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ON SPECTRAL PROPERTIES OF THE NORMALIZED LAPLACIAN AND SIGNLESS LAPLACIAN MATRICES OF THE NON-COMMUTING GRAPH OF U_{6n}

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Abstract. In this study, we investigate the spectral properties of the non-commuting graph associated with the group U_{6n}, for $n \geq 1$. Specifically, we determine the spectral radius and graph energy derived from the eigenvalues of the normalized Laplacian (NL) and normalized signless Laplacian (NSL) matrices. Our analysis reveals that the NL-energy coincides exactly with the NSL-energy for all admissible values of n, and notably, this common energy value is always an odd integer. These findings contribute to a deeper understanding of spectral invariants in algebraically defined graphs and highlight a structural symmetry between NL and NSL frameworks for this class of groups.

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1. Introduction

The non-commuting graph of a finite group G, denoted by Γ_G, is defined by taking $G \backslash Z(G)$ as its vertex set, where $Z(G)$ represents the center of G. Two distinct vertices v_p and v_q are connected by an edge if and only if they do not commute in G [1]. In this study, we consider the finite and non-abelian group U_{6n} of order $6n$, where $n \geq 1$, given by the presentation $U_{6n} = \left\langle a, b : a^{2n} = b^3 = e, a^{-1}ba = b^{-1} \right\rangle$ [8]. Accordingly, the non-commuting graph associated with this group is denoted by $\Gamma_{U_{6n}}$.

The notation of graph energy was first introduced by Gutman in 1978 [5], and is defined as the sum of the absolute values of the eigenvalues of the graph. The largest absolute eigenvalue is referred to as the spectral radius [7].

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Spectral graph theory has traditionally relied on the eigenvalue analysis of matrices associated with graphs, including the adjacency matrix, Laplacian matrix, and their normalized counterparts, to extract structural properties. Hogben [6] explored the interrelationships among these matrices within the framework of the spectral graph theory. Furthermore, Khan and Pirzada [9] investigated the spectral radius of the distance signless Laplacian matrix of graphs.

Recent developments include improvements in upper bounds of Laplacian-energy-like invariants for trees [13], as well as bounds for normalized Laplacian energy [22]. Additionally, studies on the Hermitian Laplacian matrix and classical Laplacian matrix have been discussed in [11,21].

Moreover, investigations into the energies of commuting and non-commuting graphs of dihedral groups have been carried out in [17,18], where measures such as Seidel energy and closeness energy were introduced and analyzed. In addition, Romdhini et al. [19,20] examined the non-commuting graph associated with the group U_{6n} in relation to Seidel and Laplacian matrices.

The paper is organized as follows. Section 2 presents preliminary results and existing theoretical foundations relevant to the study. Section 3 provides the main results, including explicit formulas for the spectrum and energy of $\Gamma_{U_{6n}}$ corresponding to the normalized Laplacian (NL) and normalized signless Laplacian (NSL) matrices. Finally, the concluding section discusses the obtained results and highlights the relationship between the energies derived from these two matrices.

2. PRELIMINARIES

This section summarizes essential definitions and known results that will be utilized in the subsequent analysis.

Let $\Gamma_{U_{6n}}$ be the non-commuting graph associated with the group U_{6n}.

Definition 2.1. The adjacency matrix of $\Gamma_{U_{6n}}$, denoted by $A(\Gamma_{U_{6n}}) = [a_{ij}]$ of order $n \times n$, is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \neq v_j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2. The diagonal degree matrix $D(\Gamma_{U_{6n}}) = [d_{ij}]$ is given by

$$d_{ij} = \begin{cases} d_{v_i}, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

where d_{v_i} denotes the degree of the vertex v_i, i.e., the number of vertices adjacent to v_i.

