.** This article deals with a new type of fuzzy open sets in Šostak's fuzzy topological space  $(X, \mathcal{T})$ , called  $r$ -fuzzy semi-delta-open sets for  $r \in I_0$ . An  $r$ -fuzzy semi-delta-open set  $A \in I^X$  or  $r$ - $\delta_{fs}$ -open set, is defined such that, for each fuzzy point  $x_r \in A$ , there exists an  $r$ -fuzzy open set  $A_0$  containing  $x_r$  satisfying  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A_0, r), r) \leq A$ . Some properties of  $r$ - $\delta_{fs}$ -open set are established, including the corresponding  $r$ - $\delta_{fs}$ -interior,  $r$ - $\delta_{fs}$ -closure, and basis. In addition, the functions  $r$ - $\delta_{fs}$ -open functions,  $r$ - $\delta_{fs}$ -continuous, and  $r$ - $\delta_{fs}$ -homeomorphisms are introduced, and some of their properties in Šostak's fuzzy topological spaces are established.

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## 1. INTRODUCTION

Šostak [1] introduced fuzzy topology as an extension of Chang's fuzzy topology [2]. Since then, it has been developed in many directions within the field of fuzzy topology. The notion of  $r$ -fuzzy semi-open sets was defined and studied by S. J. Lee and E. P. Lee [3] in Šostak's fuzzy topological spaces. More recently, in 2024, Singh and Gupta [4] introduced the concept of semi-delta-open sets in topological spaces. In this study, the researcher introduces a new type of  $r$ -fuzzy open set called an  $r$ -fuzzy semi-delta-open set in Šostak's fuzzy topological spaces by using the idea of semi-delta-open sets in classical topological spaces. Some of its properties are established, and its corresponding  $r$ -fuzzy semi-delta-interior,  $r$ -fuzzy semi-delta-closure, and basis are defined along with some of their properties. Moreover, this study introduces the  $r$ -fuzzy semi-delta-open functions,  $r$ -fuzzy semi-delta-continuous functions, and  $r$ -fuzzy semi-delta-homeomorphism, in Šostak's fuzzy topological spaces, and some of their properties are likewise be established.

## 2. PRELIMINARIES

Throughout this paper, nonempty sets will be denoted by  $X, Y$ , and  $Z, I = [0, 1]$  and  $I_0 = (0, 1]$ . A fuzzy set  $A$  of  $X$  is a mapping  $A : X \rightarrow I$ , and  $I^X$  denotes the family of all fuzzy sets in  $X$ . For  $\alpha \in I$ ,  $\bar{\alpha}(x) = \alpha$  for all  $x \in X$ . For any  $A \in I^X$ ,  $A^c$  denotes the complement  $\bar{1} - A$ . A fuzzy point  $x_t$  is an element of  $I^X$  such that

$$x_t(y) = \begin{cases} t, & y = x, \\ 0, & y \neq x. \end{cases}$$

for all  $y \in X$ . The set of all fuzzy points in  $X$  is denoted by  $Pt(X)$ . A fuzzy point  $x_t$  is said to be an element of a fuzzy set  $A$ , denoted by  $x_t \in A$ , if and only if  $t < A(x)$ . If  $A \subseteq X$ , the characteristic function  $\chi_A$  on  $X$  is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

**Definition 1.** [2] Let  $X$  be a nonempty set. A *Chang's fuzzy topology* on  $X$  is a family  $\tau$  of fuzzy sets which satisfies the following conditions:

- (i)  $\bar{0}, \bar{1} \in \tau$ .
- (ii) If  $A_1, A_2 \in \tau$ , then  $A_1 \wedge A_2 \in \tau$ .
- (iii) If  $\{A_\omega\}_{\omega \in \Omega} \subseteq \tau$ , then  $\bigvee_{\omega \in \Omega} A_\omega \in \tau$ .

The pair  $(X, \tau)$  is called a *Chang's fuzzy topological space*.

**Definition 2.** [1] A function  $\mathcal{T} : I^X \rightarrow I$  is called a *Šostak's fuzzy topology on  $X$*  if it satisfies the following conditions:

- (i)  $\mathcal{T}(\bar{0}) = \mathcal{T}(\bar{1}) = 1$ ,
- (ii)  $\mathcal{T}(\bigvee_{\omega \in \Omega} A_\omega) \geq \bigwedge_{\omega \in \Omega} \mathcal{T}(A_\omega)$ , for any  $\{A_\omega\}_{\omega \in \Omega} \subseteq I^X$ ,
- (iii)  $\mathcal{T}(A_1 \wedge A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)$ , for any  $A_1, A_2 \in I^X$ .

The pair  $(X, \mathcal{T})$  is called a *Šostak's fuzzy topological space*, abbreviated as SFTS.

**Remark 3.** [5] Let  $(X, \mathcal{T})$  be a SFTS. Then, for every  $r \in I_0$ ,  $\mathcal{T}_r = \{V \in I^X : \mathcal{T}(V) \geq r\}$  is a Chang's fuzzy topology on  $X$ .

**Definition 4.** [3] Let  $(X, \mathcal{T})$  be a SFTS. For  $r \in I_0$  and  $A \in I^X$ ,  $A$  is said to be

- (i) *r-fuzzy open set*, denoted by *r-f-open set*, if  $\mathcal{T}(A) \geq r$ ; and
- (ii) *r-fuzzy closed set*, denoted by *r-f-closed set*, if  $\mathcal{T}(A^c) \geq r$ .

**Definition 5.** [3] Let  $(X, \mathcal{T})$  be a SFTS. For each  $r \in I_0$  and  $A \in I^X$ , the *r-fuzzy interior* is defined by  $I_{\mathcal{T}}(A, r) = \bigvee\{V \in I^X : V \leq A, \mathcal{T}(V) \geq r\}$ .

**Definition 6.** [3] Let  $(X, \mathcal{T})$  be a SFTS. For each  $r \in I_0$  and  $A \in I^X$ , the  $r$ -fuzzy closure is defined by  $C_{\mathcal{T}}(A, r) = \bigwedge \{V \in I^X : A \leq V, \mathcal{T}(V^c) \geq r\}$ .

**Theorem 7.** [6] Let  $(X, \mathcal{T})$  be a SFTS. For  $A, B \in I^X$  and  $r, s \in I_0$ , the  $r$ -fuzzy closure,  $C_{\mathcal{T}}$ , satisfies the following conditions:  $C_{\mathcal{T}}(\bar{0}, r) = \bar{0}$ ,  $A \leq C_{\mathcal{T}}(A, r)$ ,  $C_{\mathcal{T}}(A, r) \vee C_{\mathcal{T}}(B, r) = C_{\mathcal{T}}(A \vee B, r)$ ,  $C_{\mathcal{T}}(A, r) \leq C_{\mathcal{T}}(A, s)$  if  $r \leq s$ , and  $C_{\mathcal{T}}(C_{\mathcal{T}}(A, r), r) = C_{\mathcal{T}}(A, r)$ .

**Theorem 8.** [6] Let  $(X, \mathcal{T})$  be a SFTS. For  $A, B \in I^X$  and  $r, s \in I_0$ , the  $r$ -fuzzy Interior,  $I_{\mathcal{T}}$ , satisfies the following conditions:  $I_{\mathcal{T}}(\bar{1}, r) = \bar{1}$ ,  $I_{\mathcal{T}}(A, r) \leq A$ ,  $I_{\mathcal{T}}(A, r) \wedge I_{\mathcal{T}}(B, r) = I_{\mathcal{T}}(A \wedge B, r)$ ,  $I_{\mathcal{T}}(A, r) \leq I_{\mathcal{T}}(A, s)$  if  $s \leq r$ ,  $I_{\mathcal{T}}(I_{\mathcal{T}}(A, r), r) = I_{\mathcal{T}}(A, r)$ ,  $I_{\mathcal{T}}(A^c, r) = (C_{\mathcal{T}}(A, r))^c$ , and  $C_{\mathcal{T}}(A^c, r) = (I_{\mathcal{T}}(A, r))^c$ .

**Definition 9.** [3] Let  $(X, \mathcal{T})$  be a SFTS. Let  $A \in I^X$  and  $r \in I_0$ . Then  $A$  is said to be an

- (i)  $r$ -fuzzy semiopen set if  $A \leq C_{\mathcal{T}}(I_{\mathcal{T}}(A, r), r)$ , and
- (ii)  $r$ -fuzzy semiclosed set if  $I_{\mathcal{T}}(C_{\mathcal{T}}(A, r), r) \leq A$ .

