Asia Pac. J. Math. 2026 **13**:79

OBSERVABILITY INEQUALITIES FOR COUPLED SCHRÖDINGER SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS VIA CARLEMAN ESTIMATES

ATEF SACI[1,2,*], SALAH-EDDINE REBIAI[2,3,*]

[1]Department of Mathematics, University Center of Barika, Algeria

[2]LTM, University of Batna 2, Batna, Algeria

[3]Department of Mathematics, University of Batna 2, Algeria

[*]Corresponding authors: atef.saci@cu-barika.dz, s.rebiai@univ-batna2.dz

Received Mar. 27, 2026

Abstract. We consider a system of two coupled Schrödinger equations with homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$. Under suitable regularity assumptions on the coefficients and a geometric condition on the observation subboundary Γ_1, we establish an observability inequality for the corresponding solution. More precisely, we prove that the initial energy $E(0)$ can be estimated from above by the energy localized on the observed part of the boundary over a sufficiently large time interval $[0, T]$. The proof relies on Carleman estimates for Schrödinger operators, combined with a regularization argument to handle the Neumann boundary conditions. The resulting inequality is a key tool for inverse problems and for controllability/stabilization results of coupled Schrödinger systems.

2020 Mathematics Subject Classification. 35Q40; 35R30; 93B07.

Key words and phrases. Observability inequality, coupled Schrödinger equations, Neumann boundary conditions, Carleman estimates.

1. Introduction

The concept of observability inequality plays a fundamental role in the theory of partial differential equations, particularly in the context of control theory, inverse problems, and stabilization of dynamical systems. Observability inequalities provide a quantitative estimate that bounds the energy of a system at initial time by the measurements of the solution on a portion of the boundary over a finite time interval. More precisely, for a given evolution equation, an observability inequality typically takes the form

$$C_T \mathbb{E}(0) \leq \int_0^T \int_{\Gamma_1} |\text{measurement}|^2 \, d\Gamma \, dt,$$

where $\mathbb{E}(0)$ represents the initial energy of the system and the right-hand side involves boundary observations on a subboundary Γ_1. Such inequalities are essential for establishing exact controllability

DOI: 10.28924/APJM/13-79

(via duality), uniform stabilization, and inverse problem stability. The Carleman estimate has emerged as a powerful and systematic tool for deriving observability inequalities, especially for equations with lower-order terms, variable coefficients, and in the presence of boundary conditions that do not satisfy the Lopatinski condition (such as Neumann boundary conditions). Carleman estimates are weighted L^2-estimates involving an exponential weight of the form $e^{\tau\phi(x,t)}$, where ϕ is a pseudo-convex function and $\tau > 0$ is a large parameter. Their strength lies in the ability to absorb lower-order terms and boundary contributions, often yielding observability inequalities without the need for independent uniqueness results.

The seminal work of [9] provided one of the first systematic treatments of Carleman estimates for hyperbolic equations, laying the foundation for their application to unique continuation and inverse problems. This approach was later refined and extended to Schrödinger equations in a series of important contributions. [5] and [6] developed multiplier methods for wave and Schrödinger equations, establishing regularity and continuous observability inequalities in the Dirichlet case with optimal spaces. However, these early results required lower-order terms to be handled via compactness/uniqueness arguments, leading to non-explicit constants.

A major advance came with the work of [15], who provided Carleman estimates for non-conservative Schrödinger equations on Euclidean domains. These estimates contained interior lower-order terms, but the methodology introduced the use of the multiplier $e^{\tau\phi}\nabla\phi \cdot \nabla\bar{w}$ and established a one-parameter family of estimates. This work was subsequently generalized to Riemannian manifolds by [17] and [16], extending the applicability to variable coefficient principal parts. A distinguishing feature of the work by [7] and [8] (Part I and Part II) is the elimination of lower-order terms from the Carleman estimates at the $H^1(\Omega)$-level. This is achieved through a refined pointwise estimate followed by an integral inequality that yields a Carleman estimate without interior lower-order terms. The key innovation is the use of a "pointwise" approach inspired by [9], which allows the absorption of lower-order terms via a large parameter τ. This development is critical because it yields observability inequalities with explicit constants of the form $Ce^{-C\ell^2}$, where ℓ measures the norm of the coefficients. Such explicit estimates are valuable for semilinear problems [4,20] and for computing minimal norm steering controls [3,14]. The case of Neumann boundary conditions presents particular challenges, as finite energy solutions do not produce H^1-traces on the boundary. [7,8] overcame this difficulty through a delicate regularization argument, employing a dissipative boundary problem and an approximation procedure that extends the Carleman estimates from $H^{2,2}(Q)$ to $H^{1,1}(Q)$ solutions.

The observability inequality for the Neumann case requires the control of the tangential gradient on the observed boundary Γ_1. This is accomplished via a microlocal analysis result, which together with the global uniqueness theorem yields the final continuous observability inequality.

The Riemannian setting was further developed by [16], where Carleman estimates for non-conservative Schrödinger equations on Riemannian manifolds are established. This generalizes the Euclidean results of [7,8] and parallels the earlier extension for wave equations by [18]. The use of Bochner's techniques and the Levi-Civita connection enables the treatment of variable coefficient principal parts, which correspond to elliptic operators with space-dependent coefficients in $\mathbb{R}^n$. [19] applied Carleman estimates to inverse problems for hyperbolic equations, establishing Lipschitz stability for the determination of potentials from Neumann boundary measurements. The work highlights the interplay between Carleman estimates, observability inequalities, and compactness/uniqueness arguments. Yamamoto's results demonstrate that the best possible Lipschitz stability can be achieved under optimal time conditions $T > \rho$.

In summary, the development of observability inequalities via Carleman estimates has progressed through several key phases: foundational pointwise estimates [9]; multiplier methods providing initial observability results for wave and Schrödinger equations [5,6]; Carleman estimates with lower-order terms extended to non-conservative equations and Riemannian manifolds [15,17]; the elimination of lower-order terms yielding Carleman estimates without interior lower-order terms, thereby producing explicit constants and enabling a "one-shot" derivation of observability and uniqueness [7,8]; the resolution of the Neumann boundary case through regularization and tangential trace estimates [7,8]; Riemannian generalizations to variable coefficients [16]; and applications to inverse problems demonstrating the power of these estimates for stability [13,19]. The present work fits within this framework, building upon the Carleman estimates established in [7,8] to derive observability inequalities for coupled non-conservative Schrödinger equations, which serve as the foundation for controllability and stability results.