Accordingly, the matrix $\sqrt{D(\Gamma_{U_{6n}})}$ is defined as

$$\sqrt{d_{ij}} = \begin{cases} \sqrt{d_{v_i}}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. ([3]) The normalized Laplacian matrix of $\Gamma_{U_{6n}}$ is is defined as

$$NL(\Gamma_{U_{6n}}) = \sqrt{D(\Gamma_{U_{6n}})}^{-1}(D(\Gamma_{U_{6n}}) - A(\Gamma_{U_{6n}}))\sqrt{D(\Gamma_{U_{6n}})}^{-1}.$$

Definition 2.4. ([3]) The normalized signless Laplacian matrix is given by

$$NSL(\Gamma_{U_{6n}}) = \sqrt{D(\Gamma_{U_{6n}})}^{-1}(D(\Gamma_{U_{6n}}) + A(\Gamma_{U_{6n}}))\sqrt{D(\Gamma_{U_{6n}})}^{-1}.$$

The spectrum of the normalized Laplacian matrix can be expressed as

$$\sigma_{NL}(\Gamma_{U_{6n}}) = \begin{pmatrix} \lambda_1, & \lambda_2, & \dots, & \lambda_n \\ k_1, & k_2, & \dots, & k_n \end{pmatrix},$$

where λ_i are the eigenvalues k_i denote their corresponding multiplicities. The normalized Laplacian energy is then defined by

$$E_{NL}(\Gamma_{U_{6n}}) = \Sigma_{i=1}^{n} |\lambda_i|,$$

while the spectral radius is given by

$$\rho_{NL}(\Gamma_{U_{6n}}) = max\{|\lambda| : \lambda \in \sigma_{NL}(\Gamma_{U_{6n}}).$$

Analogous definitions apply to the normalized signless Laplacian matrix.

Throughout this paper, the non-commuting graph of U_{6n} is is denoted by $\Gamma_{U_{6n}}$. Since it is a simple graph, both NL and NSL-matrices are well-defined.

We partition the group $U_{6n}\backslash Z(U_{6n})$ into the following subsets:

$$\begin{aligned} G_1 &= \left\{a^{2r+1} : 0 \le r \le n-1\right\} \\ G_2 &= \left\{a^{2r+1}b : 0 \le r \le n-1\right\} \\ G_3 &= \left\{a^{2r+1}b^2 : 0 \le r \le n-1\right\} \\ G_4 &= \left\{a^{2r}b : 0 \le r \le n-1\right\} \\ G_5 &= \left\{a^{2r}b^2 : 0 \le r \le n-1\right\}, \end{aligned}$$

each having cardinality n.

The following result is useful for simplifying determinant computations.

Theorem 2.5. ([4]) *If a square matrix be partitioned as* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *where* $|A| \neq 0$. *Then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left|D - CA^{-1}B\right|.$$

The following known result describes centralizers in U_{6n}.

Lemma 2.6. ([14]) *For $0 \leq r \leq n-1$, the following hold:*

1. $Z(U_{2n}) = \langle a^2 \rangle$.
2. $C_{U_{6n}}\left(a^{2r+1}\right) = \langle a \rangle$.
3. $C_{U_{6n}}\left(a^{2r+1}b\right) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b : 0 \leq s \leq n-1\} \rangle$.
4. $C_{U_{6n}}\left(a^{2r+1}b^2\right) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b^2 : 0 \leq s \leq n-1\} \rangle$.
5. $C_{U_{6n}}\left(a^{2r}b\right) = \langle a^2 \rangle \cdot \langle \{a^{2s}b, a^{2s}b^2 : 0 \leq s \leq n-1\} \rangle$.

As a consequence, the degrees of vertices in $\Gamma_{U_{6n}}$ are given as follows.

Corollary 2.7. ([10]) *For $n \geq 1$ and $0 \leq r \leq n-1$:*

1. $d_{a^{2r+1}} = 4n$,
2. $d_{a^{2r+1}b} = 4n$,
3. $d_{a^{2r+1}b^2} = 4n$,
4. $d_{a^{2r}b} = 3n$,
5. $d_{a^{2r}b^2} = 3n$.

It is well known that a graph may be classified based on its energy relative to the complete graph. Since $\Gamma_{U_{6n}}$ has $5n$ vertices, we adopt the following definition.

Definition 2.8. [12] A graph $\Gamma_{U_{6n}}$ with $5n$ vertices is said to be non-hypoenergetic if $E(\Gamma_{U_{6n}}) \geq 5n$.

3. MAIN RESULTS

In this section, we derive the energy of $\Gamma_{U_{6n}}$ in relation to NL and NSL matrices.

