Asia Pac. J. Math. 2026 **13**:76

AN ACCELERATED DAI-YUAN PROJECTION METHOD FOR APPROXIMATING SOLUTION OF A NONLINEAR MONOTONE EQUATION

A.O. ISHAQ[1,*], S.B. JAIYEOLA[2], R. AHMED[1], K.O. AREMU[1,3]

[1]Department of Mathematics, Usmanu Danfodiyo University Sokoto, PMB 2346, Sokoto State, Nigeria

[2]Department of Physical Sciences, Faculty of Natural and Applied Sciences, Al-Hikmah University, Ilorin, Kwara State, Nigeria

[3]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Science University, P.O. Box 60, Medunsa, 0204, Pretoria, South Africa

[*]Corresponding author: geniusolaitan@gmail.com

Received Aug. 9, 2025

Abstract. In this article, we propose a Dai-Yuan projection method for solving monotone nonlinear equations. The method is structured to avoid derivative evaluations at each iteration, thereby reducing computational complexity. To further accelerate convergence, an inertial term is incorporated into the framework. We establish global convergence of the proposed algorithm under standard assumptions. This work complements and extends existing results in the literature by combining efficiency, stability, and enhanced convergence properties.

2020 Mathematics Subject Classification. 65H10.

Key words and phrases. monotone equations; conjugate gradient methods; projection methods.

1. Introduction

. In this paper, we consider iterative method for finding the solutions of the following nonlinear monotone equation

$$K(w) = 0, \tag{1.1}$$

where K is continuous, $w \in C$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed, and convex subset of an n-dimensional Euclidean space $\mathbb{R}^n$ such that $\mathbb{R}^n$ is equipped with the Euclidean norm $\|.\|$ and the inner product $\langle u, v\rangle = u^T v$, for $u, v \in \mathbb{R}^n$.

Many effective methods have been developed lately to solve (1.1), Among them are, Newton method, Quasi-Newton method, Gauss-Newton method, Levenberg Marquardt method and a series of their variants, (see [2, 9–12, 14, 16] and references therein). These methods are very popular because of

DOI: 10.28924/APJM/13-76

their fast local convergence property. However, they are not suitable to solve large-scale optimization problems because they need the computation of Jacobian matrix or an approximation at each iteration, which makes them not suitable for handling such optimization problems. To address the drawback, matrix free approaches were developed for solving large-scale unconstrained optimization problems because of their simplicity and low storage, e.g, conjugate gradient method (CGM), spectral gradient method (SGM), and spectral conjugate gradient method (SCGM). A common one among these methods is the CGM which is as follows:

$$d_{k+1} = g_k - \sum_{i=1}^{k} \frac{g_k^T A d_i}{d_i^T A d_i} d_i \tag{1.2}$$

for $i = 1, 2, ..., k$.

Several researchers have combined the projection technique of Solodov and Svaiter [4], with CGM to solve nonlinear equations. For instance Wang et al. [3] extended the work of Solodov and Svaiter [4] and proposed a projection type method to solve monotone nonlinear equations. Also, Ma and Wang [5] presented a modification of extra-gradient method with projection for solving constrained nonlinear equations. Zheng and Zhou [6] proposed a spectral gradient projection method for solving nonlinear monotone equations. By popularizing the idea of Zheng and Zhou [6], Yu et al. [7] proposed a constrained version of the spectral gradient method for solving monotone nonlinear equations. A recent interesting projection type of CGM is the Dai-Yuan method introduced by Dai, Y.H and Yuan Y. to solve a monotone nonlinear equation. The Dai-Yuan projection method stands out among CGM projection techniques because of its strong global convergence properties and efficient descent direction, even without exact line search. Unlike other CGM methods, it effectively avoids the loss of descent direction, making it particularly suitable for tackling challenging monotone nonlinear equations. Its robustness and low computational cost per iteration further enhance its practical appeal compared to classical projection methods. Moreover, the method demonstrates stability and adaptability when extended to constrained settings, which broadens its scope of application.

