Asia Pac. J. Math. 2026 **13**:77

CHARACTERIZATIONS OF $(m, \sigma^\star)$-CONTINUOUS FUNCTIONS

BUTSAKORN KONG-IED[1], AREEYUTH SAMA-AE[2], CHAWALIT BOONPOK[1,*]

[1]Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand

[*]Corresponding author: chawalit.b@msu.ac.th

Received Mar. 16, 2026

Abstract. This paper introduces a new class of continuous functions defined from an m-space into an ideal topological space, called $(m, \sigma^\star)$-continuous functions. Moreover, several characterizations of $(m, \sigma^\star)$-continuous functions are investigated.

2020 Mathematics Subject Classification. 54C05; 54C08.

Key words and phrases. m-open set; $\tau^\star$-open set; $(m, \sigma^\star)$-continuous multifunction.

1. Introduction

The concept of ideals in topological spaces was studied by Kuratowski [18] and Vaidyanathaswamy [26]. Janković and Hamlett [16] introduced the notion of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] studied some properties of $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. As generalizations of open sets in an ideal topological space, many authors introduced the notions of semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets. Hatir and Noiri [15] introduced and investigated the notions of semi-$\mathscr{I}$-continuous functions and α-$\mathscr{I}$-continuous functions by utilizing the notions of semi-$\mathscr{I}$-open sets and α-$\mathscr{I}$-open sets, respectively. Furthermore, Hatir and Noiri [14] introduced and studied the notions of β-$\mathscr{I}$-continuous functions and β-$\mathscr{I}$-irresolute functions. On the other hand, the present authors introduced and investigated the concepts of $\star$-continuous functions [4], weakly $\star$-continuous functions [10], $\theta(\star)$-continuous functions [10], $\theta(\star)$-precontinuous functions [9], almost $\star$-precontinuous functions [9], weakly $\star$-precontinuous functions [9], pi-continuous functions [2], α-$\star$-continuous functions [3], almost α-$\star$-continuous functions [5], weakly α-$\star$-continuous functions [8], $s\beta(\star)$-continuous functions [6], weakly $s\beta(\star)$-continuous functions [7], almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [27], weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [27],

DOI: 10.28924/APJM/13-77

almost nearly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [24], almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [17] and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [17]. Popa and Noiri [23] introduced a new notion of M-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions. Moreover, Popa and Noiri [22] introduced and studied the concept of weakly M-continuous functions. Popa and Noiri [21] introduced a new notion of weakly m-continuous functions as functions from a set satisfying some minimal conditions into a topological space and considered strongly m-closed graphs, m-compactness and m-connectedness. In this paper, we introduce the concept of $(m, \sigma^\star)$-continuous functions defined between an m-space and an ideal topological space. We also investigate several characterizations and some properties concerning $(m, \sigma^\star)$-continuous functions.

2. Preliminaries

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an *ideal topological space* and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [18], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\tau^\star$-closed [16] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\tau^\star$-semi-open* [20] (*semi-$\mathscr{I}^\star$-open* [12]) if $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$. The complement of a $\tau^\star$-semi-open set is called *$\tau^\star$-semi-closed*. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all $\tau^\star$-semi-closed sets containing A is called the $\tau^\star$-semi-closure [12] of A and is denoted by $\text{sCl}^\star(A)$. The union of all $\tau^\star$-semi-open sets contained in A is called the $\tau^\star$-semi-interior [12] of A and is denoted by $\text{sInt}^\star(A)$.

Lemma 1. [12] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$.
(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called an *m-structure* [13] on X if m satisfies the following properties: (1) $\emptyset \in m$ and $X \in m$; (2) $\cup_{\alpha \in \nabla} A_\alpha \in m$ whenever $A_\alpha \in m$ for each $\alpha \in \nabla$. We call the pair (X, m) an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. Let (X, m) be an m-space and A a subset of X. The *m-closure* mCl(A) and m-*interior* mInt(A) of A are defined in [19] as follows:

$$\text{mCl} = \cap\{F \mid A \subseteq F \text{ and } X - F \in m\}$$

and $\text{mInt} = \cup\{U \mid U \subseteq A \text{ and } U \in m\}$.

Lemma 2. [13] *Let (X, m) be an m-space. Then, for a subset A of X, the following properties hold:*

(1) *$A \in m$ if and only if $A = mInt(A)$;*

(2) *A is m-closed if and only if $A = mCl(A)$;*

(3) *$mCl(A)$ is m-closed and $mInt(A)$ is m-open.*

3. Characterizations of $(m, \sigma^\star)$-continuous functions

In this section, we introduce the notion of $(m, \sigma^\star)$-continuous functions. Furthermore, several characterizations of $(m, \sigma^\star)$-continuous functions are discussed.