2. Problem setting for the observability inequality

Let $T > 0$ and let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be an open bounded domain with boundary of class C^2. Throughout this paper, we use the following notations :

$\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. $\nabla v = (\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N})$, $\Delta v = \sum_{i=1}^N \frac{\partial^2 v}{\partial x_i^2}$,

ν denotes the unit outward normal vector to $\Gamma = \partial\Omega$, $\frac{\partial v}{\partial \nu} = \nabla v \cdot \nu$ is the normal derivative.

Following [1,2,7,12,13,15], we make the following assumptions.

Assumption (H).

There exists a non-negative function $d : \overline{\Omega} \longrightarrow \mathbb{R}_+$ of class C^3, such that

(i) if we set $\nabla d = h$, then

$$\frac{\partial d}{\partial \nu} = h \cdot \nu = 0 \text{ on } \Gamma_0, \tag{1}$$

(ii) the (symmetric) Hessian matrix $\mathcal{H}_d$ of $d(.)$ is strictly positive definite on $\overline{\Omega}$, d is strictly convex in $\overline{\Omega}$. i.e. there exists $\rho > 0$ such that for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{C}^N$,

$$\operatorname{Re} \mathcal{H}_d(x)\xi \cdot \overline{\xi} \geq \rho \, |\xi|^2 , \tag{2}$$

(iii) d has no critical point in $\overline{\Omega}$:

$$\inf_{x \in \overline{\Omega}} \|\nabla d(x)\| = s > 0. \tag{3}$$

Remark 2.1. *Assumption* (H) *holds true for large classes of triples* $\{\Omega, \Gamma_0, \Gamma_1\}$*, see the appendices in* [2,7].

Let $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$, $\Sigma_1 = \Gamma_1 \times [0,T]$. We consider the following coupled system of two Schrödinger equations with homogeneous Neumann boundary conditions:

$$iw_t + \Delta w = a(x) \cdot \nabla w + n(x)w + \alpha(x) \cdot \nabla z + q(x)z \quad \text{in } Q, \tag{4a}$$

$$iz_t + \Delta z = b(x) \cdot \nabla z + m(x)z + \beta(x) \cdot \nabla w + p(x)w \quad \text{in } Q, \tag{4b}$$

$$w\Big(x, \frac{T}{2}\Big) = w_0(x) \quad \text{in } \Omega, \tag{4c}$$

$$z\Big(x, \frac{T}{2}\Big) = z_0(x) \quad \text{in } \Omega, \tag{4d}$$

$$\frac{\partial w}{\partial \nu} = 0, \quad \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \Sigma. \tag{4e}$$

Here a, b are purely imaginary. The coefficients $a, b, n, m, \alpha, \beta, q, p$ are known and satisfy

$$a, b \in \left[W^{1,\infty}(\Omega)\right]^N, \quad n, m \in W^{1,\infty}(\Omega), \quad \alpha, \beta \in [L^\infty(\Omega)]^N, \quad q, p \in W^{1,\infty}(\Omega),$$

and the initial data $(w_0, z_0) \in H^1(\Omega) \times H^1(\Omega)$.

Under Assumption (H), we aim to establish an observability inequality that provides an upper bound for the initial energy in terms of boundary measurements on Γ_1 over the time interval $[0,T]$.

Remark 2.2. *In models* (4a)–(4e)*, we regard* $t = \frac{T}{2}$ *as the initial time. This is not essential, as the change of variable* $t \to t - \frac{T}{2}$ *transforms* $t = \frac{T}{2}$ *to* $t = 0$*. However, this present choice is convenient in order to apply the Carleman estimate established in* [7] *for the Schrödinger equation, which uses the pseudo-convex function* $\varphi(x,t)$ *in* (5) *centered around* $\frac{T}{2}$.

3. Carleman estimate: an important previous result

We now present the Carleman estimate, which is the essential tool for proving the observability inequality, the main result of this paper.

3.1. **Carleman Estimate.**

We will present the Carleman estimate for a single Schrödinger equation (see [7]).

3.1.1. ***Pseudo-convex function*** ([7], *p. 46*). Choosing the strictly convex potential function $d(x)$ satisfying Assumptions (H) and $d(x) \geq d_0 > 0$, we next introduce the pseudo-convex function $\varphi(x,t)$ defined by

(5) $$\varphi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2, \; x \in \Omega, \; t \in [0.T],$$

where $T > 0$ is arbitrary and $c = c_T$ is chosen large enough as to have

(6) $$cT^2 > 4\max_{x\in\overline{\Omega}} d(x) \;\text{ so that } cT^2 > 4\max_{x\in\overline{\Omega}} d(x) + 4\delta,$$

for a suitably small $\delta > 0$, henceforth kept fixed. With $d(x)$, T and c chosen as described above, this function $\varphi(x,t)$ has the following properties:

(7) $$\varphi(x,t) \leq \varphi\left(x, \frac{T}{2}\right) \text{ for any } t > 0 \text{ and any } x \in \Omega,$$

(8) $$\varphi(x,0) = \varphi(x,T) = d(x) - c\frac{T^2}{4} \leq -\delta \text{ uniformly in } x \in \Omega.$$

There are t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that for all $x \in \Omega$

(9) $$\min_{x\in\overline{\Omega},\, t\in[t_0,t_1]} \varphi(x,t) \geq \sigma > 0.$$

Since $\varphi\left(x, \frac{T}{2}\right) = d(x) \geq d_0 > 0$. for all $x \in \Omega$ (in fact, only the weaker property: $\min_{x\in\overline{\Omega},\, t\in[t_0,t_1]} \varphi(x,t) \geq \sigma > -\delta$ is actually needed).

3.1.2. ***Carleman estimate at the*** H^1***-level*** (***Schrödinger equation***). Consider the Schrödinger equation of the form

(10) $$iw_t(x,t) + \Delta w(x,t) = G(w) + F(x,t) \text{ in } Q, \; F \in L^2(Q),$$

$$G(w) = c_1(x).\nabla w + c_0(x) w,$$

where $|c_1|, c_0 \in L^\infty(\Omega)$ so that the following pointwise estimate holds true

$$|G(w)|^2 \leq C_T\left[|\nabla w|^2 + |w|^2\right], \; \forall (x,t) \in Q,$$

at first without imposing boundary conditions. For the solutions w to (10) in the class

(11) $$w \in H^{2.2}(Q) \equiv L^2\left(0,T,H^2(\Omega)\right) \cap H^2\left(0,T,L^2(\Omega)\right),$$

(12) $$\frac{\partial w}{\partial \nu} \in L^2\left(0,T,H^{\frac{1}{2}}(\Gamma)\right), \; w_t \in L^2\left(0,T,H^1(\Omega)\right), \; w_t|_\Gamma \in L^2\left(0,T,H^{\frac{1}{2}}(\Gamma)\right).$$