Efficiency of an algorithm is often measured by the speed of convergence. In doing this, most researchers incorporates inertial extrapolation. The inertial extrapolation type algorithm was first introduced by Polyak [28] as an acceleration mechanism to tackle smooth convex minimization problems. The inertial method is a two-step iterative approach, which uses two iterates to compute the next one, effectively improving the rate [14, 5, 15]. There is a body of literature dedicated to this area, with various fast iterative algorithm sconstructed using inertial extrapolation. For example, we have inertial forward-backward splitting method [17,18], inertial Mann method [20], inertial Douglas-Rachford splitting method [24], inertial ADMM [20], inertial subgradient extragradient method [21] , inertial forward-backward-forward method [22], and inertial contraction method [19]. These strengths motivate the incorporation of an inertial term in to the Dai-Yuan projection method for solving nonlinear equations in the Euclidean space setting.

The rest of this paper is organized as follows: In Section 2, we present the inertial algorithm for solving (1.1) and some technical tools needed in this paper. In Section 3, the convergence analysis of the method is proved.

2. PRELIMINARIES AND ALGORITHM

This section presents some important definitions and Lemmas that are important for establishing the convergence of our proposed algorithm.

Definition 2.1. *Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. Then we have the following.*

K is called monotone, if

$$(Ku - Kv)^T(u - v) \geq 0, \forall u, v \in \mathbb{R}^n.$$

Lemma 2.1. *Let $K : \mathbb{R}^n$ be an Euclidean space and C be a closed and convex subset of $\mathbb{R}^n$, for each $u, v \in \mathbb{R}^n$, the following inequality holds:*

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle. \tag{2.1}$$

Lemma 2.2. *Let $C \subset \mathbb{R}^n$ be a nonempty closed and convex set, some vector $u \in \mathbb{R}^n$, the orthogonal projection of u onto C denoted $P_C(u)$ is defined by*

$$P_C(u) = argmin\left\{\|v - u\| \mid v \in C\right\},$$

where $\|u\|^2 = u^T u$.

Lemma 2.3. *P_C is called orthogonal projection of $\mathbb{R}^n$ onto C. The orthogonal projection P_C has the following fundamental properties;*

(i) $(w - P_C(w))^T(P_C(w) - z) \geq 0, \forall w \in \mathbb{R}^n, \forall z \in C,$

(ii) $\|P_C(w) - P_C(z)\| \leq \|w - z\|, \forall w, z \in \mathbb{R}^n,$

(iii) $\|P_C(w) - z\|^2 \leq \|w - z\|^2 - \|w - P_C(w)\|^2.$

Lemma 2.4. *Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. Then K is said to be Lipschitz continuous, if there exists $L > 0$ such that*

$$\|Kw - Kz\| \leq L\|w - z\|, \forall w, z \in \mathbb{R}^n. \tag{2.2}$$

Lemma 2.5. *Let $\{x_k\}$ and $\{w_k\}$ be sequences of non-negative real numbers satisfying $\sum_{k=1}^{\infty} w_k < \infty$ and $x_{k+1} \leq x_k + w_k, \quad k = 1, 2, 3....$ Then $\{x_k\}$ is a convergent sequence.*

Throughout this paper, we assume that mapping $K : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following conditions:

B_1. The feasible set C is a nonempty closed and convex subset of $\mathbb{R}^n$.

B_2. K is monotone on C and Lipschitz continuous.

B_3. The set of solution to (1.1) denoted by Γ is nonempty.

Next, we present our algorithm **(D-YPM)** as follows:

Initialization: Let $x_0 \in \mathbb{R}^n$ and parameters $\rho \in (0,1)\,, \xi \in (0,1)\,, \eta \in (0,1)\; \lambda \in (0,2)\,, r \in (0,1)\,,$ set $n = 0$

Iterative Steps: Calculate x_{n+1} as follows

Step 1: Choose α_n such that $\quad 0 \le \alpha_n \le \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} min\{\alpha, \frac{1}{k^2\|x_n - x_{n-1}\|^2}\} & x_n \neq x_{n-1} \\ \alpha, & otherwise. \end{cases} \tag{2.3}$$

Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}). \tag{2.4}$$

Step 2: If $\|K(w_n)\| \le \xi$ stop. Otherwise, generate the search direction p_n.