Definition 1. *A function $f : (X, m) \to (Y, \sigma, \mathscr{J})$ is said to be $(m, \sigma^\star)$-continuous at a point $x \in X$ if for each $\sigma^\star$-open set V of Y containing $f(x)$, there exists an m-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, m) \to (Y, \sigma, \mathscr{J})$ is said to be $(m, \sigma^\star)$-continuous if f is $(m, \sigma^\star)$-continuous at each point x of X.*

Theorem 1. *For a function $f : (X, m) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $(m, \sigma^\star)$-continuous;*

(2) *$f^{-1}(V)$ is m-open in X for every $\sigma^\star$-open set V of Y;*

(3) *$f^{-1}(K)$ is m-closed in X for every $\sigma^\star$-closed set K of Y;*

(4) *$mCl(f^{-1}(B)) \subseteq f^{-1}(Cl^\star(B))$ for every subset B of Y;*

(5) *$f(mCl(A)) \subseteq Cl^\star(f(A))$ for every subset A of X;*

(6) *$f^{-1}(Int^\star(B)) \subseteq mInt(f^{-1}(B))$ for every subset B of Y.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\sigma^\star$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is $(m, \sigma^\star)$-continuous, there exists an m-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$ and hence $x \in \text{mInt}(f^{-1}(V))$. Therefore, $f^{-1}(V) \subseteq \text{mInt}(f^{-1}(V))$. This shows that $f^{-1}(V)$ is m-open in X.

$(2) \Rightarrow (3)$: The proof is obvious.

$(3) \Rightarrow (4)$: Let B be any subset of Y. Then, $\text{Cl}^\star(B)$ is $\sigma^\star$-closed in Y and by (3), $\text{mCl}(f^{-1}(B)) \subseteq \text{mCl}(f^{-1}(\text{Cl}^\star(B))) = f^{-1}(\text{Cl}^\star(B))$.

$(4) \Rightarrow (5)$: Let A be any subset of X. By (4), we have

$$\text{mCl}(A) \subseteq \text{mCl}(f^{-1}(f(A))) \subseteq f^{-1}(\text{Cl}^\star(f(A)))$$

and so $f(\text{mCl}(A)) \subseteq \text{Cl}^\star(f(A))$.

(5) ⇒ (6): Let B be any subset of Y. Thus by (5),

$$f(\mathrm{mCl}(f^{-1}(Y-B))) \subseteq \mathrm{Cl}^\star(f(f^{-1}(Y-B)))$$
$$\subseteq \mathrm{Cl}^\star(Y-B) = Y - \mathrm{Int}^\star(B).$$

Since

$$f(\mathrm{mCl}(f^{-1}(Y-B))) = f(\mathrm{mCl}(X - F^{-}(B)))$$
$$= f(X - \mathrm{mInt}(f^{-1}(B))),$$

we have $X - \mathrm{mInt}(f^{-1}(B)) \subseteq f^{-1}(Y - \mathrm{Int}^\star(B)) = X - f^{-1}(\mathrm{Int}^\star(B))$ and hence $f^{-1}(\mathrm{Int}^\star(B)) \subseteq \mathrm{mInt}(f^{-1}(B))$.

(6) ⇒ (1): Let $x \in X$ and V be any $\sigma^\star$-open set of Y containing $f(x)$. By (6), $x \in f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$. Then, there exists an m-open set U of X containing x such that $U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$. This shows that f is $(m, \sigma^\star)$-continuous. □

A point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\delta$-*cluster point* of A if $\mathrm{Int}^\star(\mathrm{Cl}^\star(U)) \cap A \neq \emptyset$ for every $\tau^\star$-open set U of X containing x. The set of all $\star_\delta$-cluster points of A is called the $\star_\delta$-*closure* of A and is dented by $\star_\delta\mathrm{Cl}(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star_\delta$-*closed* if $\star_\delta\mathrm{Cl}(A) = A$. The complement of a $\star_\delta$-closed set is said to be $\star_\delta$-*open* [25].