Theorem 3.1. ([7], *Theorem 5.1, p. 74*). *Assume* (H) *and* $F \in L^2(Q)$, $|c_1|, c_0 \in L^\infty(\Omega)$. *Let* w *be a solution of* (10) *in the class* (11)-(12). *Then the following one parameter family of estimates holds true, for all* $\tau > 0$ *sufficiently large*

$$B|_\Sigma(w) + 4\int_Q e^{2\tau\varphi}|F|^2\, dQ$$

$$\geq \widetilde{C}_{1,\tau}\int_Q e^{2\tau\varphi}\left[|\nabla w|^2\right]dQ+\widetilde{C}_{2,\tau}\int_Q e^{2\tau\varphi}|w|^2\,dQ-C_{d;T}\tau e^{-2\tau\delta}\left[E_w(0)+E_w(T)\right], \tag{13}$$

where

$$\widetilde{C}_{1,\tau}=\delta_0\left(2\tau\rho-\frac{1}{2}\right)-4C_T,\ \widetilde{C}_{2,\tau}=4\tau^3\rho s^2(1-\delta_0)+\grave{O}\left(\tau^2\right)-4C_T. \tag{14}$$

Here $s>0$, $\delta>0$ are as in assumption (3) , (6), $0<\delta_0<1$ is some fixed number, the constant in $\grave{O}$ depends on d,c,N, while $C_{d,T}$ is a positive constant depending on $d(x)$ and T. Moreover the boundary terms $BT|_\Sigma(w)$are given explicitly as follows, where $\xi=\operatorname{Re}(w)$, $\eta=\operatorname{Im}(w)$, after recalling also assumption (1)

$$\begin{aligned} B|_\Sigma(w) &= 2\tau\int_0^T\int_{\Gamma_1}e^{2\tau\varphi}\left[2\tau^2|h|^2+\Phi\right]|w|^2\,h.\nu d\Gamma dt\\ &\quad -4c\tau\int_0^T\int_\Gamma e^{2\tau\varphi}\left(t-\frac{T}{2}\right)\left[\eta\frac{\partial\xi}{\partial\nu}-\xi\frac{\partial\eta}{\partial\nu}\right]d\Gamma dt\\ &\quad -2\tau\int_0^T\int_{\Gamma_1}e^{2\tau\varphi}\left[\xi_t\eta-\xi\eta_t\right]h.\nu d\Gamma dt\\ &\quad +\int_0^T\int_\Gamma e^{2\tau\varphi}\left[2\tau^2|h|^2-\Phi\right]\left[\overline{w}\frac{\partial w}{\partial\nu}+w\frac{\partial\overline{w}}{\partial\nu}\right]d\Gamma dt\\ &\quad +2\tau\int_0^T\int_\Gamma e^{2\tau\varphi}h\left[\nabla\overline{w}\frac{\partial w}{\partial\nu}+\nabla w\frac{\partial\overline{w}}{\partial\nu}\right]d\Gamma dt\\ &\quad -2\tau\int_0^T\int_{\Gamma_1}e^{2\tau\varphi}|\nabla w|^2\,h.\nu d\Gamma dt. \end{aligned} \tag{15}$$

Here, c is the constant in (6), while the function Φ may be taken to satisfy: either $\Phi\equiv 0$, or else $\Phi=\tau\Delta d(x)$. Moreover, the energy function E_w is defined by

$$E_w(t)=\int_\Omega\left[|\nabla w(x,t)|^2+|w(x,t)|^2\right]d\Omega. \tag{16}$$

Next, we recall a Carleman estimate for the solutions of (10) in the class

$$w\in C\left([0.T],H^1(\Omega)\right),\frac{\partial w}{\partial\nu},\ w_t\in L^2\left(0,T,L^2(\Gamma)\right). \tag{17}$$

Theorem 3.2. ([7], *Theorem 7.3.1, p. 92). With reference to (10), assume (H) and $F\in L^2\left(0,T,L^2(\Omega)\right)$ .Then, the Carleman estimate (13), (14) hold true also for a solution of (10), in the class (17).*

4. Main result: Observability inequality

The main result of this paper is the following observability inequality for a system of coupled Schrödinger equations with homogeneous Neumann boundary conditions.

We consider the following system:

$$iw_t(x,t)+\Delta w(x,t)=a(x)\cdot\nabla w+n(x)w+\alpha(x)\cdot\nabla z+q(x)z\quad\text{in } Q, \tag{18a}$$

(18b) $$iz_t(x,t) + \Delta z(x,t) = b(x)\cdot\nabla z + m(x)z + \beta(x)\cdot\nabla w + p(x)w \quad \text{in } Q,$$

(18c) $$w\Big(x,\frac{T}{2}\Big) = w_0(x) \quad \text{in } \Omega,$$

(18d) $$z\Big(x,\frac{T}{2}\Big) = z_0(x) \quad \text{in } \Omega,$$

(18e) $$\frac{\partial w}{\partial\nu} = 0, \quad \frac{\partial z}{\partial\nu} = 0 \quad \text{on } \Sigma.$$

Here the coefficients satisfy

(19) $$a,b \in \big[W^{1,\infty}(\Omega)\big]^N, \quad n,m \in W^{1,\infty}(\Omega), \quad \alpha,\beta \in [L^\infty(\Omega)]^N, \quad q,p \in W^{1,\infty}(\Omega),$$

and the initial data satisfy $(w_0,z_0) \in H^1(\Omega)\times H^1(\Omega)$. Under these conditions one has

(20) $$w,z \in C([0,T];H^1(\Omega)).$$

We assume that the solution $\{w,z\}$ of (18a)–(18e) enjoys the additional regularity

(21) $$w,z \in C([0,T];H^1(\Omega)), \quad w_t,z_t \in L^2(0,T;L^2(\Gamma)), \quad \frac{\partial w}{\partial\nu},\frac{\partial z}{\partial\nu} \in L^2(0,T;L^2(\Gamma)).$$

Theorem 4.1. (*Observability inequality*). *Assume* (*H*), (19), *and* $T>0$. *Let* $\{w,z\}$ *be a solution of* (18a)–(18e) *in the class* (21). *Then there exists a constant* $C = C(\Omega,T,q,p) > 0$ *such that*

(22) $$C_T E(0) \le \int_0^T \int_{\Gamma_1} \big(|w|^2 + |w_t|^2 + |z|^2 + |z_t|^2\big)\, d\Gamma_1\, dt,$$

where

(23) $$\begin{aligned} E(t) &= E_w(t) + E_z(t),\\ E_w(t) &= \int_\Omega \big(|\nabla w(x,t)|^2 + |w(x,t)|^2\big)\,dx,\\ E_z(t) &= \int_\Omega \big(|\nabla z(x,t)|^2 + |z(x,t)|^2\big)\,dx. \end{aligned}$$

5. Proof of Theorem 4.1 (observability inequality)

The proof of Theorem 4.1 rely heavily on Carleman estimates for Schrödinger equation.