3.1. Normalized Laplacian Energy of $\Gamma_{U_{6n}}$.

We begin by determining the characteristic polynomial of the normalized Laplacian matrix associated with $\Gamma_{U_{6n}}$.

Theorem 3.1. *Let $P_{NL(\Gamma_{U_{6n}})}(\lambda)$ denote the characteristic polynomial of $NL(\Gamma_{U_{6n}})$. Then*

$$P_{NL(\Gamma_{U_{6n}})}(\lambda) = (\lambda-1)^{5n-4}\left(\lambda - \frac{5}{4}\right)^2\left(\lambda - \frac{3}{2}\right)\lambda.$$

Proof. Since the graph $\Gamma_{U_{6n}}$ contains $5n$ vertices, the $NL(\Gamma_{U_{6n}})$ is of order $5n \times 5n$ matrix. Based on Lemma 2.6, the entries of $NL(\Gamma_{U_{6n}}) = [nl_{ij}]$ can be described as follows:

1. $nl_{ij} = 0$, for $1 \leq i \neq j \leq n$, $n+1 \leq i \neq j \leq 2n$, $2n+1 \leq i \neq j \leq 3n$, and $3n+1 \leq i \neq j \leq 5n$;
2. $nl_{ij} = -\frac{1}{2n\sqrt{3}}$, for $1 \leq i \leq 3n$, $3n+1 \leq j \leq 5n$, as well as for $3n+1 \leq i \leq 5n$, $1 \leq j \leq 3n$;
3. $nl_{ij} = -\frac{1}{4n}$, for $1 \leq i \leq n$, $n+1 \leq j \leq 3n$, for $n+1 \leq i \leq 3n$, $1 \leq j \leq n$, for $n+1 \leq i \leq 2n$, $2n+1 \leq j \leq 3n$, and for $2n+1 \leq i \leq 3n$, $n+1 \leq j \leq 2n$;
4. $nl_{ij} = 1$, for $i = j$.

We can construct $NL(\Gamma_{U_{6n}})$ as follows:

$$\begin{array}{c|ccccccccccccccc}
 & a & \cdots & a^{2n-1} & ab & \cdots & a^{2n-1}b & ab^2 & \cdots & a^{2n-1}b^2 & b & \cdots & a^{2(n-1)}b & b^2 & \cdots & a^{2(n-1)}b^2 \\
\hline
a & 1 & \cdots & 0 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & \frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1} & 0 & \cdots & 1 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
ab & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & 1 & \cdots & 0 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & 0 & \cdots & 1 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
ab^2 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & 1 & \cdots & 0 & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b^2 & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & -\frac{1}{4n} & \cdots & -\frac{1}{4n} & 0 & \cdots & 1 & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} \\
b & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2(n-1)}b & -\frac{1}{2n\sqrt{3}} & -\cdots & \frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & 0 & \cdots & 1 & 0 & \cdots & 0 \\
b^2 & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2(n-1)}b^2 & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & -\frac{1}{2n\sqrt{3}} & \cdots & -\frac{1}{2n\sqrt{3}} & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}.$$

Furthermore, $NL(\Gamma_{U_{6n}})$ can be decomposed into 25 block submatrices, arranged as follows:

$$NL(\Gamma_{U_{6n}}) = \begin{pmatrix}
I_n & -\frac{1}{4n}J_n & -\frac{1}{4n}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\
-\frac{1}{4n}J_n & I_n & -\frac{1}{4n}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\
-\frac{1}{4n}J_n & -\frac{1}{4n}J_n & I_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\
-\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & I_n & 0_n \\
-\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & 0_n & I_n
\end{pmatrix}.$$

By applying $P_{NL(\Gamma_{U_{6n}})}(\lambda) = |\lambda I_{5n} - NL(\Gamma_{U_{6n}})|$, we obtain

$$P_{NL(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix}
(\lambda-1)I_n & \frac{1}{4n}J_n & \frac{1}{4n}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{4n}J_n & (\lambda-1)I_n & \frac{1}{4n}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{4n}J_n & \frac{1}{4n}J_n & (\lambda-1)I_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & (\lambda-1)I_n & 0_n \\
\frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & 0_n & (\lambda-1)I_n
\end{vmatrix}.$$

To derive the expression for $P_{NL(\Gamma_{U_{6n}})}(\lambda)$, it is essential to apply a sequence of elementary row and column transformations. Let R_i and C_i denote the operations performed on the i-th row and i-th column of the matrix, respectively. The computation proceeds according to the following steps:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{3n}$, for $1 \le i \le 2n$.