Definition 2. [25] *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\tau^\star$-semi-regular if for each $\tau^\star$-semi-closed set F and each $x \notin F$, there exist disjoint $\tau^\star$-semi-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 3. [25] *Let $(X, \tau, \mathscr{I})$ be a $\tau^\star$-semi-regular space. Then, the following properties hold:*

(1) *$\mathrm{Cl}^\star(A) = \star_\delta\mathrm{Cl}(A)$ for every subset A of X.*
(2) *Every $\tau^\star$-open set is $\star_\delta$-open.*

Theorem 2. *For a function $f : (X, m) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $(m, \sigma^\star)$-continuous;*
(2) *$f^{-1}(\star_\delta\mathrm{Cl}(B))$ is m-closed in X for every subset B of Y;*
(3) *$f^{-1}(K)$ is m-closed in X for every $\star_\delta$-closed set K of Y;*
(4) *$f^{-1}(V)$ is m-open in X for every $\star_\delta$-open set V of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 3, $\star_\delta\mathrm{Cl}(B)$ is $\sigma^\star$-closed in Y. Since f is $(m, \sigma^\star)$-continuous, by Theorem 1 we have $f^{-1}(\star_\delta\mathrm{Cl}(B))$ is m-closed in X.

(2) ⇒ (3): Let K be any $\star_\delta$-closed set of Y. Then, $\star_\delta\mathrm{Cl}(K) = K$ and by (2), $f^{-1}(K)$ is m-closed in X.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let V be any $\sigma^\star$-open set of Y. Since $(Y, \sigma, \mathscr{J})$ is $\sigma^\star$-semi-regular, V is $\star_\delta$-open in Y, by (4) we have $f^{-1}(V)$ is m-open in X. Thus, f is $(m, \sigma^\star)$-continuous by Theorem 1. □

Recall that a point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\theta$*-cluster point* [11] of A if $\text{Cl}^\star(U) \cap A \neq \emptyset$ for every $\tau^\star$-open set U of X containing x. The set of all $\star_\theta$-cluster points of A is called the $\star_\theta$*-closure* [11] of A and is dented by $\star_\theta\text{Cl}(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star_\theta$*-closed* [11] if $\star_\theta\text{Cl}(A) = A$. The complement of a $\star_\theta$-closed set is called $\star_\theta$*-open*.

Definition 3. [25] *An ideal topological space $(X, \tau, \mathscr{I})$ is called $\tau^\star$-regular if for each $\tau^\star$-closed set F and each $x \notin F$, there exist disjoint $\tau^\star$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 4. [25] *Let $(X, \tau, \mathscr{I})$ be a $\tau^\star$-regular space. Then, the following properties hold:*

(1) *$Cl^\star(A) = \star_\theta Cl(A)$ for every subset A of X.*
(2) *Every $\tau^\star$-open set is $\star_\theta$-open.*

Theorem 3. *For a function $f : (X, m) \to (Y, \sigma, \mathscr{J})$, where $(Y, \sigma, \mathscr{J})$ is $\sigma^\star$-regular, the following properties are equivalent:*

(1) *f is $(m, \sigma^\star)$-continuous;*
(2) *$f^{-1}(\star_\theta Cl(B))$ is m-closed in X for every subset B of Y;*
(3) *$f^{-1}(K)$ is m-closed in X for every $\star_\theta$-closed set K of Y;*
(4) *$f^{-1}(V)$ is m-open in X for every $\star_\theta$-open set V of Y.*

Proof. $(1) \Rightarrow (2)$: Let B be any subset of Y. By Lemma 4, $\star_\theta\text{Cl}(B)$ is $\sigma^\star$-closed in Y. Since f is $(m, \sigma^\star)$-continuous, by Theorem 1 we have $f^{-1}(\star_\theta\text{Cl}(B))$ is m-closed in X.

$(2) \Rightarrow (3)$: Let K be any $\star_\theta$-closed set of Y. Then, $\star_\theta\text{Cl}(K) = K$. Thus by (2), $f^{-1}(K)$ is m-closed in X.

$(3) \Rightarrow (4)$: The proof is obvious.

$(4) \Rightarrow (1)$: Let V be any $\sigma^\star$-open set of Y. Since $(Y, \sigma, \mathscr{J})$ is $\sigma^\star$-regular, we have V is $\star_\theta$-open in Y and by (4), $f^{-1}(V)$ is m-open in X. Thus, f is $(m, \sigma^\star)$-continuous by Theorem 1. □

Acknowledgements. This research project was financially supported by Mahasarakham University.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.E. Abd El-Monsef, E.F. Lashien, A.A. Nasef, On $\mathscr{I}$-Open Sets and $\mathscr{I}$-Continuity, Kyungpook Math. J. 32 (1992), 21–30.

[2] C. Boonpok, $\mathscr{m}$-Continuity and Weak $\mathscr{m}$-Continuity, Carpathian Math. Publ. 17 (2025), 171–186.

[3] C. Boonpok, J. Khampakdee, Upper and Lower α-$\star$-Continuity, Eur. J. Pure Appl. Math. 17 (2024), 201–211. https://doi.org/10.29020/nybg.ejpam.v17i1.4858.