**Definition 10.** [3] Let  $(X, \mathcal{T})$  be a SFTS. For each  $r \in I_0$  and  $A \in I^X$ , the  $r$ -fuzzy semiinterior is defined by  $sI_{\mathcal{T}}(A, r) = \bigvee \{V \in I^X \mid V \leq A, V \text{ is } r\text{-fuzzy semiopen}\}$ .

**Definition 11.** [3] Let  $(X, \mathcal{T})$  be a SFTS. For each  $r \in I_0$  and  $A \in I^X$ , the  $r$ -fuzzy semiclosure is defined by  $sC_{\mathcal{T}}(A, r) = \bigwedge \{V \in I^X \mid A \leq V, V \text{ is } r\text{-fuzzy semiclosed}\}$ .

### 3. SOME RESULTS OF $r$ - $\delta_{fs}$ -OPEN SET

**Definition 12.** Let  $(X, \mathcal{T})$  be a SFTS,  $x \in X$ ,  $A \in I^X$ , and  $r \in I_0$ . Then  $A$  is said to be an  $r$ -fuzzy semi-delta-open set, denoted by  $r$ - $\delta_{fs}$ -open set, if for all fuzzy point  $x_r \in A$ , there exists an  $r$ -fuzzy open set  $A_0$  containing  $x_r$ , such that  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A_0, r), r) \leq A$ .

A fuzzy set  $A \in I^X$  is said to be an  $r$ - $\delta_{fs}$ -closed set if and only if  $A^c$  is an  $r$ - $\delta_{fs}$ -open set in SFTS.

**Example 13.** Let  $(X, \mathcal{T})$  be a SFTS, where  $X = \{x, y, z\}$  and  $A_1, A_2, A_3, A_4 \in I^X$  be fuzzy sets of  $X$  defined as follows:

$$\begin{aligned} A_1(x) = 0.4, A_1(y) = 0.5, A_1(z) = 0.2; & \quad A_3(x) = 0.7, A_3(y) = 0.4, A_3(z) = 0.8; \\ A_2(x) = 0.6, A_2(y) = 0.6, A_2(z) = 0.8; & \quad A_4(x) = 0.6, A_4(y) = 0.4, A_4(z) = 0.8, \end{aligned}$$

and the function  $\mathcal{T} : I^X \rightarrow I$  is defined as follows:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A = A_1, \\ \frac{1}{4}, & \text{if } A = A_2, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $r = \frac{1}{4}$  and choose the fuzzy set  $A_1$ . By Definition 12,  $A_1$  is a  $\frac{1}{4}$ - $\delta_{fs}$ -open set in  $X$ .

**Remark 14.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then  $\bar{0}$  and  $\bar{1}$  are  $r$ - $\delta_{fs}$ -open in  $X$ .

**Theorem 15.** Let  $(X, \mathcal{T})$  be a SFTS,  $A, B \in I^X$ ,  $x \in X$ , and  $r \in I_0$ . If  $A \leq B$  and  $A$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ , then  $B$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .

*Proof.* Let  $A \leq B$  and  $A$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ . Then by Definition 12, for all fuzzy point  $x_r \in A$ , there exists an  $r$ - $f$ -open set  $A_0$  containing  $x_r$  such that  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A_0, r), r) \leq A \leq B$ . Thus,  $B$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .  $\square$

**Theorem 16.** Let  $(X, \mathcal{T})$  be a SFTS,  $A \in I^X$ , and a fixed  $r \in I_0$ . Then  $A$  is an  $r$ - $\delta_{fs}$ -open set if and only if for every fuzzy point  $x_r \in A$ , there exists a nonempty family  $\mathcal{U}_{x_r} \subseteq I^X$  of  $r$ -fuzzy open sets such that  $x_r \in U$  for all  $U \in \mathcal{U}_{x_r}$ , and  $\bigvee_{U \in \mathcal{U}_{x_r}} I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \leq A$ .

*Proof.* Let  $A \in I^X$  be an  $r$ - $\delta_{fs}$ -open set and  $x_r \in A$ . By Definition 12, there exists an  $r$ -fuzzy open set  $U_0$  containing  $x_r$  such that  $I_{\mathcal{T}}(sC_{\mathcal{T}}(U_0, r), r) \leq A$ . Define  $\mathcal{U}_{x_r} = \{U \in I^X : U \text{ is an } r\text{-fuzzy open set, } x_r \in U, I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \leq A\}$ . Clearly,  $\mathcal{U}_{x_r} \neq \emptyset$  since  $U_0 \in \mathcal{U}_{x_r}$ . Note that for every  $U \in \mathcal{U}_{x_r}$ ,  $I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \leq A$ . Hence,  $\bigvee_{U \in \mathcal{U}_{x_r}} I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \leq A$ .

Conversely, assume that for every fuzzy point  $x_r \in A$ , there exists a nonempty family  $\mathcal{U}_{x_r} \subseteq I^X$  of  $r$ - $f$ -open sets such that  $x_r \in U$  for all  $U \in \mathcal{U}_{x_r}$ , and  $\bigvee_{U \in \mathcal{U}_{x_r}} I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \leq A$ . Let  $x_r \in A$ . Since  $\mathcal{U}_{x_r} \neq \emptyset$ , choose any  $U_0 \in \mathcal{U}_{x_r}$ . By hypothesis,  $x_r \in U_0$  and

$$\begin{aligned} I_{\mathcal{T}}(sC_{\mathcal{T}}(U_0, r), r) &\leq \bigvee_{U \in \mathcal{U}_{x_r}} I_{\mathcal{T}}(sC_{\mathcal{T}}(U, r), r) \\ &\leq A. \end{aligned}$$

More precisely,  $A$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ .  $\square$

**Theorem 17.** Let  $(X, \mathcal{T})$  be a SFTS,  $A \in I^X$ ,  $x \in X$ , and  $r \in I_0$ . If for each  $r$ - $f$ -open set  $A$  of  $X$  satisfies  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A, r), r) \leq A$ , then for each  $r$ - $f$ -open set of  $X$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .

*Proof.* Let  $A$  be an  $r$ - $f$ -open set of  $X$  and it satisfies  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A, r), r) \leq A$ . Now,  $x_r \in A$  implies  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A, r), r) \leq A$ . By Definition 12, this shows that  $A$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .  $\square$

**Theorem 18.** Let  $(X, \mathcal{T})$  be a SFTS,  $A \in I^X$ ,  $r \in I_0$ . If  $\{A_{\omega}\}_{\omega \in \Omega}$  is a family of  $r$ - $\delta_{fs}$ -open sets of  $X$ , then  $\bigvee_{\omega \in \Omega} A_{\omega}$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .

*Proof.* Let  $\{A_{\omega}\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -open sets of  $X$ . Note that  $A_{\omega}$  is an  $r$ - $\delta_{fs}$ -open, then for any  $\omega \in \Omega$ . By Definition 12, for all fuzzy points  $x_r \in A_{\omega}$ , where  $x \in X$ , there exists an  $r$ - $f$ -open set  $A_0$  such that  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A_0, r), r) \leq A_{\omega}$ . Now, since  $x_r \in A_{\omega}$  and  $A_{\omega} \leq \bigvee_{\omega \in \Omega} A_{\omega}$ ,

$$r \leq A_{\omega}(x) \leq \sup_{\omega \in \Omega} \{A_{\omega}(x) : x \in X\}$$

$$= \bigvee_{\omega \in \Omega} A_{\omega}(x).$$

So,  $x_r \in \bigvee_{\omega \in \Omega} A_{\omega}$ . Thus, for all  $x_r \in \bigvee_{\omega \in \Omega} A_{\omega}$ , there exists an  $r$ - $f$ -open set  $A_0$  such that  $I_{\mathcal{T}}(sC_{\mathcal{T}}(A_0, r), r) \leq \bigvee_{\omega \in \Omega} A_{\omega}$ . As a result,  $\bigvee_{\omega \in \Omega} A_{\omega}$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .  $\square$

**Theorem 19.** Let  $(X, \mathcal{T})$  be a SFTS,  $A \in I^X$ ,  $r \in I_0$ . If  $\{A_{\omega}\}_{\omega \in \Omega}$  is a family of  $r$ - $\delta_{fs}$ -closed sets of  $X$ , then  $\bigwedge_{\omega \in \Omega} A_{\omega}$  is an  $r$ - $\delta_{fs}$ -closed set of  $X$ .