$$p_0 = -K(w_0), \quad p_n = -K(w_n) + \mu p_{n-1}. \quad n \ge 1, \tag{2.5}$$

where

$$\mu_n = \frac{\|K(w_n)\|^2}{\max\{p_{n-1}^T y_{n-1}, t\|p_{n-1}\|\|K(w_n)\|\}}, \quad t > 0, \tag{2.6}$$

$$y_{n-1} = K(w_n) - K(w_{n-1}).$$

Step 3: Choose $t_n = \eta r^i$, where i is the least non-negative integer such that

$$-K(w_n + \eta r^i p_n)p_n \ge \delta t_n \|p_n\|^2. \tag{2.7}$$

Compute the trial points

$$v_n = w_n + t_n p_n.$$

Step 4: If $v_n \in C \quad and \quad \|K(v_n)\| \le \xi$. Stop and set $w_{n+1} = v_n$,otherwise compute the next iterate by

$$x_{n+1} = P_C[w_n - \lambda\beta_n K(v_n)], \tag{2.8}$$

where

$$\beta_n = \frac{K(v_n)^T(w_n - v_n)}{\|K(v_n)\|^2}. \tag{2.9}$$

set $n = n + 1$ and return to step1.

Remark 1: Observe that from Step 1 of Algorithm **D-YPM**, that for all n

$$\sum_{n=0}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 \quad < \quad \infty. \tag{2.10}$$

Lemma 2.6. *Let p_n be generated by Algorithm* **D-YPM**, *Then p_n always satisfies the sufficient condition, that is,*

$$K(w_n)^T p_n = -c\|K(w_n)\|^2. \tag{2.11}$$

Proof. The proof is in two parts, For $n = 0$, it clearly holds, i.e

from (2.5)

$$p_0 = -K(w_0).$$

Multiply both side by $K(w_0)^T$, we get

$$\begin{aligned} K(w_0)^T p_0 &= -K(w_0)K(w_0)^T \\ K(w_0)^T &= -\|K(w_0)\|^2. \end{aligned} \tag{2.12}$$

For $n \geq 1$, by multiplying (2.5) in Step 2 by $K(w_n)^T$, we have

$$\begin{aligned} K(w_n)^T p_n &= -K(w_n)K(w_n)^T + \mu K(w_n)^T p_{n-1} \\ &= -\|K(w_n)\|^2 + \frac{\|K(w_n)\|^2}{\max\{p_{n-1}^T y_{n-1}, t\|p_{n-1}\|\|K(w_n)\|\}} . p_{n-1} K(w_n)^T \\ &\leq -\|K(w_n)\|^2 + \frac{\|K(w_n)\|^2}{t\|p_{n-1}\|K(w_n)\|} . \|K(w_n)\|\|p_{n-1}\| \\ &= -\|K(w_n)\|^2 + \frac{\|K(w_n)\|^2}{t} \\ &= -(1 - \frac{1}{t})\|K(w_n)\|^2. \end{aligned}$$

□

Remark 2: By (2.12) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |K(w_n)^T p_n| &= |-\|K(w_n)\|^2| \\ \|K(w_n)\|\|p_n\| &= \|K(w_n)\|^2 \\ \|K(w_n)\| &\leq \|p_n\|. \end{aligned} \tag{2.13}$$

3. CONVERGENCE ANALYSIS

Lemma 3.1. *Let Assumptions $B_1 - B_3$ hold and the sequence x_n and v_n be generated by Algorithm* **D-YPM**. *if $x^* \in \Gamma$. Then the following inequality holds*

$$\|x_{n+1} - x_n\| \leq \|w_n - x^*\| - \lambda(2-\lambda)\|\delta^2 \frac{\|w_n - v_n\|^4}{\|K(v_n)\|^2}. \tag{3.1}$$

Moreover,

$$\lim_{k\to\infty} \|w_n - v_n\|^4 = 0. \tag{3.2}$$

Proof. Let $x^* \in \Gamma$ then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|p_C[w_n - \lambda\beta_n K(v_n)] - x^*\|^2 \\ &\le \|w_n - \lambda\beta_n K(v_n) - x^*\|^2 \\ &= \|w_n - x^*\|^2 - 2\lambda\beta_n K(v_n)^T(w_n - x^*) + \lambda^2\beta_n^2\|K(v_n)\|^2. \end{aligned} \tag{3.3}$$

By assumption B_1, we have

$$\begin{aligned} K(v_n)^T(w_n - x^*) &= K(v_n)^T(w_n - v_n + v_n - x^*) \\ &= K(v_n)^T(w_n - v_n) + K(v_n)^T(v_n - x^*) \\ &\ge K(v_n)^T(w_n - v_n) + K(x^*)^T(v_n - x^*). \end{aligned} \tag{3.4}$$