Step 1.

proposition 5.1. (*Carleman estimates*). *Let* $\{w,z\}$ *be the solution of the system* (18*a*)-(18*e*) *in the class* (11) , (12) , *and we assume* $\varphi(x,t)$ *by* (5). *Then, for all* $\tau > 0$ *large, we have*

(24) $$\begin{aligned} &K_{3.\tau.data}\int_Q e^{2\tau\varphi}\Big[|\nabla w|^2 + |w|^2 + |\nabla z|^2 + |z|^2\Big]\,dQ - C_{d;T}\tau e^{-2\tau\delta}\left[E(T) + E(0)\right]\\ \le\ & B|_\Sigma(w) + B|_\Sigma(z) + const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\Big[\|w\|_{C([0.T],\,L^2(\Omega))} + \|z\|_{C([0.T],\,L^2(\Omega))}\Big], \end{aligned}$$

where

$$K_{3.\tau.data} = \min\Big\{\Big(\widetilde{C}_{1,\tau} - 4c_{\alpha,q,\beta,p}\Big),\ \Big(\widetilde{C}_{2,\tau} - 4c_{\alpha,q,\beta,p}\Big)\Big\},$$

$$c_{\alpha,q,\beta,p} = \max(c_{\alpha,q}, c_{\beta,p})$$

$$E(t) = E_w(t) + E_z(t), \tag{25}$$

Here $E_w(t)$, $E_z(t)$ are as in (23), also we have $B|_\Sigma(w)$ and $B|_\Sigma(z)$ defined as in (15).

Proof of Proposition 5.1. By applying the Carleman estimates (13) of Theorem 3.1 to the w-equation (18a) with $F(.,.) = \alpha(.).\nabla z + q(.)z(.,.) \in L^2(0,T,H^1(\Omega))$

$$\begin{aligned} B|_\Sigma(w) + 4\int_Q e^{2\tau\varphi}|\alpha(x).\nabla z + q(x)z(x,t)|^2 dQ \;\; &\geq \;\; \widetilde{C}_{1,\tau}\int_Q e^{2\tau\varphi}\left[|\nabla w|^2\right]dQ + \widetilde{C}_{2,\tau}\int_Q e^{2\tau\varphi}|w|^2 dQ \\ &\quad -C_{d;T}\tau e^{-2\tau\delta}\left[E_w(0) + E_w(T)\right]. \end{aligned} \tag{26}$$

From (19), $|\alpha|, q \in L^\infty(\Omega)$, and consequently

$$\int_Q e^{2\tau\varphi}|\alpha(x).\nabla z + q(x)z(x,t)|^2 dQ \leq c_{\alpha,q}\int_Q e^{2\tau\varphi}|\nabla z + z(x,t)|^2 dQ \tag{27}$$

Substituting (27) on the left hand side of (26) yields the following estimate

$$\begin{aligned} &\widetilde{C}_{1,\tau}\int_Q e^{2\tau\varphi}\left[|\nabla w|^2\right]dQ + \widetilde{C}_{2,\tau}\int_Q e^{2\tau\varphi}|w|^2 dQ - C_{d;T}\tau e^{-2\tau\delta}\left[E_w(0) + E_w(T)\right] \\ &\leq \;\; B|_\Sigma(w) + 4c_{\alpha,q}\int_Q e^{2\tau\varphi}|\nabla z + z(x,t)|^2 dQ + const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\|w\|_{C([0.T],\,L^2(\Omega))}\right]. \end{aligned} \tag{28}$$

where we added the coefficient $const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\|w\|_{C([0.T],\,L^2(\Omega))}\right]$ to the right-hand side of (28).

Also, by applying the Carleman estimates (13) of Theorem 3.1 to the z-equation (18b) with $F(.,.) = \beta(.).\nabla w + p(.)w(.,.) \in L^2(0,T,H^1(\Omega))$

$$\begin{aligned} B|_\Sigma(z) + 4\int_Q e^{2\tau\varphi}|\beta(x).\nabla w + p(x)w(x,t)|^2 dQ \;\; &\geq \;\; \widetilde{C}_{1,\tau}\int_Q e^{2\tau\varphi}\left[|\nabla z|^2\right]dQ + \widetilde{C}_{2,\tau}\int_Q e^{2\tau\varphi}|z|^2 dQ \\ &\quad -C_{d;T}\tau e^{-2\tau\delta}\left[E_z(0) + E_z(T)\right]. \end{aligned} \tag{29}$$

From (19), $|\beta|, p \in L^\infty(\Omega)$, and consequently

$$\int_Q e^{2\tau\varphi}|\beta(x).\nabla w + p(x)w(x,t)|^2 dQ \leq c_{\beta,p}\int_Q e^{2\tau\varphi}|\nabla w + w(x,t)|^2 dQ \tag{30}$$

Substituting (30) on the left hand side of (29) yields the following estimate

$$\begin{aligned} &\widetilde{C}_{1,\tau}\int_Q e^{2\tau\varphi}\left[|\nabla z|^2\right]dQ + \widetilde{C}_{2,\tau}\int_Q e^{2\tau\varphi}|z|^2 dQ - C_{d;T}\tau e^{-2\tau\delta}\left[E_z(0) + E_z(T)\right] \\ &\leq \;\; B|_\Sigma(z) + 4c_{\beta,p}\int_Q e^{2\tau\varphi}|\nabla w + w(x,t)|^2 dQ + const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\|z\|_{C([0.T],\,L^2(\Omega))}\right], \end{aligned} \tag{31}$$

where we added the coefficient $const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\|z\|_{C([0.T],\,L^2(\Omega))}\right]$ to the right-hand side of (31).

Summing up (28) and (31), to obtain

$$\begin{aligned} &\left(\widetilde{C}_{1,\tau} - 4c_{\alpha,q,\beta,p}\right)\int_Q e^{2\tau\varphi}\left[|\nabla w|^2 + |\nabla z|^2\right]dQ + \left(\widetilde{C}_{2,\tau} - 4c_{\alpha,q,\beta,p}\right)\int_Q e^{2\tau\varphi}\left[|w|^2 + |z|^2\right]dQ \\ &-C_{d;T}\tau e^{-2\tau\delta}\left[E(T) + E(0)\right] \end{aligned}$$

$$\leq \quad B|_{\Sigma}(w) + B|_{\Sigma}(z) + const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\|w\|_{C([0.T],\,L^2(\Omega))} + \|z\|_{C([0.T],\,L^2(\Omega))}\right]$$

This gives (24) of Proposition 5.1, as desired.