[4] C. Boonpok, On Some Spaces via Topological Ideals, Open Math. 21 (2023), 20230118. https://doi.org/10.1515/math-2023-0118.

[5] C. Boonpok and N. Srisarakham, Almost α-$\star$-Continuity for Multifunctions, Int. J. Anal. Appl. 21 (2023), 107. https://doi.org/10.28924/2291-8639-21-2023-107.

[6] C. Boonpok, P. Pue-on, Upper and Lower $s\beta(\star)$-Continuous Multifunctions, Eur. J. Pure Appl. Math. 16 (2023), 1634–1646. https://doi.org/10.29020/nybg.ejpam.v16i3.4732.

[7] C. Boonpok, J. Khampakdee, Upper and Lower Weak $s\beta(\star)$-Continuity, Eur. J. Pure Appl. Math. 16 (2023), 2544–2556. https://doi.org/10.29020/nybg.ejpam.v16i4.4734.

[8] C. Boonpok, P. Pue-on, Upper and Lower Weakly α-$\star$-Continuous Multifunctions, Int. J. Anal. Appl. 21 (2023), 90. https://doi.org/10.28924/2291-8639-21-2023-90.

[9] C. Boonpok, $\theta(\star)$-Precontinuity, Mathematica, 65 (2023), 31–42. https://doi.org/10.24193/mathcluj.2023.1.04.

[10] C. Boonpok, Weak Openness and Weak Continuity in Ideal Topological Spaces, Mathematica 64 (2022), 173–185.

[11] C. Boonpok, P. Pue-On, Continuity for Multifunctions in Ideal Topological Spaces, WSEAS Trans. Math. 19 (2021), 624–631. https://doi.org/10.37394/23206.2020.19.69.

[12] C. Boonpok, Weak Quasi Continuity for Multifunctions in Ideal Topological Spaces, Adv. Math.: Sci. J. 9 (2020), 339–355.

[13] F. Cammaroto, T. Noiri, On Λ_m-sets and related topological spaces, Acta Math. Hung. 109 (2005), 261–279. https://doi.org/10.1007/s10474-005-0245-4.

[14] E. Hatir, T. Noiri, On β-$\mathscr{I}$-Open Sets and a Decomposition of Almost $\mathscr{I}$-Continuity, Bull. Malays. Math. Sci. Soc. 29 (2006), 119–124.

[15] E. Hatir, T. Noiri, On Decompositions of Continuity via Idealization, Acta Math. Hung. 96 (2002), 341–349. https://doi.org/10.1023/a:1019760901169.

[16] D. Jankovic, T.R. Hamlet, New Topologies from Old via Ideals, Am. Math. Mon. 97 (1990), 295–310. https://doi.org/10.2307/2324512.

[17] B. Kong-ied, A. Sama-Ae, C. Boonpok, Almost Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous and Weakly Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions, Eur. J. Pure Appl. Math. 18 (2025), 6572. https://doi.org/10.29020/nybg.ejpam.v18i3.6572.

[18] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.

[19] H. Maki, K.C. Rao, A.N. Gani, On Generalizing Semi-Open Sets and Preopen Sets, Pure Appl. Math. Sci. 49 (1999), 17–29.

[20] T. Noiri, V. Popa, On (mI, nJ)-Continuous Multifunctions, Rom. J. Math. Comput. Sci. 15 (2025), 1–8.

[21] V. Popa, T. Noiri, On Weakly m-Continuous Functions, Mathematica 45 (2003), 53–67.

[22] V. Popa, T. Noiri, A Unified Theory of Weak Continuity for Functions, Rend. Circ. Mat. Palermo 51 (2002), 439–464.

[23] V. Popa, T. Noiri, On M-Continuous Functions, Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II 18 (2000), 31–41.

[24] P. Pue-on, A. Sama-Ae, C. Boonpok, On Almost Nearly $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions, Int. J. Anal. Appl. 24 (2026), 111.

[25] M. Thongmoon, A. Sama-Ae, C. Boonpok, $(m, \sigma^\star)$-Continuity for Multifunctions, Asia Pac. J. Math. 13 (2026), 58. https://doi.org/10.28924/APJM/13-58.

[26] R. Vaidyanathaswamy, The Localization Theory in Set-Topology, Proc. Indian Acad. Sci. 20 (1945), 51–61.

[27] N. Viriyapong, A. Sama-Ae, C. Boonpok, On Almost $\tau^\star(\sigma_1, \sigma_2)$-Continuity and Weak $\tau^\star(\sigma_1, \sigma_2)$-Continuity, Eur. J. Pure Appl. Math. 18 (2025), 6568. https://doi.org/10.29020/nybg.ejpam.v18i3.6568.