Since Γ is a solution set and $x^* \in \Gamma$, then

$$K(v_n)^T(w_n - x^*) = K(v_n)^T(w_n - v_n). \tag{3.5}$$

Now, from (3.5), we have that

$$\begin{aligned} K(v_n)^T(w_n - x^*) &= K(w_n + t_n p_n)^T(-t_n p_n) \\ &= \delta t_n^2 \frac{\|w_n - v_n\|^2}{t_n^2}. \end{aligned}$$

This implies that

$$K(v_n)^T(w_n - x^*) = K(v_n)^T(w_n - v_n) = \delta\|w_n - v_n\|^{\cdot} \tag{3.6}$$

Combining (3.3) and (3.5), we have

$$\|x_{n+1} - x_n\|^2 \le \|w_n - x^*\|^2 - \lambda(2 - \lambda)\delta^2 \frac{\|w_n - v_n\|^4}{\|K(v_n)\|^2}, \tag{3.7}$$

and thus

$$\|x_{n+1} - x^*\| \le \|w_n - x^*\|. \tag{3.8}$$

This implies that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \|w_n - x^*\| \\ &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\| \\ &\le \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.9}$$

Noting from (2.10), it follows from Lemma (2.6) that the sequence $\|x_n - x^*\|$ is a convergent, therefore we can deduce that $\|x_n - x^*\|$ is bounded by a positive number say M_0, so by the definition of inertial term (2.4), Lemma (2.2), eqn (3.7) and the boundedness of x_n we have that

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n(x_n - x_{n-1})^T(x_n - x^* + \alpha_n(x_n - x_{n-1})) \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|\|(x_n - x^*) + \alpha_n(x_n - x_{n-1})\| \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|(\|x_n - x^*\| + \|\alpha_n(x_n - x_{n-1})\|) \\
&\leq \|x_n - x^*\|^2 + 2M_0\alpha_n\|x_n - x_{n-1}\| + 4M_0\alpha_n\|x_n - x_{n-1}\| \\
&\leq \|x_n - x^*\|^2 + 6M_0\alpha_n\|x_n - x_{n-1}\|. \qquad (3.10)
\end{aligned}$$

Combining (3.7) and (3.10), it follows that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + 6M_0\alpha_n\|x_n - x_{n-1}\| - \lambda(2-\lambda)\delta^2\frac{\|w_n - v_n\|^4}{\|K(v_n)\|^2}. \qquad (3.11)$$

Thus, we have

$$\lambda(2-\lambda)\delta^2\frac{\|w_n - v_n\|^4}{\|K(v_n)\|^2} \leq \|x_n - x^*\| + 6M_0\alpha_n\|x_n - x_{n-1}\| - \|x_{n+1} - x^*\|^2. \qquad (3.12)$$

Since x_n is bounded, it is easy to see that w_n is also bounded. Building upon the assumption of Lipschitz constant for the mapping K, we can conclude the existence of a positive constant denoted by ϵ_k, that is

$$\|K(w_n)\| \leq \epsilon_k \quad \forall k, \qquad (3.13)$$

using (3.16) and Assumption B_2 yields

$$\begin{aligned}
K(v_n)^T(w_n - v_n) &= (K(v_n) - K(w_n))^T(w_n - v_n) + K(w_n)^T(w_n - v_n) \\
&\leq \|K(w_n)\|\|w_n - v_n\| \\
&\leq \epsilon_k\|w_n - v_n\|. \qquad (3.14)
\end{aligned}$$

Combining the above with (3.6), we can deduce that $\|w_n - v_n\| \leq \frac{\epsilon_n}{\delta}$. Then, we can obtain

$$\|v_n\| \leq \frac{\epsilon_k}{\delta} + \|w_n\|. \qquad (3.15)$$

Since w_n is bounded, therefore v_n is also bounded. Given the Lipschitz constant of the mapping K and boundedness of v_n, it follows that the sequence $K(v_n)$ is also bounded. Consequently, if a positive number $\bar{w} > 0$ such that $\|K(v_n)\| \leq \bar{w}$. By combining the information with the (3.9) and summing over $k = 1, 2, 3, \ldots$ we have

$$\begin{aligned}
\delta^2\lambda(2-\lambda)\sum_{n=0}^{\infty}\|w_n - v_n\|^4 &\leq \|K(v_n)\|^2\sum_{n=0}^{\infty}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 6M_0\alpha_n\|x_n - x_{n-1}\|) \\
&< \infty.
\end{aligned}$$