Step 2.

proposition 5.2. *Let $\{w,z\}$ be the solution of the system (18a)-(18e) in the class (11), (12). Then, for $0 < t \leq T$, we have*

$$e^{-Kt}E(0) - \tilde{A}(T) \leq E(t) \leq \left[E(0) + \tilde{A}(T)\right]e^{Kt}, \tag{32}$$

where $E(t)$ is defined by (23), and

$$\tilde{A}(T) = 2\left\{\int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right| \left[|w_t| + \frac{1}{2}\left|\frac{\partial w}{\partial \nu}\right| \left|r_1\cdot\nu + \acute{O}_c\right| + |n|\,|w| + |w| + |\alpha\cdot\nabla z| + |q|\,|z|\right] d\Gamma dr\right\}$$
$$+ 2\left\{\int_0^T \int_\Gamma \left|\frac{\partial z}{\partial \nu}\right| \left[|z_t| + \frac{1}{2}\left|\frac{\partial z}{\partial \nu}\right| \left|r_1\cdot\nu + \acute{O}_c\right| + |m|\,|z| + |z| + |\beta\cdot\nabla w| + |p|\,|w|\right] d\Gamma dr\right\},$$

$$K = \max\left\{\left(\acute{O} + c_{\beta,p}\right), \left(\acute{O} + c_{\alpha,q}\right)\right\}. \tag{33}$$

Proof of proposition 5.2.

Step (i). We apply equality (Lemma 6.1, part (v)) in [7] to equation (18a) where $f(.,.) = \alpha(.).\nabla z + q(.)z \in L^2\left(0,T,H^1(\Omega)\right)$, $F(w) = a(x).\nabla w + n(x)w$, with $a = -ir_1$, then for all $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} E_w(t) &= E_w(s) + 2\ \mathrm{Re}\left\{\int_s^t \int_\Gamma \frac{\partial \overline{w}}{\partial \nu}\left[w_t + inw - i\left(\alpha(x).\nabla z + q(x)z\right)\right] d\Gamma dr\right\} \\ &+ \int_s^t \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right|^2 r_1.\nu\ d\Gamma dr + 2\ \mathrm{Re}\left\{i\int_s^t \int_\Gamma \frac{\partial w}{\partial \nu}\overline{w} d\Gamma dr\right\} \\ &+ \acute{O}_c \int_s^t \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right|^2 d\Gamma dr + \acute{O}\left\{\int_s^t E_w(r)\,dr + \|\alpha(x).\nabla z + q(x)z\|_{L^2(s,t,H^1(\Omega))}\right\}, \end{aligned} \tag{34}$$

Step (ii). We apply equality (Lemma 6.1, Part (v)) in [7] to equation (18b) where $f(.,.) = \beta(.).\nabla w + p(.)w \in L^2\left(0,T,H^1(\Omega)\right)$, $F(z) = b(x).\nabla z + m(x)z$, with $b = -ir_1$, then for all $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} E_z(t) &= E_z(s) + 2\ \mathrm{Re}\left\{\int_s^t \int_\Gamma \frac{\partial \overline{z}}{\partial \nu}\left[z_t + imz - i\left(\beta(x).\nabla w + p(x)w\right)\right] d\Gamma dr\right\} \\ &+ \int_s^t \int_\Gamma \left|\frac{\partial z}{\partial \nu}\right|^2 r_1.\nu\ d\Gamma dr + 2\ \mathrm{Re}\left\{i\int_s^t \int_\Gamma \frac{\partial z}{\partial \nu}\overline{z} d\Gamma dr\right\} \\ &+ \acute{O}_c \int_s^t \int_\Gamma \left|\frac{\partial z}{\partial \nu}\right|^2 d\Gamma dr + \acute{O}\left\{\int_s^t E_z(r)\,dr + \|\beta(x).\nabla w + p(x)w\|_{L^2(s,t,H^1(\Omega))}\right\}, \end{aligned} \tag{35}$$

Step (iii). By summing (34) and (35), we find by (25) for $E(t)$

$$E(t) = E(s) + 2\ \mathrm{Re}\left\{\int_s^t \int_\Gamma \frac{\partial \overline{w}}{\partial \nu}\left[w_t + inw - i\left(\alpha(x).\nabla z + q(x)z\right)\right] d\Gamma dr\right\}$$

$$+2\ \mathrm{Re}\left\{\int_s^t\int_\Gamma \frac{\partial\overline{z}}{\partial\nu}\left[z_t+imz-i\left(\beta\left(x\right).\nabla w+p\left(x\right)w\right)\right]d\Gamma dr\right\}$$

$$\int_s^t\int_\Gamma\left|\frac{\partial w}{\partial\nu}\right|^2 r_1.\nu\ d\Gamma dr+2\ \mathrm{Re}\left\{i\int_s^t\int_\Gamma\frac{\partial w}{\partial\nu}\overline{w}d\Gamma dr\right\}$$

$$+\int_s^t\int_\Gamma\left|\frac{\partial z}{\partial\nu}\right|^2 r_1.\nu\ d\Gamma dr+2\ \mathrm{Re}\left\{i\int_s^t\int_\Gamma\frac{\partial z}{\partial\nu}\overline{z}d\Gamma dr\right\}$$

$$+\acute{O}_c\int_s^t\int_\Gamma\left|\frac{\partial w}{\partial\nu}\right|^2\ d\Gamma dr+\acute{O}\left\{\int_s^t E_w\left(r\right)dr+\left\|\alpha\left(x\right).\nabla z+q\left(x\right)z\right\|_{L^2(s,t,H^1(\Omega))}\right\}$$

$$+\acute{O}_c\int_s^t\int_\Gamma\left|\frac{\partial z}{\partial\nu}\right|^2\ d\Gamma dr+\acute{O}\left\{\int_s^t E_z\left(r\right)dr+\left\|\beta\left(x\right).\nabla w+p\left(x\right)w\right\|_{L^2(s,t,H^1(\Omega))}\right\} \tag{36}$$