*Proof.* Let  $\{A_{\omega}\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -closed sets of  $X$ . Since  $A_i$  is an  $r$ - $\delta_{fs}$ -closed for all  $\omega \in \Omega$ , by Definition 12 and Theorem 18,  $A_{\omega}^c$  is  $r$ - $\delta_{fs}$ -open for all  $\omega \in \Omega$  and  $\bigvee_{\omega \in \Omega} A_{\omega}^c$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ . Thus,  $(\bigvee_{\omega \in \Omega} A_{\omega}^c)^c = \bigwedge_{\omega \in \Omega} A_{\omega}$  is an  $r$ - $\delta_{fs}$ -closed set of  $X$ .  $\square$

**Definition 20.** Let  $(X, \mathcal{T})$  be SFTS and  $r \in (0, 1]$ . A collection  $\mathcal{B} \subseteq [0, 1]^X$  of  $r$ - $\delta_{fs}$ -open sets in  $X$  is called a *basis* if every  $r$ - $\delta_{fs}$ -open set  $A \in [0, 1]^X$  can be expressed as  $A = \bigvee_{B \in \mathcal{B}'} B$ , where  $\mathcal{B}' \subseteq \mathcal{B}$ .

**Example 21.** Let  $(X, \mathcal{T})$  be a SFTS, where  $X = \{x, y\}$  and  $A_1, A_2, A_3, A_4 \in I^X$  be fuzzy sets of  $X$  defined as follows:

$$A_1(a) = 0.2, A_1(b) = 0.1; \quad A_3(a) = 0.7, A_3(b) = 0.7;$$

$$A_2(a) = 0.3, A_2(b) = 0.5; \quad A_4(a) = 0.2, A_4(b) = 0.8,$$

and the function  $\mathcal{T} : I^X \rightarrow I$ , is defined as follows:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A \in \{A_1, A_2, A_3\} \\ 0, & \text{otherwise.} \end{cases}$$

Let  $r = \frac{1}{2}$  and  $\mathcal{B} \subseteq I^X$  be the collection of all  $\frac{1}{2}$ - $\delta_{fs}$ -open sets in  $(X, \mathcal{T})$ . Then by Definition 20,  $\mathcal{B}$  is a basis of  $(X, \mathcal{T})$ .

**Remark 22.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Let  $\mathcal{B}$  be a basis consisting of  $r$ - $\delta_{fs}$ -open sets in  $X$ . Then every  $r$ - $\delta_{fs}$ -open set can be expressed as a supremum of some elements of  $\mathcal{B}$

**Theorem 23.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{T}$  if for all  $r$ - $\delta_{fs}$ -open set  $U \in [0, 1]^X$ , a fuzzy point  $x_r \in U$ , where  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x_r \in B \leq U$ .

*Proof.* Let  $U \in I^X$  be an  $r$ - $\delta_{fs}$ -open set in  $(X, \mathcal{T})$  and a fuzzy point  $x_r \in U$ , where  $x \in X$ . Then, by Remark 22,  $U = \bigvee_{B \in \mathcal{B}} B$ . So, there exists  $B \in \mathcal{B}$  such that  $x_r \in B \leq U$ .  $\square$

**Definition 24.** Let  $(X, \mathcal{T})$  be a SFTS. For each  $r \in I_0$  and  $A \in I^X$ , (i) the  $r$ -fuzzy semi-delta-interior, denoted by  $r$ - $\delta_{fs}I_{\mathcal{T}}$ , is defined by

$$\delta_{fs}I_{\mathcal{T}}(A, r) = \bigvee \{V \in I^X : V \leq A, V \text{ is an } r\text{-}\delta_{fs}\text{-open}\}$$

and (ii) the  $r$ -fuzzy semi-delta-closure, denoted by  $r\text{-}\delta_{fs}C_{\mathcal{T}}$ , is defined by

$$\delta_{fs}C_{\mathcal{T}}(A, r) = \bigwedge \{V \in I^X : A \leq V, V \text{ is an } r\text{-}\delta_{fs}\text{-closed}\}.$$

**Example 25.** Let  $(X, \mathcal{T})$  be a SFTS, where  $X = \{x, y, z\}$  and  $A_1, A_2, A_3, A_4 \in I^X$  be fuzzy sets of  $X$  defined as follows:

$$A_1(x) = 0.4, A_1(y) = 0.5, A_1(z) = 0.2; \quad A_3(x) = 0.7, A_3(y) = 0.4, A_3(z) = 0.8;$$

$$A_2(x) = 0.6, A_2(y) = 0.6, A_2(z) = 0.8; \quad A_4(x) = 0.6, A_4(y) = 0.4, A_4(z) = 0.8;$$

and the function  $\mathcal{T} : I^X \rightarrow I$  is defined as follows:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A = A_1, \\ \frac{1}{4}, & \text{if } A = A_2, \\ 0, & \text{otherwise} \end{cases}$$

Let  $r = \frac{1}{4}$ . (i) Choose  $A_1^c$ . Then by Definition 24 (i),  $\delta_{fs}I_{\mathcal{T}}(A_1^c, \frac{1}{4}) = A_1$ . (ii) Choose  $A_2^c$ . Then by Definition 24 (ii),  $\delta_{fs}C_{\mathcal{T}}(A_2^c, \frac{1}{4}) = A_4$ .

**Theorem 26.** Let  $(X, \mathcal{T})$  be a SFTS. Let  $A \in I^X$  and  $r \in I_0$ . Then  $A$  is an  $r\text{-}\delta_{fs}$ -open set of  $X$  if and only if  $A = \delta_{fs}I_{\mathcal{T}}(A, r)$ .

*Proof.* Let  $A \in I^X$  be an  $r\text{-}\delta_{fs}$ -open set and  $\mathcal{U}$  be the family of  $r\text{-}\delta_{fs}$ -open sets in  $X$  contained in  $A$ . Since  $A \leq A$ ,  $A \in \mathcal{U}$ . By Definition 24 (i),  $A = \bigvee \mathcal{U} = \delta_{fs}I_{\mathcal{T}}(A, r)$ .

Conversely, let  $A = \delta_{fs}I_{\mathcal{T}}(A, r)$  and let  $\mathcal{U}$  be the family of  $r\text{-}\delta_{fs}$ -open sets in  $X$  contained in  $A$ . Then  $\bigvee \mathcal{U} = \delta_{fs}I_{\mathcal{T}}(A, r)$ . So,  $A = \bigvee \mathcal{U}$  and  $A \in \mathcal{U}$ . Hence,  $A$  is an  $r\text{-}\delta_{fs}$ -open set of  $X$ .  $\square$

**Theorem 27.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then for every  $A \in I^X$ ,  $r\text{-}\delta_{fs}I_{\mathcal{T}}(A, r)$  is an  $r\text{-}\delta_{fs}$ -open set in  $X$ .

*Proof.* Let  $A \in I^X$  and  $r \in I_0$ . Let  $\{U_{\omega}\}_{\omega \in \Omega}$  be a family of  $r\text{-}\delta_{fs}$ -open sets in  $X$  such that  $U_{\omega} \leq A$  for each  $\omega \in \Omega$ . Then  $\delta_{fs}I_{\mathcal{T}}(A, r) = \bigvee_{U_{\omega} \leq A} U_{\omega}$ . Hence, by Theorem 18,  $\delta_{fs}I_{\mathcal{T}}(A, r)$  is an  $r\text{-}\delta_{fs}$ -open set in  $X$ .  $\square$

**Theorem 28.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . For every  $A, B \in I^X$ , the following hold:

- (i)  $\delta_{fs}I_{\mathcal{T}}(A, r) \leq A$ ;
- (ii) if  $A \leq B$ , then  $\delta_{fs}I_{\mathcal{T}}(A, r) \leq \delta_{fs}I_{\mathcal{T}}(B, r)$ ;
- (iii)  $\delta_{fs}I_{\mathcal{T}}(\delta_{fs}I_{\mathcal{T}}(A, r), r) = \delta_{fs}I_{\mathcal{T}}(A, r)$ ; and
- (iv)  $\delta_{fs}I_{\mathcal{T}}(A \wedge B, r) \leq \delta_{fs}I_{\mathcal{T}}(A, r) \wedge \delta_{fs}I_{\mathcal{T}}(B, r)$ .