(2) $R_{j-i} \longrightarrow R_{j-i} - R_{3n-i}$, for $1 \le i \le n$ and $j = n, 2n$.

(3) $R_{3n-i} \longrightarrow R_{3n-i} - R_{3n}$, for $1 \le i \le n-1$.

(4) $C_{2n+i} \longrightarrow C_{2n+i} + C_{j+i}$, for $1 \le i \le n$ and $j = 0, n$.

(5) $C_{3n+1} \longrightarrow C_{3n+1} + C_{3n+2} + C_{3n+3} + \ldots + C_{4n} + C_{4n+1} + \ldots + C_{5n}$.

(6) $R_{j+i} \longrightarrow R_{j+i} - R_{n+j}$, for $1 \leq i \leq n-1$ and $j = 0, n$.

(7) $C_j \longrightarrow C_j + C_{j-1} + C_{j-2} + \ldots + C_{j-(n-1)}$, $j = n, 2n, 3n$.

(8) $R_{3n} \longrightarrow R_{3n} - \frac{1}{2n(\lambda-1)\sqrt{3}} R_{3n+2} - \frac{1}{2n(\lambda-1)\sqrt{3}} R_{3n+3} - \ldots + \frac{1}{2n(\lambda-1)\sqrt{3}} R_{4n} - \frac{1}{2n(\lambda-1)\sqrt{3}} R_{4n+1} - \ldots - \frac{1}{2n(\lambda-1)\sqrt{3}} R_{5n}$.

Thus, $P_{NL(\Gamma_{U_{6n}})}(\lambda)$ can be expressed in the form of the following determinant:

$$\begin{vmatrix}
(\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
-\frac{1}{4n}J_{1\times(n-1)} & \lambda-\frac{5}{4} & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} & 0 & 0 & 0_{1\times(n-1)} & 0_{1\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
0_{1\times(n-1)} & 0 & -\frac{1}{4n}J_{1\times(n-1)} & \lambda-\frac{5}{4} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\
\frac{1}{4n}J_{1\times(n-1)} & \frac{1}{4} & \frac{1}{4n}J_{1\times(n-1)} & \frac{1}{4} & \frac{1}{2n}J_{1\times(n-1)} & \lambda-\frac{1}{2} & \frac{1}{\sqrt{3}} & 0_{1\times(n-1)} & 0_{1\times n} \\
\frac{1}{2n\sqrt{3}}J_{1\times(n-1)} & \frac{1}{2\sqrt{3}} & \frac{1}{2n\sqrt{3}}J_{1\times(n-1)} & \frac{1}{2\sqrt{3}} & \frac{3}{2n\sqrt{3}}J_{1\times(n-1)} & \frac{3}{2\sqrt{3}} & \lambda-1 & 0_{1\times(n-1)} & 0_{1\times n} \\
0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times n} \\
0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & (\lambda-1)I_{n\times n}
\end{vmatrix}. \tag{3.1}$$

By applying Theorem 2.5, $P_{NL(\Gamma_{U_{6n}})}(\lambda)$ can be expressed as follows:

$$P_{NL(\Gamma_{U_{6n}})}(\lambda) = (\lambda-1)^{5n-4}\left(\lambda-\frac{5}{4}\right)^2\left(\lambda-\frac{3}{2}\right)\lambda.$$

□

Theorem 3.2. *The $NL-$spectrum of $\Gamma_{U_{6n}}$ is given by*

$$Spec(\Gamma_{U_{6n}}) = \begin{pmatrix} \frac{3}{2} & \frac{5}{4} & 1 & 0 \\ 1 & 2 & 5n-4 & 1 \end{pmatrix}.$$