Step (iv). Using (19) and (23), via (inequality (6.1), [2]) we have

$$\begin{aligned}\left\|\beta\left(x\right).\nabla w+p\left(x\right)w\right\|_{L^2(s,t,H^1(\Omega))} &\le c_{\beta,p}\int_s^t\left[\int_\Omega\left(\left|w\right|^2+\left|\nabla w\right|^2\right)d\Omega\right]dr\\ &\le c_{\beta,p}\int_s^t E_w\left(r\right)dr,\end{aligned} \tag{37}$$

$$\begin{aligned}\left\|\alpha\left(x\right).\nabla z+q\left(x\right)z\right\|_{L^2(s,t,H^1(\Omega))} &\le c_{\alpha,q}\int_s^t\left[\int_\Omega\left(\left|z\right|^2+\left|\nabla z\right|^2\right)d\Omega\right]dr\\ &\le c_{\alpha,q}\int_s^t E_z\left(r\right)dr,\end{aligned} \tag{38}$$

By inserting (37) and (38) in (36), we obtain

$$\left|E\left(t\right)-E\left(s\right)\right|\le\tilde{A}\left(T\right)+K\int_s^t E\left(r\right)dr, \tag{39}$$

where $\tilde{A}\left(T\right)$ and K is defined by (33).

Step (v). We apply Gronwall inequality to the two inequalities that result from (39), then for $0\le s\le t\le T$, we find

$$E\left(t\right)\le\left[E\left(s\right)+\tilde{A}\left(T\right)\right]e^{K(t-s)},\ E\left(s\right)\le\left[E\left(t\right)+\tilde{A}\left(T\right)\right]e^{K(t-s)}, \tag{40}$$

for $s=0$ in (40) we find (32), as desired.

Step 3.

proposition 5.3. *Let $\{w,z\}$ be the solution of the system (18a)-(18e) in the class (11), (12). Then, for τ large, we have*

$$\begin{aligned}&\left\{\left[K_{3.\tau.data}K_{t_0,t_1}e^{2\tau\sigma}-C_{d;T}\tau\left(1+e^{KT}\right)e^{-2\tau\delta}\right]\right\}E\left(0\right)\\ \le\ &BT|_\Sigma\left(w,z\right)+const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2)}\left[\left\|w\right\|_{C([0.T],\,L^2(\Omega))}+\left\|z\right\|_{C([0.T],\,L^2(\Omega))}\right],\end{aligned} \tag{41}$$

where taking τ sufficiently large makes the coefficient $\{\}$ in front of $E\left(0\right)$ positive in (41), where

$$BT|_\Sigma\left(w,z\right)=B|_\Sigma\left(w\right)+B|_\Sigma\left(z\right)+\left\{\left[K_{3.\tau.data}\left(t_1-t_0\right)e^{2\tau\sigma}+C_{d;T}\tau e^{KT}e^{-2\tau\delta}\right]\right\}\tilde{A}\left(T\right), \tag{42}$$

$$K_{t_0,t_1} = -\frac{1}{K}\left[e^{-Kt_1} - e^{-Kt_0}\right], \tag{43}$$

also $K_{3.\tau.data}$ is defined by (25) and K is defined by (33).

Proof of Proposition 5.3. Using the left inequality in (32), we calculate by (33)

$$\begin{aligned}\int_{t_0}^{t_1} E(t)\,dt &\geq \int_{t_0}^{t_1}\left[e^{-Kt}E(0) - \tilde{A}(T)\right]dt \\ &= K_{t_0,t_1}E(0) - (t_1 - t_0)\tilde{A}(T),\end{aligned} \tag{44}$$

Moreover, by (9) for $\varphi(x,t)$ and (23), we estimate with reference to (24)

$$\int_Q e^{2\tau\varphi}\left[|w_t|^2 + |\nabla w|^2 + |w|^2 + |\nabla z|^2 + |z|^2\right]dQ > e^{2\tau\sigma}\int_{t_0}^{t_1} E(t)\,dt. \tag{45}$$

Using now the right inequality in (32), we find

$$E(T) + E(0) \leq \left(1 + e^{KT}\right)E(0) + \tilde{A}(T)\,e^{KT}. \tag{46}$$

We now use (44) into(45) and substitute the result, along with (46), into the left of (24), this way we obtain (41).

Step 4. We will extend the estimate (41) for the solution $\{w,z\}$ of class $H^{2.2}(Q) \times H^{2.2}(Q)$, to the solution of class

$$\begin{gathered} w, z \in C\left([0,T], H^1(\Omega)\right) \\ w_t, z_t \in L^2\left(0,T,L^2(\Gamma)\right), \frac{\partial w}{\partial \nu}, \frac{\partial z}{\partial \nu} \in L^2\left(0,T,L^2(\Gamma)\right). \end{gathered} \tag{47}$$

We use the same approach as in ([2], paragraph 8) et ([7], paragraph 7).

Step 5. We now absorb the tangential traces from the boundary terms $BT|_{\Sigma}(w,z)$ in (41).

proposition 5.4. *Let $\{w,z\}$ be the solution of the system (18a)-(18e) in the class (47). Then, there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$, such that*

$$\begin{aligned} &\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}\left[|\nabla_{\tan} w|^2 + |\nabla_{\tan} z|^2\right]d\Gamma_1 dt \\ \leq\ & C_{\epsilon,\epsilon_0,T}\{\int_0^T\int_{\Gamma_1}\left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2\right]d\Gamma_1 dt + \\ & \|\nabla w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|\nabla z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}\}, \end{aligned} \tag{48}$$

where $T > 0$, Γ_1 non-empty part of the boundaries Γ, $\epsilon > \epsilon_0 > 0$ and ϵ_0 arbitrarily small.

Proof of Proposition 5.4.

Step (i). We apply (8.10) of Theorem (8.3) in [7] to the w-equation (18a) with $f(x,t) = \alpha(x).\nabla z + q(x)z$, then

$$\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}|\nabla_{\tan}w|^2\,d\Gamma_1 dt$$
$$\leq C_{\epsilon,\epsilon_0,T}\left\{\int_0^T\int_{\Gamma_1}\left[\left(\frac{\partial w}{\partial\nu}\right)^2+w_t^2\right]d\Gamma_1 dt\right.$$
$$\left.+\|w\|^2_{L^2\left(0,T,H^{\frac{1}{2}+\epsilon_0}(\Omega)\right)}+\|\alpha(x)\cdot\nabla z+q(x)z\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q)}\right\}. \tag{49}$$