*Proof.*

- (i) Let  $A \in I^X$  be a fuzzy set in  $X$  and  $\{V_\omega\}_{\omega \in \Omega}$  be the collection of  $r$ - $\delta_{fs}$ -open sets of  $X$  such that  $V \leq A$ . This implies

$$\begin{aligned} \delta_{fs}I_{\mathcal{T}}(A, r) &= \bigvee_{\substack{\omega \in \Omega \\ V \leq A}} V \\ &\leq A. \end{aligned}$$

- (ii) Let  $A \leq B$ . Let  $O_A$  be the set of all  $r$ - $\delta_{fs}$ -open sets contained in  $A$  and  $O_B$  be the set of all  $r$ - $\delta_{fs}$ -open sets contained in  $B$ . Suppose that  $V \in O_A$ , then  $V \leq A$ , and since  $A \leq B$ ,  $V \leq B$  and  $V \in O_B$ . This implies that  $O_A \subseteq O_B$ , and hence,

$$\begin{aligned} \delta_{fs}I_{\mathcal{T}}(A, r) &= \bigvee O_A \\ &\leq \bigvee O_B \\ &= \delta_{fs}I_{\mathcal{T}}(B, r). \end{aligned}$$

- (iii) Let  $A \in I^X$ . Then by Theorem 27,  $\delta_{fs}I_{\mathcal{T}}(A, r)$  is an  $r$ - $\delta_{fs}$ -open. This implies by Theorem 26,

$$\delta_{fs}I_{\mathcal{T}}(\delta_{fs}I_{\mathcal{T}}(A, r), r) = \delta_{fs}I_{\mathcal{T}}(A, r).$$

- (iv) Let  $A, B \in I^X$ . Since  $A \wedge B \leq A$  and  $A \wedge B \leq B$ , by Theorem 28 (ii),  $\delta_{fs}I_{\mathcal{T}}(A \wedge B, r) \leq \delta_{fs}I_{\mathcal{T}}(A, r)$  and  $\delta_{fs}I_{\mathcal{T}}(A \wedge B, r) \leq \delta_{fs}I_{\mathcal{T}}(B, r)$ . Hence,

$$\delta_{fs}I_{\mathcal{T}}(A \wedge B, r) \leq \delta_{fs}I_{\mathcal{T}}(A, r) \wedge \delta_{fs}I_{\mathcal{T}}(B, r).$$

□

**Theorem 29.** Let  $(X, \mathcal{T})$  be a SFTS. Let  $A \in I^X$  and  $r \in I_0$ . Then  $A$  is an  $r$ - $\delta_{fs}$ -closed set of  $X$  if and only if  $A = \delta_{fs}C_{\mathcal{T}}(A, r)$ .

*Proof.* Let  $A \in I^X$  be an  $r$ - $\delta_{fs}$ -closed set and  $\mathcal{V}$  be the family of  $r$ - $\delta_{fs}$ -closed sets in  $X$  containing  $A$ . Since  $A \leq A$ ,  $A \in \mathcal{V}$ . By Definition 24 (ii),  $A = \bigwedge \mathcal{U} = \delta_{fs}C_{\mathcal{T}}(A, r)$ .

Conversely, let  $A = \delta_{fs}C_{\mathcal{T}}(A, r)$  and let  $\mathcal{V}$  be the family of  $r$ - $\delta_{fs}$ -closed sets in  $X$  containing  $A$ . Then  $\bigwedge \mathcal{U} = \delta_{fs}I_{\mathcal{T}}(A, r)$ . So,  $A = \bigwedge \mathcal{V}$  and  $A \in \mathcal{V}$ . Hence,  $A$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ . □

**Theorem 30.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then for every  $A \in I^X$ ,  $r$ - $\delta_{fs}C_{\mathcal{T}}(A, r)$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ .

*Proof.* Let  $A \in I^X$  and  $r \in I_0$ . Let  $\{U_\omega\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -closed sets in  $X$  such that  $A \leq U_\omega$  for each  $\omega \in \Omega$ . Then  $\delta_{fs}C_{\mathcal{T}}(A, r) = \bigwedge_{\substack{\omega \in \Omega \\ U_\omega \leq A}} U_\omega$ . Hence, by Theorem 19,  $\delta_{fs}C_{\mathcal{T}}(A, r)$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ . □

**Theorem 31.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in (0, 1]$ . For every  $A, B \in [0, 1]^X$ , the following hold:

- (i)  $A \leq \delta_{fs}C_{\mathcal{T}}(A, r)$ ;
- (ii) if  $A \leq B$ , then  $\delta_{fs}C_{\mathcal{T}}(A, r) \leq \delta_{fs}C_{\mathcal{T}}(B, r)$ ;
- (iii)  $\delta_{fs}C_{\mathcal{T}}(\delta_{fs}C_{\mathcal{T}}(A, r), r) = \delta_{fs}C_{\mathcal{T}}(A, r)$ ; and
- (iv)  $\delta_{fs}C_{\mathcal{T}}(A, r) \vee \delta_{fs}C_{\mathcal{T}}(B, r) \leq \delta_{fs}C_{\mathcal{T}}(A \vee B, r)$ .

*Proof.*

- (i) Let  $A \in I^X$  be a fuzzy set in  $X$  and  $\{V_{\omega}\}_{\omega \in \Omega}$  be the collection of  $r$ - $\delta_{fs}$ -closed sets of  $X$  such that  $V \geq A$ . This implies

$$\begin{aligned} A &\leq \bigwedge_{\substack{\omega \in \Omega \\ V \leq A}} V \\ &= \delta_{fs}C_{\mathcal{T}}(A, r). \end{aligned}$$

- (ii) Let  $A \leq B$ . Let  $C_A$  be the set of all  $r$ - $\delta_{fs}$ -closed sets containing  $A$  and  $C_B$  be the set of all  $r$ - $\delta_{fs}$ -closed sets containing  $B$ . Suppose that  $V \in C_B$ , then  $B \leq V$ , and since  $A \leq B$ ,  $A \leq V$  and  $V \in C_A$ . This implies that  $C_B \subseteq C_A$ , and hence,

$$\begin{aligned} \delta_{fs}C_{\mathcal{T}}(A, r) &= \bigwedge C_A \\ &\leq \bigwedge C_B \\ &= \delta_{fs}C_{\mathcal{T}}(B, r). \end{aligned}$$

- (iii) Let  $A \in I^X$ . By Theorem 30,  $\delta_{fs}C_{\mathcal{T}}(A, r)$  is an  $r$ - $\delta_{fs}$ -closed. This implies by Theorem 29,  $\delta_{fs}C_{\mathcal{T}}(\delta_{fs}C_{\mathcal{T}}(A, r), r) = \delta_{fs}C_{\mathcal{T}}(A, r)$ .
- (iv) Let  $A, B \in I^X$ . Since  $A \leq A \vee B$  and  $B \leq A \vee B$ , by Theorem 31 (ii),  $\delta_{fs}C_{\mathcal{T}}(A, r) \leq \delta_{fs}C_{\mathcal{T}}(A \vee B, r)$  and  $\delta_{fs}C_{\mathcal{T}}(B, r) \leq \delta_{fs}C_{\mathcal{T}}(A \vee B, r)$ . This implies that

$$\delta_{fs}C_{\mathcal{T}}(A, r) \vee \delta_{fs}C_{\mathcal{T}}(B, r) \leq \delta_{fs}C_{\mathcal{T}}(A \vee B, r).$$

□

**Remark 32.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then for each  $A \in I^X$ ,  $\delta_{fs}I_{\mathcal{T}}(A, r) \leq \delta_{fs}C_{\mathcal{T}}(A, r)$ .

**Theorem 33.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . For any family  $\{A_{\omega}\}_{\omega \in \Omega} \subseteq I^X$ ,

- (i)  $\bigvee_{\omega \in \Omega} \delta_{fs}C_{\mathcal{T}}(A_{\omega}, r) \leq \delta_{fs}C_{\mathcal{T}}\left(\bigvee_{\omega \in \Omega} A_{\omega}, r\right)$ ; and
- (ii)  $\delta_{fs}I_{\mathcal{T}}\left(\bigwedge_{\omega \in \Omega} A_{\omega}, r\right) \leq \bigwedge_{\omega \in \Omega} \delta_{fs}I_{\mathcal{T}}(A_{\omega}, r)$ .