Proof. The roots of $P_{NL(\Gamma_{U_{6n}})}(\lambda) = 0$ are the eigenvalues of $\Gamma_{U_{6n}}$. From Theorem 3.1, the eigenvalues are $\lambda_1 = \frac{3}{2}$ of multiplicity 1, $\lambda_2 = -\frac{5}{4}$ with multiplicity 2, $\lambda_3 = 1$ of multiplicity $5n-4$, and $\lambda_4 = -0$ of multiplicity 1. Therefore, the spectrum of $\Gamma_{U_{6n}}$ associated to $NL-$matrix is

$$Spec(\Gamma_{U_{6n}}) = \begin{pmatrix} \frac{3}{2} & \frac{5}{4} & 1 & 0 \\ 1 & 2 & 5n-4 & 1 \end{pmatrix}.$$

□

Theorem 3.3. *The $NL-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_{NL}(\Gamma_{U_{6n}}) = \frac{3}{2}.$$

Proof. It is clear by Theorem 3.2, the largest absolute λ_i for $i = 1, 2, 3, 4$ is 1. □

Theorem 3.4. *The $NL-$energy of $\Gamma_{U_{6n}}$ is*

$$E_{NL}(\Gamma_{U_{6n}}) = 5n.$$

Proof. By applying the result of Theorem 3.2, the NL-energy of $\Gamma_{U_{6n}}$ is obtained as follows:

$$E_{NL}(\Gamma_{U_{6n}}) = (1)\left|\frac{3}{2}\right| + (2)\left|\frac{5}{4}\right| + (5n-4)\,|1| + (1)\,|0| = 5n.$$

□

3.2. Normalized Signless Laplacian Energy of $\Gamma_{U_{6n}}$.

Theorem 3.5. *Let $P_{NSL(\Gamma_{U_{6n}})}(\lambda)$ be the characteristic polynomial of $NSL(\Gamma_{U_{6n}})$. Then*

$$P_{NSL(\Gamma_{U_{6n}})}(\lambda) = (\lambda-1)^{5n-4}\left(\lambda-\frac{3}{4}\right)^2(\lambda-2)\left(\lambda-\frac{1}{2}\right).$$

Proof. Since $\Gamma_{U_{6n}}$ has $5n$ vertices, then $NSL(\Gamma_{U_{6n}})$ is a $5n\times 5n$ matrix. According to Lemma 2.6, then the $(i,j)-$entries of $NSL(\Gamma_{U_{6n}}) = [nsl_{ij}]$ are:

(1) $nsl_{ij} = 0$, for $1 \le i \ne j \le n$, $n+1 \le i \ne j \le 2n$, $2n+1 \le i \ne j \le 3n$, and $3n+1 \le i \ne j \le 5n$;

(2) $nsl_{ij} = \frac{1}{2n\sqrt{3}}$, for $1 \le i \le 3n$, $3n+1 \le j \le 5n$, and for $3n+1 \le i \le 5n$, $1 \le j \le 3n$;

(3) $nsl_{ij} = \frac{1}{4n}$, for $1 \le i \le n$, $n+1 \le j \le 3n$, for $n+1 \le i \le 3n$, $1 \le j \le n$, for $n+1 \le i \le 2n$, $2n+1 \le j \le 3n$, and for $2n+1 \le i \le 3n$, $n+1 \le j \le 2n$;

(4) $nsl_{ij} = 1$, for $i = j$.

We can construct $NSL(\Gamma_{U_{6n}})$ as follows:

$$\begin{array}{c|ccccccccccccccc|}
 & a & \cdots & a^{2n-1} & ab & \cdots & a^{2n-1}b & ab^2 & \cdots & a^{2n-1}b^2 & b & \cdots & a^{2(n-1)}b & b^2 & \cdots & a^{2(n-1)}b^2 \\
\hline
a & 1 & \cdots & 0 & \frac{1}{4n} & \cdots & \frac{1}{4n} & -\frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1} & 0 & \cdots & 1 & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
ab & \frac{1}{4n} & \cdots & \frac{1}{4n} & 1 & \cdots & 0 & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b & \frac{1}{4n} & \cdots & \frac{1}{4n} & 0 & \cdots & 1 & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
ab^2 & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{4n} & \cdots & \frac{1}{4n} & 1 & \cdots & 0 & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b^2 & \frac{1}{4n} & \cdots & \frac{1}{4n} & \frac{1}{4n} & \cdots & \frac{1}{4n} & 0 & \cdots & 1 & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} \\
b & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2(n-1)}b & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & 0 & \cdots & 1 & 0 & \cdots & 0 \\
b^2 & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{2(n-1)}b^2 & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & \frac{1}{2n\sqrt{3}} & \cdots & \frac{1}{2n\sqrt{3}} & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}.$$