Step (ii). We apply (8.10) of Theorem (8.3) in [7] to the z-equation (18b) with $f(x,t)=\beta(x).\nabla w+p(x)w$, then

$$\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}|\nabla_{\tan}z|^2\,d\Gamma_1 dt$$
$$\leq C_{\epsilon,\epsilon_0,T}\left\{\int_0^T\int_{\Gamma_1}\left[\left(\frac{\partial z}{\partial\nu}\right)^2+z_t^2\right]d\Gamma_1 dt\right.$$
$$\left.+\|z\|^2_{L^2\left(0,T,H^{\frac{1}{2}+\epsilon_0}(\Omega)\right)}+\|\beta(x)\cdot\nabla w+p(x)w\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q)}\right\}. \tag{50}$$

Step (iii). Next we sum up (49) and (50) and use

$$\begin{aligned}\|\alpha(x).\nabla z+q(x)z\|_{H^{-\frac{1}{2}+\epsilon_0}(Q)} &\leq c_{\alpha,q}\left[\|\nabla z\|_{H^{\frac{1}{2}+\epsilon_0}(Q)}+\|z\|_{,H^{\frac{1}{2}+\epsilon_0}(Q)}\right],\\ \|\beta(x).\nabla w+p(x)w\|_{H^{-\frac{1}{2}+\epsilon_0}(Q)} &\leq c_{\beta,p}\left[\|\nabla w\|_{H^{\frac{1}{2}+\epsilon_0}(Q)}+\|w\|_{,H^{\frac{1}{2}+\epsilon_0}(Q)}\right].\end{aligned}$$

Hence we get (48).

Step 6.

proposition 5.5. *Let $\{w,z\}$ be the solution of the system (18a)-(18e) in the class (47). Then, for τ sufficiently large, there exists a positive $const_{\varphi,\tau}$ such that*

$$\begin{aligned} const_{\varphi,\tau}E(0) &\leq \int_0^T\int_{\Gamma_1}\left[w_t^2+w^2+|z|^2+|z_t|^2\right]d\Gamma_1 dt+\\ & const_{T,\tau,\varphi,\epsilon_0}\left[\|\nabla w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}+\|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}+\|\nabla z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}+\|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}\right].\end{aligned} \tag{51}$$

Proof of Proposition5.5. For fixed $\epsilon>0$ small we apply estimate (41) of Proposition 5.3 over the interval $[\epsilon,T-\epsilon]$, rather than over $[0,T]$. We obtain

$$\begin{aligned} & K_{\tau,\epsilon}E(\epsilon)\\ \leq\ & BT|_{[\epsilon,T-\epsilon]\times\Gamma}(w,z)+\\ & const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2),\epsilon}\left[\|w\|_{C([0.T],\,L^2(\Omega))}+\|z\|_{C([0.T],\,L^2(\Omega))}\right].\end{aligned} \tag{52}$$

Using the left inequality in (32) with $t=\epsilon$ we have

$$e^{-K\epsilon}E\left(0\right)-\tilde{A}\left(T\right)\leq E\left(\epsilon\right). \tag{53}$$

We next insert (53) into the left of (52), we find

$$\begin{aligned} & K_{\tau,\epsilon}\left(e^{-K\epsilon}\right)E\left(0\right) \\ \leq\ & BT|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w,z\right)+K_{\tau,\epsilon}\tilde{A}\left(T\right)+ \\ & const_{T,\tau,\beta,\sigma,\grave{O}(\tau^2),\epsilon}\left[\|w\|_{C([0.T],\,L^2(\Omega))}+\|z\|_{C([0.T],\,L^2(\Omega))}\right]. \end{aligned} \tag{54}$$

We see via the definition $BT|\left(w,z\right)$ on $[\epsilon,T-\epsilon]\times\Gamma$ in (42) that

$$\begin{aligned} & BT|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w,z\right)+K_{\tau,\epsilon}\tilde{A}\left(T\right) \\ =\ & B|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w\right)+B|_{[\epsilon,T-\epsilon]\times\Gamma}\left(z\right)+ \\ & \left\{\left[K_{3.\tau.data}\left(t_1-t_0\right)e^{2\tau\sigma}+C_{d;T}\tau e^{KT}e^{-2\tau\delta}\right]\right\}\tilde{A}\left(T\right)+K_{\tau,\epsilon}\tilde{A}\left(T\right) \\ \leq\ & BT|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w\right)+B|_{[\epsilon,T-\epsilon]\times\Gamma}\left(z\right)+\tilde{K}\tilde{A}\left(T\right), \\ \tilde{K}\ =\ & \max\left\{\left[K_{3.\tau.data}\left(t_1-t_0\right)e^{2\tau\sigma}+C_{d;T}\tau e^{KT}e^{-2\tau\delta}\right],\ K_{\tau,\epsilon}\right\}, \end{aligned} \tag{55}$$

since $\frac{\partial w}{\partial\nu}|_{\Sigma}=0$, $\frac{\partial z}{\partial\nu}|_{\Sigma}=0$ by (18e), And because $|\nabla M|^2=\left|\frac{\partial M}{\partial\nu}\right|^2+|\nabla_{\tan}M|^2$, then from (55) we have via the definition of $B|_{\Sigma}\left(w\right)$ and $B|_{\Sigma}\left(z\right)$ in (15), and via the definition of $\tilde{A}\left(T\right)$ in (33)

$$\begin{aligned} & BT|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w,z\right)+K_{\tau,\epsilon}\tilde{A}(T) \\ & \leq BT|_{[\epsilon,T-\epsilon]\times\Gamma}\left(w\right)+B|_{[\epsilon,T-\epsilon]\times\Gamma}\left(z\right)+\tilde{K}\tilde{A}(T) \\ & =2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}\left[2\tau^2|h|^2+\Phi\right]|w|^2h\cdot\nu\,d\Gamma dt \\ & \quad-2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}\left[\xi_t\eta-\xi\eta_t\right]h\cdot\nu\,d\Gamma dt \\ & \quad+2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}|\nabla_{\tan}w|^2h\cdot\nu\,d\Gamma dt \\ & \quad+2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}\left[2\tau^2|h|^2+\Phi\right]|z|^2h\cdot\nu\,d\Gamma dt \\ & \quad-2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}\left[\xi_t\eta-\xi\eta_t\right]h\cdot\nu\,d\Gamma dt \\ & \quad-2\tau\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}e^{2\tau\varphi}|\nabla_{\tan}z|^2h\cdot\nu\,d\Gamma dt \\ & \leq C_{\varphi,\tau}\left\{\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}\left[|w|^2+|w_t|^2+|\nabla_{\tan}w|^2\right]d\Gamma_1dt\right. \\ & \qquad\left.+\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}\left[|z|^2+|z_t|^2+|\nabla_{\tan}z|^2\right]d\Gamma_1dt\right\}. \end{aligned} \tag{56}$$

By (48), and always $\frac{\partial w}{\partial \nu}|_\Sigma = 0$, $\frac{\partial z}{\partial \nu}|_\Sigma = 0$, then from (56) we have

$$\begin{aligned} & BT|_{[\epsilon,T-\epsilon]\times\Gamma}(w,z) + K_{\varphi,\tau,\epsilon}\tilde{A}(T) \\ \leq\ & C_{\varphi,\tau,T,\epsilon,\epsilon_0}\{\int_0^T\int_{\Gamma_1}\left[|w_t|^2+|w|^2+|z|^2+|z_t|^2\right]d\Gamma_1 dt + \\ & \|\nabla w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|\nabla z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}\}, \end{aligned} \tag{57}$$

inserting (57) into the right of (54) yields (51), as desired.