*Proof.*

- (i) Let  $\{A_\omega\}_{\omega \in \Omega} \subseteq I^X$  be any family of fuzzy sets in  $X$  and  $r \in I_0$ . Note that for each  $\beta \in \Omega$ ,  $A_\beta \leq \bigvee_{\omega \in \Omega} A_\omega$ . By Theorem 31 (ii),  $\delta_{fs}C_{\mathcal{T}}(A_\beta, r) \leq \delta_{fs}C_{\mathcal{T}}(\bigvee_{\omega \in \Omega} A_\omega, r)$ . Consequently,

$$\bigvee_{\omega \in \Omega} \delta_{fs}C_{\mathcal{T}}(A_\omega, r) \leq \delta_{fs}C_{\mathcal{T}}\left(\bigvee_{\omega \in \Omega} A_\omega, r\right).$$

- (ii) Let  $\{A_\omega\}_{\omega \in \Omega} \subseteq I^X$  any family of fuzzy sets in  $X$  and  $r \in I_0$ . Note that for each  $\beta \in \Omega$ ,  $\bigwedge_{i \in I} A_\omega \leq A_\beta$ . By Theorem 28 (ii),  $\delta_{fs}I_{\mathcal{T}}(\bigwedge_{\omega \in \Omega} A_\omega, r) \leq \delta_{fs}I_{\mathcal{T}}(A_\beta, r)$ . As a result,

$$\delta_{fs}I_{\mathcal{T}}\left(\bigwedge_{\omega \in \Omega} A_\omega, r\right) \leq \bigwedge_{\omega \in \Omega} \delta_{fs}I_{\mathcal{T}}(A_\omega, r).$$

□

**Theorem 34.** Let  $(X, \mathcal{T})$  be a SFTS and  $r \in I_0$ . Then for every  $A \in I^X$ ,

- (i)  $\delta_{fs}I_{\mathcal{T}}(A^c, r) = (\delta_{fs}C_{\mathcal{T}}(A, r))^c$ ; and  
(ii)  $\delta_{fs}C_{\mathcal{T}}(A^c, r) = (\delta_{fs}I_{\mathcal{T}}(A, r))^c$ .

*Proof.*

- (i) Let  $A \in I^X$  be a fuzzy set in  $X$  and  $r \in I_0$ . Let  $\{V_\omega\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -closed sets in  $X$  and for each  $\omega \in \Omega$ ,  $V_\omega \geq A$ . Then,  $\delta_{fs}C_{\mathcal{T}}(A, r) = \bigwedge_{\substack{\omega \in \Omega \\ V_\omega \geq A}} V_\omega$ . Complementing both sides yield

$$\begin{aligned} (\delta_{fs}C_{\mathcal{T}}(A, r))^c &= \left( \bigwedge_{\substack{\omega \in \Omega \\ V_\omega \geq A}} V_\omega \right)^c \\ &= \bigvee_{\substack{\omega \in \Omega \\ V_\omega \leq A^c}} V_\omega^c, \end{aligned}$$

where  $V_\omega^c$  is an  $r$ - $\delta_{fs}$ -open set in  $X$  for each  $\omega \in \Omega$ . Now, let  $U_\omega = V_\omega^c \in I^X$  for each  $\omega \in \Omega$ . Then

$$\begin{aligned} (\delta_{fs}C_{\mathcal{T}}(A, r))^c &= \bigvee_{\substack{\omega \in \Omega \\ U_\omega \leq A^c}} U_\omega \\ &= \delta_{fs}I_{\mathcal{T}}(A^c, r) \end{aligned}$$

- (ii) Let  $A \in I^X$  be a fuzzy set in  $X$  and  $r \in I_0$ . Let  $U = A^c \in I^X$ , by Theorem 34 (i),  $\delta_{fs}I_{\mathcal{T}}(U^c, r) = (\delta_{fs}C_{\mathcal{T}}(U, r))^c$ . Taking complement of both sides then simplify,

$$\begin{aligned} (\delta_{fs}I_{\mathcal{T}}(U^c, r))^c &= ((\delta_{fs}C_{\mathcal{T}}(U, r))^c)^c \\ &= \delta_{fs}C_{\mathcal{T}}(U, r). \end{aligned}$$

As a result,

$$(\delta_{fs}I_{\mathcal{T}}((A, r))^c) = (\delta_{fs}I_{\mathcal{T}}((A^c)^c, r))^c$$

$$= \delta_{fs} C_{\mathcal{T}}(A^c, r).$$

□

#### 4. SOME RESULTS OF $r$ - $\delta_{fs}$ -OPEN FUNCTION, $r$ - $\delta_{fs}$ -CONTINUITY, AND $r$ - $\delta_{fs}$ -HOMEOMORPHISM

**Definition 35.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . Then  $f$  is called an  $r$ -fuzzy semi-delta-open function, denoted by  $r$ - $\delta_{fs}$ -open function, if  $f(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$  for each  $r$ - $\delta_{fs}$ -open set  $A$  in  $X$ .

**Example 36.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs, where  $X = Y = \{x, y, z\}$  and the fuzzy sets  $A_1, A_2, A_3 \in I^X, I^Y$  are defined by the following:

$$A_1(x) = 0.4, A_1(y) = 0.2, A_1(z) = 0.4;$$

$$A_2(x) = 0.5, A_2(y) = 0.5, A_2(z) = 0.4;$$

$$A_3(x) = 0.3, A_3(y) = 0.2, A_3(z) = 0.6.$$

And the functions  $\mathcal{T} : I^X \rightarrow I$  and  $\mathcal{N} : I^Y \rightarrow I$  are defined by the following:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A = A_3 \\ 0, & \text{otherwise.} \end{cases} \quad \mathcal{N}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{2}{3}, & \text{if } A = A_1, \\ \frac{1}{2}, & \text{if } A = A_2, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $r = \frac{1}{2}$  and  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be an identity function. Then by Definition 35,  $f$  is a  $\frac{1}{2}$ - $\delta_{fs}$ -open function.

**Theorem 37.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . Then  $f$  is an  $r$ - $\delta_{fs}$ -open function if and only if for every collection  $\{A_\omega\}_{\omega \in \Omega}$  of  $r$ - $\delta_{fs}$ -open sets of  $X$ ,  $f(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f(A_\omega)$  is an  $r$ - $\delta_{fs}$ -open in  $Y$ .

*Proof.* Let  $\{A_\omega\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -open sets of  $X$ . By Theorem 18,  $\bigvee_{\omega \in \Omega} A_\omega$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ . Since  $f$  is an  $r$ - $\delta_{fs}$ -open function, it follows that  $f(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f(A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set of  $Y$ .

Conversely, assume that for every family  $\{A_\omega\}_{\omega \in \Omega}$  of  $r$ - $\delta_{fs}$ -open sets in  $X$ ,  $\bigvee_{\omega \in \Omega} f(A_\omega) = f(\bigvee_{\omega \in \Omega} A_\omega)$  is an  $r$ - $\delta_{fs}$ -open in  $Y$ . Let  $\mathcal{B}$  be a basis of  $(X, \mathcal{T})$  consisting of  $r$ - $\delta_{fs}$ -open set in  $X$  and  $A \in I^X$  be any  $r$ - $\delta_{fs}$ -open set in  $X$ . Then, by Remark 22, there exists a subfamily  $\{A_\omega\}_{\omega \in \Omega} \subseteq \mathcal{B}$  such that  $A = \bigvee_{\omega \in \Omega} A_\omega$ . By hypothesis,  $f(\bigvee_{\omega \in \Omega} A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$  and  $f(A) = f(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f(A_\omega)$ . Indeed,  $f(A)$

is an  $r$ - $\delta_{fs}$ -open in  $Y$ . Since  $A$  is an arbitrary  $r$ - $\delta_{fs}$ -open set in  $X$ , by Definition 35,  $f$  is an  $r$ - $\delta_{fs}$ -open function.  $\square$

**Theorem 38.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . Then  $f$  is an  $r$ - $\delta_{fs}$ -open function if and only if for every  $A \in I^X$ ,

$$f(\delta_{fs}I_{\mathcal{T}}(A, r)) \leq \delta_{fs}I_{\mathcal{N}}(f(A), r).$$

*Proof.* Let  $f$  be an  $r$ - $\delta_{fs}$ -open function. Let  $A \in I^X$ . By Theorem 28 (i)  $\delta_{fs}I_{\mathcal{T}}(A, r) \leq A$ , and so,  $f(\delta_{fs}I_{\mathcal{T}}(A, r)) \leq f(A)$ . By Theorem 18,  $\delta_{fs}I_{\mathcal{T}}(A, r)$  is an  $\delta_{fs}$ -open set in  $X$ , and since  $f$  is an  $\delta_{fs}$ -open function,  $f(\delta_{fs}I_{\mathcal{T}}(A, r))$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ . Now, let

$$\mathcal{F} = \{V \in [0, 1]^Y : V \leq f(A), V \text{ is an } r\text{-}\delta_{fs}\text{-open set in } Y\}$$