Furthermore, $NSL(\Gamma_{U_{6n}})$ can be decomposed into 25 block submatrices, arranged as follows:

$$NSL(\Gamma_{U_{6n}}) = \begin{pmatrix}
I_n & \frac{1}{4n}J_n & \frac{1}{4n}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{4n}J_n & I_n & \frac{1}{4n}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{4n}J_n & \frac{1}{4n}J_n & I_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n \\
\frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & I_n & 0_n \\
\frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & \frac{1}{2n\sqrt{3}}J_n & 0_n & I_n
\end{pmatrix}.$$

By applying $P_{NSL(\Gamma_{U_{6n}})}(\lambda) = |\lambda I_{5n} - NSL(\Gamma_{U_{6n}})|$, we obtain

$$P_{NSL(\Gamma_{U_{6n}})}(\lambda) = \begin{vmatrix} (\lambda-1)I_n & -\frac{1}{4n}J_n & -\frac{1}{4n}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\ -\frac{1}{4n}J_n & (\lambda-1)I_n & -\frac{1}{4n}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\ -\frac{1}{4n}J_n & -\frac{1}{4n}J_n & (\lambda-1)I_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n \\ -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & (\lambda-1)I_n & 0_n \\ -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & -\frac{1}{2n\sqrt{3}}J_n & 0_n & (\lambda-1)I_n \end{vmatrix}.$$

To obtain the explicit form of $P_{NSL(\Gamma_{U_{6n}})}(\lambda)$, a sequence of elementary row and column operations is required. Let R_i and C_i denote be the the operations applied to the $i-$th row and $i-$th column, respectively, of the matrix associated with $P_{NSL(\Gamma_{U_{6n}})}(\lambda)$. The procedure is carried out through the following steps:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{3n}$, for $1 \leq i \leq 2n$.
(2) $R_{j-i} \longrightarrow R_{j-i} - R_{3n-i}$, for $1 \leq i \leq n$ and $j = n, 2n$.
(3) $R_{3n-i} \longrightarrow R_{3n-i} - R_{3n}$, for $1 \leq i \leq n-1$.
(4) $C_{2n+i} \longrightarrow C_{2n+i} + C_{j+i}$, for $1 \leq i \leq n$ and $j = 0, n$.
(5) $C_{3n+1} \longrightarrow C_{3n+1} + C_{3n+2} + C_{3n+3} + \ldots + C_{4n} + C_{4n+1} + \ldots + C_{5n}$.
(6) $R_{j+i} \longrightarrow R_{j+i} - R_{n+j}$, for $1 \leq i \leq n-1$ and $j = 0, n$.
(7) $C_j \longrightarrow C_j + C_{j-1} + C_{j-2} + \ldots + C_{j-(n-1)}$, $j = n, 2n, 3n$.
(8) $R_{3n} \longrightarrow R_{3n} + \frac{1}{2n(\lambda-1)\sqrt{3}}R_{3n+2} + \frac{1}{2n(\lambda-1)\sqrt{3}}R_{3n+3} + \ldots + \frac{1}{2n(\lambda-1)\sqrt{3}}R_{4n} + \frac{1}{2n(\lambda-1)\sqrt{3}}R_{4n+1} + \ldots + \frac{1}{2n(\lambda-1)\sqrt{3}}R_{5n}$.