This means that

$$\lim_{n\to\infty}\|w_n - v_n\| = 0. \qquad (3.16)$$

□

Lemma 3.2. *Let w_n and v_n be the sequence generated by Algorithm* **D-YPM**. *then we have for all n*

$$t_n \geq t_* > 0, \tag{3.17}$$

where

$$t_* = \frac{\rho\|K(w_n)\|^2}{(L+\delta)\|p_n\|^2}.$$

Proof. By (2.7) we have that

$$-K(w_n + \eta r^i p_n)^T \quad \geq \quad \delta t_n \|p_n\|^2.$$

If $t_n \neq \eta$ the line search procedure implies that $\bar{t}_n = \frac{t_n}{\eta}$ does not satisfy the condition in (2.7). By contradiction, we have

$$-K(w_n + \eta^{-1} t_n p_n)^T p_n < \delta \eta^{-1} t_n \|p_n\|^2. \tag{3.18}$$

By Lipschitz continuity and (3.16)

$$\begin{aligned}
\|K(w_n)\|^2 &= -K(w_n)^T p_n + K(w_n + \eta^{-1} t_n p_n)^T p_n - K(w_n + \eta^{-1} t_n p_n)^T p_n \\
&= (K(w_n + \eta^{-1} t_n p_n) - K(w_n))^T p_n - K(w_n + \eta^{-1} t_n p_n)^T p_n \\
&\leq \|K(w_n + \eta^{-1} t_n p_n) - K(w_n)\| \|p_n\| - K(w_n + \eta^{-1} t_n p_n)^T p_n \\
&\leq L\|\eta^{-1} t_n p_n\| \|p_n\| + \delta \eta^{-1} t_n \|p_n\|^2 \\
&\leq L\eta^{-1} t_n \|p_n\|^2 + \delta \rho^{-1} t_n \|p_n\|^2 \\
&\leq \eta^{-1} t_n (L+\delta) \|p_n\|^2. \qquad (3.19)
\end{aligned}$$

The above inequality leads to the desired inequality □

Lemma 3.3. *Let ϵ_0 be a constant such that for all $n \geq 0$, $\|K(w_n)\| \geq c\epsilon_o$, then the search direction p_n generated by Algorithm* **D-YPM** *is bounded. that is $\|p_n\| \leq m_2$.*

Recall from sufficient decent condition

$$\begin{aligned}
-K(w_n)^T p_n &\geq \|K(w_n)\|^2 \\
|-K(w_n)^T p_n| &\geq c\|K(w_n)\|^2 \\
\|K(w_n)\| \|p_n\| &\geq c\|K(w_n)\|^2,
\end{aligned}$$

which implies

$$\|p_n\| \geq c\|K(w_n)\| \geq c\epsilon. \tag{3.20}$$

Now, we establish that the search direction p_n is bounded, we have two :

Case 1: If the $\max\{p_{n-1}^T y_{n-1}, t\|p_{n-1}\|\|K(w_n)\|\}$ is $t\|p_{n-1}\|\|K(w_n)\|$, we have as follows:

$$p_n = -K(w_n) + \mu p_{n-1}.$$

Then

$$\begin{aligned} \|p_n\| &= \left\| -K(w_n) + \frac{\|K(w_n)\|^2}{\max\{p_{n-1}^T y_{n-1}, t\|p_{n-1}\|\|K(w_n)\|} \|p_{n-1} \right\| \\ &\leq \|K(w_n)\| + \frac{\|K(w_n)\|^2}{t\|p_{n-1}\|\|K(w_n)\|} \|p_{n-1}\| \\ &\leq \|K(w_n)\| + \frac{\|k(w_n)\|^2}{t\|K(w_n)\|} \\ &= \|K(w_n)\| + \frac{\|K(w_n)\|}{t} \\ &= \left(1 + \frac{1}{t}\right) \|k(w_n)\|. \end{aligned}$$