Then  $f(\delta_{fs}I_{\mathcal{T}}(A, r)) \in \mathcal{F}$ , and hence,

$$\begin{aligned} f(\delta_{fs}I_{\mathcal{T}}(A, r)) &\leq \bigvee_{V \in \mathcal{F}} V \\ &= \delta_{fs}I_{\mathcal{N}}(f(A), r). \end{aligned}$$

Conversely, assume that for every  $A \in I^X$ ,  $f(\delta_{fs}I_{\mathcal{T}}(A, r)) \leq \delta_{fs}I_{\mathcal{N}}(f(A), r)$ . Let  $U \in I^X$  be an  $r$ - $\delta_{fs}$ -open set. By Theorem 26,  $\delta_{fs}I_{\mathcal{T}}(U, r) = U$ . So,

$$\begin{aligned} f(U) &= f(\delta_{fs}I_{\mathcal{T}}(U, r)) \\ &\leq \delta_{fs}I_{\mathcal{N}}(f(U), r) \end{aligned}$$

Theorem 28 (i) yields  $f(U) = \delta_{fs}I_{\mathcal{N}}(f(U), r)$ . By Theorem 27,  $f(U)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ . Since  $U \in I^X$  was an arbitrary chosen  $r$ - $\delta_{fs}$ -open set, it follows that by Definition 35,  $f$  is an  $r$ - $\delta_{fs}$ -open function.  $\square$

**Theorem 39.** Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{N})$ , and  $(Z, \mathcal{K})$  be a SFTSs, and  $r \in (0, 1]$ . If  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  is an  $r$ - $\delta_{fs}$ -open function and  $g : (Y, \mathcal{N}) \rightarrow (Z, \mathcal{K})$  is an  $r$ - $\delta_{fs}$ -open function, then the composition  $g \circ f$  is an  $r$ - $\delta_{fs}$ -open function.

*Proof.* Let  $A \in I^X$  be an  $r$ - $\delta_{fs}$ -open set. Since  $f$  is an  $r$ - $\delta_{fs}$ -open function,  $f(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ . Again, since  $g$  is an  $r$ - $\delta_{fs}$ -open function and  $f(A) \in I^Y$ ,  $g(f(A))$  is  $r$ - $\delta_{fs}$ -open in  $Z$ . Thus,  $g(f(A)) = (g \circ f)(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $Z$ , and hence  $g \circ f$  is an  $r$ - $\delta_{fs}$ -open function.  $\square$

**Definition 40.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs. Let  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function and  $r \in I_0$ . Then  $f$  is called a  $r$ -fuzzy semi-delta-continuous function, denoted by  $r$ - $\delta_{fs}$ -continuous function, if  $f^{-1}(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$  for each  $r$ - $\delta_{fs}$ -open set  $A$  in  $Y$ .

**Example 41.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs, where  $X = Y = \{x, y, z\}$ , and the fuzzy sets  $A_1, A_2, A_3, A_4 \in I^X$  and  $A_5, A_6 \in I^Y$  are defined by the following:

$$\begin{aligned} A_1(x) &= 0.4, A_1(y) = 0.5, A_1(z) = 0.2; & A_4(x) &= 0.6, A_4(y) = 0.4, A_4(z) = 0.8; \\ A_2(x) &= 0.6, A_2(y) = 0.6, A_2(z) = 0.8; & A_5(x) &= 0.4, A_5(y) = 0.2, A_5(z) = 0.5; \\ A_3(x) &= 0.7, A_3(y) = 0.4, A_3(z) = 0.8; & A_6(x) &= 0.6, A_6(y) = 0.8, A_6(z) = 0.6. \end{aligned}$$

And the functions  $\mathcal{T} : I^X \rightarrow I$  and  $\mathcal{N} : I^Y \rightarrow I$  are defined by the following:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A = A_1, \\ \frac{1}{4}, & \text{if } A = A_2, \\ 0, & \text{otherwise.} \end{cases} \quad \mathcal{N}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{4}, & \text{if } A \in \{A_5, A_6\} \\ 0, & \text{otherwise.} \end{cases}$$

Let  $r = \frac{1}{4}$  and  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be defined by  $f(x) = x, f(y) = z$ , and  $f(z) = y$ . Then by Definition 40,  $f$  is a  $\frac{1}{4}$ - $\delta_{fs}$ -continuous function.

**Theorem 42.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . Then  $f$  is an  $r$ - $\delta_{fs}$ -continuous if and only if for every family  $\{A_\omega\}_{\omega \in \Omega} \subseteq I^Y$  of  $r$ - $\delta_{fs}$ -open sets of  $Y$ ,  $f^{-1}(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f^{-1}(A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ .

*Proof.* Let  $\{A_\omega\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -open sets of  $Y$ . Then by Theorem 18,  $\bigvee_{\omega \in \Omega} A_\omega$  is an  $r$ - $\delta_{fs}$ -open set of  $Y$ . By continuity of  $f$ ,  $f^{-1}(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f^{-1}(A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set of  $X$ .

Conversely, assume that for every family  $\{A_\omega\}_{\omega \in \Omega}$  of  $r$ - $\delta_{fs}$ -open sets in  $Y$ ,  $\bigvee_{\omega \in \Omega} f^{-1}(A_\omega) = f^{-1}(\bigvee_{\omega \in \Omega} A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ . Let  $\mathcal{B}$  be a basis of  $(Y, \mathcal{N})$  consisting of  $r$ - $\delta_{fs}$ -open set in  $Y$  and  $A \in I^Y$  be any  $r$ - $\delta_{fs}$ -open set in  $Y$ . Then, by Remark 22, there exists a subfamily  $\{A_\omega\}_{\omega \in \Omega} \subseteq \mathcal{B}$  such that  $A = \bigvee_{\omega \in \Omega} A_\omega$ . By hypothesis,  $f^{-1}(\bigvee_{\omega \in \Omega} A_\omega)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$  and  $f^{-1}(A) = f^{-1}(\bigvee_{\omega \in \Omega} A_\omega) = \bigvee_{\omega \in \Omega} f^{-1}(A_\omega)$ . Indeed,  $f(A)$  is an  $r$ - $\delta_{fs}$ -open in  $X$ . Since  $A$  is an arbitrary  $r$ - $\delta_{fs}$ -open set in  $Y$ , by Definition 40,  $f$  is an  $r$ - $\delta_{fs}$ -continuous.  $\square$

**Theorem 43.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . If  $f$  is an  $r$ - $\delta_{sf}$ -continuous and a fuzzy set  $A \in I^Y$  is an  $r$ - $\delta_{fs}$ -closed set in  $Y$ , then  $f^{-1}(A)$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ .

*Proof.* Let  $f$  be an  $r$ - $\delta_{sf}$ -continuous and  $A \in I^Y$  be an  $r$ - $\delta_{fs}$ -closed set in  $Y$ . Then  $A^c$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ . Since  $f$  is an  $r$ - $\delta_{sf}$ -continuous,  $f^{-1}(A^c)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ . Note that  $f^{-1}(A^c) = (f^{-1}(A))^c$ . Thus,  $(f^{-1}(A))^c$  is an  $r$ - $\delta_{fs}$ -open fuzzy set in  $X$ . Thus, it follows that  $f^{-1}(A)$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ .  $\square$

**Theorem 44.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . If  $f$  is an  $r$ - $\delta_{fs}$ -continuous, then for every family  $\{A_\omega\}_{\omega \in \Omega}$  of  $r$ - $\delta_{fs}$ -closed sets of  $Y$ ,  $f^{-1}(\bigwedge_{\omega \in \Omega} A_\omega) = \bigwedge_{\omega \in \Omega} f^{-1}(A_\omega)$ , is an  $r$ - $\delta_{fs}$ -closed in  $X$ .