Thus, $P_{NSL(\Gamma_{U_{6n}})}(\lambda)$ can be expressed in the form of the following determinant:

$$\begin{vmatrix} (\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ -\frac{1}{4n}J_{1\times(n-1)} & \lambda-\frac{3}{4} & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} & 0 & 0 & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{1\times(n-1)} & 0 & -\frac{1}{4n}J_{1\times(n-1)} & \lambda-\frac{3}{4} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times n} \\ -\frac{1}{4n}J_{1\times(n-1)} & -\frac{1}{4} & -\frac{1}{4n}J_{1\times(n-1)} & -\frac{1}{4} & -\frac{1}{2n}J_{1\times(n-1)} & \lambda-\frac{3}{2} & -\frac{1}{\sqrt{3}} & 0_{1\times(n-1)} & 0_{1\times n} \\ -\frac{1}{2n\sqrt{3}}J_{1\times(n-1)} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2n\sqrt{3}}J_{1\times(n-1)} & -\frac{1}{2\sqrt{3}} & -\frac{3}{2\sqrt{3}}J_{1\times(n-1)} & -\frac{3}{2\sqrt{3}} & \lambda-1 & 0_{1\times(n-1)} & 0_{1\times n} \\ 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & 0_{n\times(n-1)} & 0_{n\times 1} & (\lambda-1)I_{n\times n} \end{vmatrix}. \tag{3.2}$$

By Theorem 2.5, $P_{NSL(\Gamma_{U_{6n}})}(\lambda)$ can be stated as

$$P_{NSL(\Gamma_{U_{6n}})}(\lambda) = (\lambda-1)^{5n-4}\left(\lambda - \frac{3}{4}\right)^2 (\lambda-2)\left(\lambda - \frac{1}{2}\right).$$

□

Theorem 3.6. *The $NSL-$spectrum of $\Gamma_{U_{6n}}$ is*

$$Spec(\Gamma_{U_{6n}}) = \begin{pmatrix} 2 & 1 & \frac{3}{4} & \frac{1}{2} \\ 1 & 5n-4 & 2 & 1 \end{pmatrix}.$$

Proof. The roots of $P_{NSL(\Gamma_{U_{6n}})}(\lambda) = 0$ are the eigenvalues of $\Gamma_{U_{6n}}$. From Theorem 3.5, the eigenvalues are $\lambda_1 = 2$ of multiplicity 1, $\lambda_2 = 1$ with multiplicity $5n - 4$, $\lambda_3 = \frac{3}{4}$ of multiplicity 2, and $\lambda_4 = \frac{1}{2}$ of multiplicity 1. Therefore, the spectrum of $\Gamma_{U_{6n}}$ associated to $NSL-$matrix is

$$Spec(\Gamma_{U_{6n}}) = \begin{pmatrix} 2 & 1 & \frac{3}{4} & \frac{1}{2} \\ 1 & 5n-4 & 2 & 1 \end{pmatrix}.$$

□

Theorem 3.7. *The $NSL-$spectral radius of $\Gamma_{U_{6n}}$ is*

$$\rho_{NSL}(\Gamma_{U_{6n}}) = 2.$$

Proof. It is clear by Theorem 3.6, the largest absolute λ_i for $i = 1, 2, 3, 4$ is 1. □

Theorem 3.8. *The $NSL-$energy of $\Gamma_{U_{6n}}$ is*

$$E_{NSL}(\Gamma_{U_{6n}}) = 5n.$$

Proof. Using the result of Theorem 3.6, then the NSL-energy of $\Gamma_{U_{6n}}$ as follows:

$$E_{NSL}(\Gamma_{U_{6n}}) = (1)\,|2| + (5n-4)\,|1| + (2)\left|\frac{3}{4}\right| + (1)\left|\frac{1}{2}\right| = 5n.$$

□

4. Discussion

Based on the results established in Theorems 3.4 and 3.8, together with Definition 2.8, we obtain the following consequences.

Corollary 4.1. *The graph $\Gamma_{U_{6n}}$ is non-hypoenergetic with respect to both NL and NSL energies.*

Corollary 4.2. *The energies $E_{NL}(\Gamma_{U_{6n}})$ and $E_{NSL}(\Gamma_{U_{6n}})$ are odd integers.*

The result stated in Corollary 4.2 (1) is consistent with the established findings reported in [2] and [15]. Furthermore, by directly comparing the results in Theorem 3.4 and 3.8, we obtain the following immediate consequence.

Corollary 4.3. $E_{NL}(\Gamma_{U_{6n}}) = E_{NSL}(\Gamma_{U_{6n}})$.

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