Setting $M_2 = 1 + \frac{1}{t}$ it follows that $\forall n \geq 0$ we obtain

$$\|p_n\| \leq M_2 \epsilon \tag{3.21}$$

Theorem 3.4. (*Global convergence*) *Suppose that assumption B_1, B_2 and B_3 hold. Let x_n be the sequence generated by Algorithm* **D-YPM**, *then*

$$\lim_{n \to \infty} inf \|K(w_n)\| = 0. \tag{3.22}$$

Proof. Suppose (3.22) does not hold, there exist a constant $\epsilon > 0$ such that

$$\|K(w_n)\| \geq \epsilon_0 \quad n \geq 0. \tag{3.23}$$

By (2.12), we know that

$$\|K(w_n)\| \leq \|p_n\|.$$

This shows that

$$\|p_n\| \geq \|K(w_n)\| \geq \epsilon_0 \quad \forall n \geq 0. \tag{3.24}$$

Next we show that p_n is bounded. From

$$p_n = -K(w_n) + \mu p_{n-1},$$

we have that

$$\begin{aligned} \|p_n\| &= \left\| -K(w_n) + \frac{\|K(w_n)\|^2}{\max\{p_{n-1}^T y_{n-1}, t\|p_{n-1}\|\|K(w_n)\|} \|p_{n-1} \right\| \\ &\leq \|K(w_n)\| + \frac{\|K(w_n)\|^2}{t\|p_{n-1}\|\|K(w_n)\|} \|p_{n-1}\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|K(w_n)\| + \frac{\|k(w_n)\|^2}{t\|K(w_n)\|} \\
&= \|K(w_n)\| + \frac{\|K(w_n)\|}{t} \\
&= \left(1 + \frac{1}{t}\right)\|k(w_n)\|.
\end{aligned}
$$

Setting $M_1 = 1 + \frac{1}{t}$ it follows that $\forall n \geq 0$

$$\|p_n\| \leq M_1\|K(w_n)\|, \tag{3.25}$$

which together with (3.17) means that $\lim_{n\to\infty} t_n = 0$. For ease of notation, we define $t_n = \eta r^{ik}$. Thus from (2.7), for sufficiently large n, the following inequality holds

$$K(w_n + \eta r^{ik} p_n)^T p_n < \delta\eta r^{ik}\|p_n\|^2. \tag{3.26}$$

Combining with the boundedness of w_n, there exists a subsequence of $\{w_n\}$ and $\{p_n\}$ such that, taking limits on (3.26) yields

$$K(\bar{w})^T \bar{p} < 0,$$

where $\bar{w}$, $\bar{p}$ are limit points of corresponding subsequences, Also by taking limit on both side of (2.7) we obtain

$$K(\bar{w})^T \bar{p} \geq \|K(\bar{w})\|^2.$$

From the above two inequalities, we get that $\|K(\bar{w})\| = 0$, which contradicts (3.24). Furthermore, since K is monotone and (3.23) holds, then the sequence $\{w_n\}$ has some accumulation point say w_* for which $K(w_*) = 0$ that is, w_* is a solution of (1.1). From (3.5), it holds that $\{\|w_n - v_n\|\}$ converges and since w_* is an accumulation point of $\{w_n\}$ we must have $\{w_n\}$ converges to w_*. □

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Nocedal, S.J. Wright, Numerical Optimization, Springer New York, 2006. https://doi.org/10.1007/978-0-387-40065-5.

[2] A.H. Ibrahim, P. Kumam, A. Kamandi, A.B. Abubakar, An Efficient Hybrid Conjugate Gradient Method for Unconstrained Optimization, Optim. Methods Softw. 37 (2022), 1370–1383. https://doi.org/10.1080/10556788.2021.1998490.

[3] C. Wang, Y. Wang, C. Xu, A Projection Method for a System of Nonlinear Monotone Equations with Convex Constraints, Math. Methods Oper. Res. 66 (2007), 33–46. https://doi.org/10.1007/s00186-006-0140-y.

[4] M.V. Solodov, B.F. Svaiter, A Globally Convergent Inexact Newton Method for Systems of Monotone Equations, in: Applied Optimization, Springer US, Boston, MA, 1998: pp. 355–369. `https://doi.org/10.1007/978-1-4757-6388-1_18`.

[5] F. Ma, C. Wang, Modified Projection Method for Solving a System of Monotone Equations with Convex Constraints, J. Appl. Math. Comput. 34 (2009), 47–56. `https://doi.org/10.1007/s12190-009-0305-y`.