*Proof.* Let  $\{A_\omega\}_{\omega \in \Omega}$  be a family of  $r$ - $\delta_{fs}$ -closed sets of  $Y$ . Then by Theorem 19,  $\bigwedge_{\omega \in \Omega} A_\omega$  is an  $r$ - $\delta_{fs}$ -closed set of  $Y$ . Since  $f$  is an  $r$ - $\delta_{fs}$ -continuous, by Theorem 42,  $f^{-1}(\bigwedge_{\omega \in \Omega} A_\omega) = \bigwedge_{\omega \in \Omega} f^{-1}(A_\omega)$ , is an  $r$ - $\delta_{fs}$ -closed set of  $X$ .  $\square$

**Theorem 45.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . Then  $f$  is an  $r$ - $\delta_{fs}$ -continuous if and only if for every  $A \in I^Y$ ,

$$f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r)) \leq \delta_{fs}I_{\mathcal{T}}(f^{-1}(A), r).$$

*Proof.* Let  $f$  be an  $r$ - $\delta_{fs}$ -continuous and let  $A \in I^Y$ . By Theorem 28 (i),  $\delta_{fs}I_{\mathcal{N}}(A, r) \leq A$ . Then  $f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r)) \leq f^{-1}(A)$ . By Theorem 27,  $\delta_{fs}I_{\mathcal{N}}(A, r)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ , and since  $f$  is an  $r$ - $\delta_{fs}$ -continuous,  $f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r))$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ . Now, let

$$\mathcal{F} = \{V \in I^X : V \leq f^{-1}(A), V \text{ is an } r\text{-}\delta_{fs}\text{-open set in } X\}.$$

Then  $f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r)) \in \mathcal{F}$ . Hence,

$$\begin{aligned} f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r)) &\leq \bigvee_{V \in \mathcal{F}} V \\ &= \delta_{fs}I_{\mathcal{T}}(f^{-1}(A), r). \end{aligned}$$

Conversely, assume that for every  $A \in I^Y$ ,  $f^{-1}(\delta_{fs}I_{\mathcal{N}}(A, r)) \leq \delta_{fs}I_{\mathcal{T}}(f^{-1}(A), r)$ . Let  $U \in I^Y$  be an  $r$ - $\delta_{fs}$ -open set in  $Y$ . By Theorem 26,  $\delta_{fs}I_{\mathcal{N}}(U, r) = U$ . So,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\delta_{fs}I_{\mathcal{N}}(U, r)) \\ &\leq \delta_{fs}I_{\mathcal{T}}(f^{-1}(U), r). \end{aligned}$$

By Theorem 28 (i) yields  $f^{-1}(U) = r$ - $\delta_{fs}I_{\mathcal{T}}(f^{-1}(U), r)$ . By Theorem 26,  $f^{-1}(U)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ . Since  $U$  was arbitrary chosen  $r$ - $\delta_{fs}$ -open set in  $Y$ , it follows that by Definition 40,  $f$  is an  $r$ - $\delta_{fs}$ -continuous.  $\square$

**Theorem 46.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function, and  $r \in I_0$ . If  $f$  is an  $r$ - $\delta_{fs}$ -continuous, then for every fuzzy set  $A \in I^Y$ ,

$$\delta_{fs}C_{\mathcal{T}}(f^{-1}(A), r) \leq f^{-1}(\delta_{fs}C_{\mathcal{N}}(A, r)).$$

*Proof.* Let  $f$  be an  $r$ - $\delta_{fs}$ -continuous and  $A \in I^Y$ . By Theorem 31 (i),  $A \leq \delta_{fs}C_{\mathcal{N}}(A, r)$ . Then  $f^{-1}(A) \leq f^{-1}(\delta_{fs}C_{\mathcal{N}}(A, r))$ . Note that by Theorem 30,  $\delta_{fs}C_{\mathcal{N}}(A, r)$  is an  $r$ - $\delta_{fs}$ -closed set in  $Y$ , and since  $f$  is an

$r$ - $\delta_{fs}$ -continuous, by Theorem 43,  $f^{-1}(\delta_{fs}C_{\mathcal{N}}(A, r))$  is an  $r$ - $\delta_{fs}$ -closed set in  $X$ . Now, let

$$\mathcal{F} = \{V \in I^X : f^{-1}(A) \leq V, V \text{ is an } r\text{-}\delta_{fs}\text{-closed set}\}.$$

Then,  $f^{-1}(\delta_{fs}C_{\mathcal{N}}(A, r)) \in \mathcal{F}$ . Thus,

$$\begin{aligned} \delta_{fs}C_{\mathcal{T}}(f^{-1}(A), r) &= \bigwedge_{V \in \mathcal{F}} V \\ &\leq f^{-1}(\delta_{fs}C_{\mathcal{N}}(A, r)). \end{aligned}$$

□

**Theorem 47.** Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{N})$ , and  $(Z, \mathcal{K})$  be a SFTSs, and  $r \in I_0$ . If  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  is an  $r$ - $\delta_{fs}$ -continuous and  $g : (Y, \mathcal{N}) \rightarrow (Z, \mathcal{K})$  is an  $r$ - $\delta_{fs}$ -continuous, then the composition  $g \circ f$  is an  $r$ - $\delta_{fs}$ -continuous.

*Proof.* Let  $A \in I^Z$  be an  $r$ - $\delta_{fs}$ -open set. Since  $g$  is an  $r$ - $\delta_{fs}$ -continuous,  $g^{-1}(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $Y$ . Also, since  $f$  is an  $r$ - $\delta_{fs}$ -continuous and  $g^{-1}(A) \in I^Y$ ,  $f^{-1}(g^{-1}(A))$  is an  $r$ - $\delta_{fs}$ -open in  $X$ . Thus,  $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$  is an  $r$ - $\delta_{fs}$ -open set in  $X$ , and hence  $g \circ f$  is an  $r$ - $\delta_{fs}$ -continuous. □

**Definition 48.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs. Let  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function and  $r \in I_0$ . Then  $f$  is called an  $r$ -fuzzy semi-delta-homeomorphism, denoted by  $r$ - $\delta_{fs}$ -homeomorphism, if  $f$  is bijective,  $r$ - $\delta_{fs}$ -continuous, and  $r$ - $\delta_{fs}$ -open function.

**Example 49.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs, where  $X = Y = \{x, y, z\}$ , the fuzzy sets  $A_1, A_2 \in I^X$  and  $A_3, A_4 \in I^Y$  are defined by the following:

$$\begin{aligned} A_1(x) = 0.4, A_1(y) = 0.2, A_1(z) = 0.5; & \quad A_3(x) = 0.4, A_3(y) = 0.5, A_3(z) = 0.2; \\ A_2(x) = 0.6, A_2(y) = 0.8, A_2(z) = 0.6; & \quad A_4(x) = 0.6, A_4(y) = 0.6, A_4(z) = 0.8. \end{aligned}$$

And the functions  $\mathcal{T} : I^X \rightarrow I$  and  $\mathcal{N} : I^Y \rightarrow I$  are defined by the following:

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{4}, & \text{if } A \in \{A_1, A_2\} \\ 0, & \text{otherwise.} \end{cases} \quad \mathcal{N}(A) = \begin{cases} 1, & \text{if } A \in \{\bar{0}, \bar{1}\}, \\ \frac{1}{2}, & \text{if } A = A_3, \\ \frac{1}{4}, & \text{if } A = A_4, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $r = \frac{1}{4}$  and  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be defined by  $f(x) = x$ ,  $f(y) = z$ , and  $f(z) = y$ . Clearly,  $f$  is bijective, and by Definition 35 and 40,  $f$  is an  $\frac{1}{4}$ - $\delta_{fs}$ -open function and  $\frac{1}{4}$ - $\delta_{fs}$ -continuous, respectively. Therefore, by Definition 48,  $f$  is a  $\frac{1}{4}$ - $\delta_{fs}$ -homeomorphism.

**Remark 50.** Every  $r\text{-}\delta_{f_s}$ -homeomorphism  $f$  is both an  $r\text{-}\delta_{f_s}$ -continuous, and  $r\text{-}\delta_{f_s}$ -open function.

**Theorem 51.** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{N})$  be a SFTSs,  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  be a function,  $A \in I^X$ , and  $r \in I_0$ . If  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, then the following properties hold:

- (i)  $f^{-1}$  is an  $r\text{-}\delta_{f_s}$ -continuous;
- (ii)  $A$  is an  $r\text{-}\delta_{f_s}$ -open in  $X$  if and only if  $f(A)$  is an  $r\text{-}\delta_{f_s}$ -open in  $Y$ ;
- (iii) if  $A$  is an  $r\text{-}\delta_{f_s}$ -closed in  $X$ , then  $f(A)$  is an  $r\text{-}\delta_{f_s}$ -closed in  $Y$
- (iv)  $f(\delta_{f_s}I_{\mathcal{T}}(A, r)) = r\text{-}\delta_{f_s}I_{\mathcal{N}}(f(A), r)$ ; and
- (v)  $f^{-1}(\delta_{f_s}C_{\mathcal{T}}(A, r)) = \delta_{f_s}C_{\mathcal{N}}(f^{-1}(A), r)$ .