Step 7 Now after creating (51), we can remove the terms $\|\nabla w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|w\|_{H^{\frac{1}{2}+\epsilon_0}(Q)}$ and $\|\nabla z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|_{H^{\frac{1}{2}+\epsilon_0}(Q)}$ in estimate (51) by applying compactness/uniqueness argument as in the section below, or as in [10,11] and thus obtain the desired final estimate (22).

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I. Lasiecka, R. Triggiani, Carleman Estimates and Exact Boundary Controllability for a System of Coupled, Nonconservative Second-Order Hyperbolic Equations, in: Partial Differential Equation Methods in Control and Shape Analysis, CRC Press, 1997: pp. 231–260. https://doi.org/10.1201/9781482273618-15.

[2] I. Lasiecka, R. Triggiani, X. Zhang, Nonconservative Wave Equations with Unobserved Neumann B.C.: Global Uniqueness and Observability in One Shot, Contemp. Math. 268 (2000), 227–325. https://doi.org/10.1090/conm/268/04315.

[3] I. Lasiecka, R. Triggiani, Exact Controllability of the Wave Equation with Neumann Boundary Control, Appl. Math. Optim. 19 (1989), 243–290. https://doi.org/10.1007/bf01448201.

[4] I. Lasiecka, R. Triggiani, Exact Controllability of Semilinear Abstract Systems with Application to Waves and Plates Boundary Control Problems, Appl. Math. Optim. 23 (1991), 109–154. https://doi.org/10.1007/bf01442394.

[5] I. Lasiecka, R. Triggiani, Optimal Regularity, Exact Controllability and Uniform Stabilization of Schrödinger Equations with Dirichlet Control, Differ. Integral Equ. 5 (1992), 521–535. https://doi.org/10.57262/die/1370979316.

[6] I. Lasiecka, R. Triggiani, Uniform Stabilization of the Wave Equation with Dirichlet or Neumann Feedback Control Without Geometrical Conditions, Appl. Math. Optim. 25 (1992), 189–224. https://doi.org/10.1007/bf01182480.

[7] I. Lasiecka, R. Triggiani, X. Zhang, Global Uniqueness, Observability and Stabilization of Nonconservative Schrödinger Equations via Pointwise Carleman Estimates. Part I: $H^1(\Omega)$-Estimates, J. Inverse Ill-posed Probl. 12 (2004), 43–123. https://doi.org/10.1515/156939404773972761.

[8] I. Lasiecka, R. Triggiani, X. Zhang, Global Uniqueness, Observability and Stabilization of Nonconservative Schrödinger Equations via Pointwise Carleman Estimates. Part II: $L_2(\Omega)$-Estimates, J. Inverse Ill-posed Probl. 12 (2004), 183–231. https://doi.org/10.1515/1569394042530919.

[9] M. Lavrentev, V. Romanov, S. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, American Mathematical Society, 1986. https://doi.org/10.1090/mmono/064.

[10] J.L. Lions, Exact Controllability, Stabilization and Perturbations for Distributed Systems, SIAM Rev. 30 (1988), 1–68. https://doi.org/10.1137/1030001.

[11] W. Littman, Near Optimal Time Boundary Controllability for a Class of Hyperbolic Equations, in: I. Lasiecka, R. Triggiani, (eds) Control Problems for Systems Described by Partial Differential Equations and Applications. Lecture Notes in Control and Information Sciences, Vol 97. Springer, Berlin, pp. 307–312, (1987). https://doi.org/10.1007/BFb0038763.

[12] S. Liu, R. Triggiani, Global Uniqueness in Determining Electric Potentials for a System of Strongly Coupled Schrödinger Equations with Magnetic Potential Terms, J. Inverse Ill-Posed Probl. 19 (2011), 223–254. https://doi.org/10.1515/jiip.2011.030.

[13] S. Liu, R. Triggiani, Recovering of Damping Coefficients for a System of Coupled Wave Equations with Neumann Boundary Conditions: Uniqueness and Stability, Chin. Ann. Math. Ser. B 32 (2011), 669–698. https://doi.org/10.1007/s11401-011-0672-1.

[14] R. Triggiani, Exact Boundary Controllability on $L_2(\Omega) \times H^{-1}(\Omega)$ of the Wave Equation with Dirichlet Boundary Control Acting on a Portion of the Boundary??, and Related Problems, Appl. Math. Optim. 18 (1988), 241–277. https://doi.org/10.1007/bf01443625.

[15] R. Triggiani, Carleman Estimate and Exact Boundary Controllability for a System of Coupled Non-conservative Schrödinger Equations, Rend. Istit. Mat. Univ. Trieste 28 (1996), 453–504.

[16] I. Lasiecka, R. Triggiani, X. Zhang, Carleman Estimates at the $H^1(\Omega)$- and $L^2(\Omega)$-Level for Nonconservative Schrödinger Equations with Unobserved Neumann B.C., Arch. Inequal. Appl. 2 (2004), 215–338.

[17] R. Triggiani, P.F. Yao, Inverse/Observability Estimates for Schrödinger Equations with Variable Coefficients, Control Cybernet. 28 (1999), 627–664.

[18] T. Roberto, P.F. Yao, Carleman Estimates with No Lower-Order Terms for General Riemann Wave Equations. Global Uniqueness and Observability in One Shot, Appl. Math. Optim. 46 (2002), 331–375. https://doi.org/10.1007/s00245-002-0751-5.

[19] M. Yamamoto, Uniqueness and Stability in Multidimensional Hyperbolic Inverse Problems, J. Math. Pures Appl. 78 (1999), 65–98. https://doi.org/10.1016/s0021-7824(99)80010-5.

[20] X. Zhang, Explicit Observability Inequalities for the Wave Equation with Lower Order Terms by Means of Carleman Inequalities, SIAM J. Control. Optim. 39 (2000), 812–834. https://doi.org/10.1137/s0363012999350298.