[6] L. Zhang, W. Zhou, Spectral Gradient Projection Method for Solving Nonlinear Monotone Equations, J. Comput. Appl. Math. 196 (2006), 478–484. `https://doi.org/10.1016/j.cam.2005.10.002`.

[7] Z. Yu, J. Lin, J. Sun, Y. Xiao, L. Liu, et al., Spectral Gradient Projection Method for Monotone Nonlinear Equations with Convex Constraints, Appl. Numer. Math. 59 (2009), 2416–2423. `https://doi.org/10.1016/j.apnum.2009.04.004`.

[8] P. Gao, C. He, An Efficient Three-Term Conjugate Gradient Method for Nonlinear Monotone Equations with Convex Constraints, Calcolo 55 (2018), 53. `https://doi.org/10.1007/s10092-018-0291-2`.

[9] Z. Dai, H. Zhu, A Modified Hestenes-Stiefel-Type Derivative-Free Method for Large-Scale Nonlinear Monotone Equations, Mathematics 8 (2020), 168. `https://doi.org/10.3390/math8020168`.

[10] A.H. Ibrahim, P. Kumam, A.B. Abubakar, J. Abubakar, A Method with Inertial Extrapolation Step for Convex Constrained Monotone Equations, J. Inequal. Appl. 2021 (2021), 189. `https://doi.org/10.1186/s13660-021-02719-3`.

[11] A.H. Ibrahim, J. Deepho, A. Bala Abubakar, A. Adamu, A Three-Term Polak-Ribière-Polyak Derivative-Free Method and Its Application to Image Restoration, Sci. Afr. 13 (2021), e00880. `https://doi.org/10.1016/j.sciaf.2021.e00880`.

[12] A.B. Abubakar, P. Kumam, A.H. Ibrahim, P. Chaipunya, S.A. Rano, New Hybrid Three-Term Spectral-Conjugate Gradient Method for Finding Solutions of Nonlinear Monotone Operator Equations with Applications, Math. Comput. Simul. 201 (2022), 670–683. `https://doi.org/10.1016/j.matcom.2021.07.005`.

[13] A.H. Ibrahim, P. Kumam, Re-Modified Derivative-Free Iterative Method for Nonlinear Monotone Equations with Convex Constraints, Ain Shams Eng. J. 12 (2021), 2205–2210. `https://doi.org/10.1016/j.asej.2020.11.009`.

[14] Y. Zheng, B. Zheng, Two New Dai–Liao–Type Conjugate Gradient Methods for Unconstrained Optimization Problems, J. Optim. Theory Appl. 175 (2017), 502–509. `https://doi.org/10.1007/s10957-017-1140-1`.

[15] B. Polyak, Some Methods of Speeding up the Convergence of Iteration Methods, USSR Comput. Math. Math. Phys. 4 (1964), 1–17. `https://doi.org/10.1016/0041-5553(64)90137-5`.

[16] J. Abubakar, P. Kumam, A. Hassan Ibrahim, A. Padcharoen, Relaxed Inertial Tseng's Type Method for Solving the Inclusion Problem with Application to Image Restoration, Mathematics 8 (2020), 818. `https://doi.org/10.3390/math8050818`.

[17] H. Attouch, J. Peypouquet, P. Redont, A Dynamical Approach to an Inertial Forward-Backward Algorithm for Convex Minimization, SIAM J. Optim. 24 (2014), 232–256. `https://doi.org/10.1137/130910294`.

[18] D.A. Lorenz, T. Pock, An Inertial Forward-Backward Algorithm for Monotone Inclusions, J. Math. Imaging Vis. 51 (2014), 311–325. `https://doi.org/10.1007/s10851-014-0523-2`.

[19] Q.L. Dong, Y.J. Cho, L.L. Zhong, T.M. Rassias, Inertial Projection and Contraction Algorithms for Variational Inequalities, J. Glob. Optim. 70 (2017), 687–704. `https://doi.org/10.1007/s10898-017-0506-0`.

[20] D.V. Thong, D. Van Hieu, An Inertial Method for Solving Split Common Fixed Point Problems, J. Fixed Point Theory Appl. 19 (2017), 3029–3051. `https://doi.org/10.1007/s11784-017-0464-7`.

[21] D.V. Thong, D. Van Hieu, Modified Subgradient Extragradient Method for Variational Inequality Problems, Numer. Algorithms 79 (2017), 597–610. `https://doi.org/10.1007/s11075-017-0452-4`.