*Proof.*

- (i) Assume, that  $f^{-1}$  is not an  $r\text{-}\delta_{f_s}$ -continuous. Then there exists an  $r\text{-}\delta_{f_s}$ -open set  $A \in I^X$  such that  $(f^{-1})^{-1}(A)$  is not  $r\text{-}\delta_{f_s}$ -open in  $Y$ . Note that  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, then  $f$  is bijective, and so,  $(f^{-1})^{-1}(A) = f(A)$ . Thus,  $f(A)$  is not an  $r\text{-}\delta_{f_s}$ -open in  $Y$ . However, since  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, by Remark 50,  $f$  is an  $r\text{-}\delta_{f_s}$ -open function. Since,  $A$  is an  $r\text{-}\delta_{f_s}$ -open set in  $X$ ,  $f(A)$  is an  $r\text{-}\delta_{f_s}$ -open in  $Y$ , a contradiction
- (ii) Assume that  $A \in I^X$  is an  $r\text{-}\delta_{f_s}$ -open in  $X$ . Since  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism,  $f$  is an  $r\text{-}\delta_{f_s}$ -open function. Hence,  $f(A)$  is an  $r\text{-}\delta_{f_s}$ -open set in  $Y$ .  
Conversely, assume that  $f(A) \in I^Y$  is an  $r\text{-}\delta_{f_s}$ -open in  $Y$ . Since  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism,  $f$  is an  $r\text{-}\delta_{f_s}$ -continuous function. Therefore,  $f^{-1}(f(A))$  is an  $r\text{-}\delta_{f_s}$ -open in  $X$ . Now, since  $f$  is bijective,  $A = f^{-1}(f(A))$ , and hence,  $A$  is an  $r\text{-}\delta_{f_s}$ -open set in  $X$ .
- (iii) Let  $A \in I^X$  be an  $r\text{-}\delta_{f_s}$ -closed set in  $X$ . Then  $A^c$  is an  $r\text{-}\delta_{f_s}$ -open set in  $X$ . Since  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, by Remark 50 and Definition 35,  $f$  is an  $r\text{-}\delta_{f_s}$ -open function and  $f(A^c)$  is an  $r\text{-}\delta_{f_s}$ -open set in  $Y$ . Since  $f$  is bijective, then  $(f(A^c))^c = ((f(A))^c)^c = f(A)$  is an  $r\text{-}\delta_{f_s}$ -closed in  $Y$ .
- (iv) Let  $A \in I^X$  be any fuzzy set. Note that  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, by Remark 50,  $f$  is also an  $r\text{-}\delta_{f_s}$ -open function. By Theorem 38,  $f(\delta_{f_s}I_{\mathcal{T}}(A, r)) \leq \delta_{f_s}I_{\mathcal{N}}(f(A), r)$ . Conversely, let  $\mathcal{V} = \{V \in I^Y : V \leq f(A), V \text{ is an } r\text{-}\delta_{f_s}\text{-open}\}$ , where  $A \in I^X$ . Again, by Remark 50,  $f$  is an  $r\text{-}\delta_{f_s}$ -continuous. Hence, for each  $V \in \mathcal{V}$ ,  $f^{-1}(V)$  is an  $r\text{-}\delta_{f_s}$ -open set in  $X$ . Since  $V \leq f(A)$ , for each  $V \in \mathcal{V}$ , and  $f$  is bijective,  $f^{-1}(V) \leq f^{-1}(f(A)) = A$  implying  $f^{-1}(V) \leq A$ . Now, let  $\mathcal{U} = \{U \in I^X : U \leq A, U \text{ is an } r\text{-}\delta_{f_s}\text{-open in } X\}$ . Thus, for each  $V \in \mathcal{V}$ ,  $f^{-1}(V) \in \mathcal{U}$  and by Definition 24 (i),

$$\begin{aligned} f^{-1}(V) &\leq \bigvee_{U \in \mathcal{U}} U \\ &= r\text{-}\delta_{f_s}I_{\mathcal{T}}(A, r). \end{aligned}$$

This means that for each  $V \in \mathcal{V}$ ,  $f^{-1}(V) \leq r\text{-}\delta_{f_s}I_{\mathcal{T}}(A, r)$ . Applying  $f$  to both sides and by bijectivity of  $f$ , for each  $V \in \mathcal{V}$ ,

$$\begin{aligned} V &= f(f^{-1}(V)) \\ &\leq f(\delta_{f_s}I_{\mathcal{T}}(A, r)). \end{aligned}$$

So, every  $V \in \mathcal{V}$ , satisfies  $V \leq f(\delta_{f_s}I_{\mathcal{T}}(A, r))$ . As a result,

$$\begin{aligned} \delta_{f_s}I_{\mathcal{N}}(f(A), r) &= \bigvee_{V \in \mathcal{V}} V \\ &\leq f(\delta_{f_s}I_{\mathcal{T}}(A, r)). \end{aligned}$$

(v) Since  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism. By Remark 50,  $f$  is an  $r\text{-}\delta_{f_s}$ -continuous. Then by Theorem 46,  $\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r) \leq f^{-1}(\delta_{f_s}C_{\mathcal{N}}(A, r))$ . On the other hand, note that  $f^{-1}(A) \in I^X$ , by Theorem 30 (i),  $f^{-1}(A) \leq \delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)$ . Since  $f$  is bijective, applying  $f$  to both sides yield

$$\begin{aligned} A &= f(f^{-1}(A)) \\ &\leq f(\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)). \end{aligned}$$

Note that by Theorem 30,  $\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)$  is an  $r\text{-}\delta_{f_s}$ -closed set in  $X$ , and again,  $f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism, then by Theorem 51 (iii),  $f(\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r))$  is an  $r\text{-}\delta_{f_s}$ -closed set in  $Y$ . Now, let  $\{V_{\omega}\}_{\omega \in \Omega} \subseteq I^Y$  be the collection of  $r\text{-}\delta_{f_s}$ -closed set in  $Y$  where  $V \geq A$ . Then,  $f(\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)) \in \{V_{\omega}\}_{\omega \in \Omega}$ . Then

$$\begin{aligned} \delta_{f_s}C_{\mathcal{N}}(A, r) &= \bigwedge_{\substack{\omega \in \Omega \\ V_{\omega} \geq A}} V_{\omega} \\ &\leq f(\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)). \end{aligned}$$

Since  $f$  is bijective, applying  $f^{-1}$  both sides yield

$$\begin{aligned} f^{-1}(\delta_{f_s}C_{\mathcal{N}}(A, r)) &\leq f^{-1}(f(\delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r))) \\ &= \delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r). \end{aligned}$$

As a result,  $f^{-1}(\delta_{f_s}C_{\mathcal{N}}(A, r)) = \delta_{f_s}C_{\mathcal{T}}(f^{-1}(A), r)$ .

□

**Theorem 52.** Let  $(X, \mathcal{T})$ ,  $(Y, \mathcal{N})$ , and  $(Z, \mathcal{K})$  be a SFTSs, and  $r \in I_0$ . If  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{N})$  and  $g : (Y, \mathcal{N}) \rightarrow (Z, \mathcal{K})$  are  $r\text{-}\delta_{f_s}$ -homeomorphisms, then  $g \circ f$  is an  $r\text{-}\delta_{f_s}$ -homeomorphism.

*Proof.* Let  $f$  and  $g$  be an  $r\text{-}\delta_{f_s}$ -homeomorphisms. Then both  $f$  and  $g$  are bijections and their composition  $g \circ f$  is also a bijection. Next, note that  $f$  and  $g$  are also  $r\text{-}\delta_{f_s}$ -open functions, by Theorem 39,  $g \circ f$  is an

$r$ - $\delta_{f_s}$ -open function. Lastly, note that since  $f$  and  $g$  are also  $r$ - $\delta_{f_s}$ -continuous, by Theorem 47,  $g \circ f$  is an  $r$ - $\delta_{f_s}$ -continuous. As a result, by Definition 48,  $g \circ f$  is an  $r$ - $\delta_{f_s}$ -homeomorphism.  $\square$

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