Asia Pac. J. Math. 2026 **13**:81

THE ALPHA LOGARITHM WEIBULL DISTRIBUTION: STATISTICAL PROPERTIES, MULTIPLE ESTIMATION METHODS, AND APPLICATIONS TO LIFETIME DATA

AYAT K. SAGHEER, HAZIM G. KALT*

Department of Mathematics, College of Education for Pure Science, University of Kerbala, Iraq

*Corresponding author: hazim.galit@uokerbala.edu.iq

Received Mar. 27, 2026

Abstract. In this paper, we present and investigate a new three-parameter continuous probability distribution denoted by the Alpha Logarithm Weibull (ALW) distribution, which is obtained by applying the Alpha Logarithm-G (ALG) family transformation to the classical Weibull baseline distribution. The new shape parameter α provides additional flexibility beyond the standard Weibull distribution, enabling the model to fit increasing, decreasing, bathtub-shaped, and unimodal (upside-down bathtub) hazard rate functions that are unattainable by the standard Weibull distribution. A complete set of mathematically rigorous properties is established, including the probability density function (PDF), cumulative distribution function (CDF), survival function, hazard rate function, quantile function, raw and incomplete moments, moment generating function, Rényi entropy, and stress-strength reliability. For parameter estimation, four methods are proposed, maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE) and Anderson–Darling estimation (ADE). A comprehensive Monte Carlo simulation study using five parameter configurations assesses the performance of estimators through bias and mean squared error (MSE); the results confirm consistency, asymptotic unbiasedness, and numerical stability of all methods. The practical utility of the ALW distribution is demonstrated on four heterogeneous real datasets drawn from materials engineering, biomedical survival analysis, and mechanical reliability. In every case, the ALW model outperforms six competing distributions with respect to standard goodness-of-fit criteria (AIC, BIC, CAIC, HQIC) and achieves the highest Kolmogorov–Smirnov p-value.

2020 Mathematics Subject Classification. 62E10; 62N05; 60E05.

Key words and phrases. alpha logarithm-G family; Weibull distribution; multiple estimation methods; stress-strength reliability; Rényi entropy; lifetime data.

1. Introduction

The statistical analysis of lifetime, reliability, and survival data plays a central role in engineering, biomedicine, actuarial science, and environmental studies. The classical two-parameter Weibull distribution [1] has served as the workhorse of lifetime modelling for over seven decades owing to its closed-form expressions, tractable inference, and its ability to accommodate decreasing, constant, and

DOI: 10.28924/APJM/13-81

increasing failure rates through its shape parameter. Despite these advantages, the standard Weibull model is unable to capture more complex hazard behaviours such as bathtub-shaped or unimodal hazard rates, which frequently arise in practice. This limitation has motivated a vast literature on Weibull extensions and generalisations.

Representative contributions include the mixture Weibull–Generalised Gamma distribution [2], the Maxwell–Weibull distribution [3], the Gull Alpha Power Transform Weibull distribution [4], the Power Exponential Weibull distribution [5], and the Alpha Power Type II-G family applied to the Weibull baseline [6]. In a complementary direction, Mohsin et al. [7] recently introduced the Alpha Logarithm-G (ALG) family by embedding a logarithmic transformation into the CDF of any parent distribution and demonstrated its effectiveness on the exponential baseline. The ALG construction provides a simple and mathematically convenient framework that retains closed-form expressions for important reliability quantities while providing real added flexibility.

Research gap. Although the ALG family has shown considerable promise, its application to the Weibull baseline—the most widely used lifetime distribution—has not been explored. Moreover, existing Weibull extensions rarely provide a simultaneous treatment of (i) a comprehensive set of theoretical properties including entropy and stress-strength reliability; (ii) a rigorous comparison of multiple estimation methods beyond MLE; and (iii) validation on heterogeneous datasets spanning engineering and biomedical domains. The present work fills this gap by introducing the ALW distribution and providing all of the above.

Main contributions.

(1) A new three-parameter distribution, the ALW, is introduced.

(2) A comprehensive set of closed-form or series-based mathematical properties is derived, including raw and incomplete moments, Rényi entropy, and stress-strength reliability $R = P(X_1 > X_2)$.

(3) Four estimation methods (MLE, LSE, WLSE, ADE) are developed and compared via an extensive Monte Carlo simulation.

(4) The ALW model is applied to four real datasets and shown to outperform six competing distributions.

The remainder of the paper is organised as follows. Section 2 defines the ALW distribution and presents its fundamental functions. Section 3 derives the statistical properties. Section 4 develops the four estimation methods. Section 5 reports the simulation study. Section 6 presents the real-data applications. Section 7 concludes.

2. The Alpha Logarithm Weibull Distribution

2.1. The Alpha Logarithm-G Family. Mohsin et al. [7] defined the CDF of the ALG family as

$$F_{\mathrm{ALG}}(x;\, \alpha, \boldsymbol{\rho}) = \frac{\ln\bigl(\alpha\, G(x;\boldsymbol{\rho}) + 1\bigr)}{\ln(\alpha+1)}, \quad \alpha > 0, \tag{1}$$

where $G(x;\boldsymbol{\rho})$ is the baseline CDF with parameter vector $\boldsymbol{\rho}$, and α is an additional shape parameter controlling the deviation from the baseline. The corresponding PDF is

$$f_{\mathrm{ALG}}(x;\, \alpha, \boldsymbol{\rho}) = \frac{\alpha\, g(x;\boldsymbol{\rho})}{\ln(\alpha+1)\, [\alpha\, G(x;\boldsymbol{\rho}) + 1]}, \tag{2}$$

where $g(x;\boldsymbol{\rho}) = G'(x;\boldsymbol{\rho})$ is the baseline PDF. When $\alpha \to 0^+$, an application of L'Hôpital's rule shows that $F_{\mathrm{ALG}}(x) \to G(x)$, so the baseline model is recovered in the limiting case.

2.2. Baseline: Weibull Distribution. The two-parameter Weibull distribution [1] has CDF and PDF

$$G(x;\beta,\vartheta) = 1 - e^{-x^{\vartheta}\beta^{-\vartheta}}, \quad x \ge 0;\ \beta, \vartheta > 0, \tag{3}$$

$$g(x;\beta,\vartheta) = \vartheta\, \beta^{-\vartheta}\, x^{\vartheta-1}\, e^{-x^{\vartheta}\beta^{-\vartheta}}, \tag{4}$$

where $\beta > 0$ is the scale parameter and $\vartheta > 0$ is the shape parameter governing hazard-rate monotonicity: the hazard is decreasing for $\vartheta < 1$, constant for $\vartheta = 1$, and increasing for $\vartheta > 1$.

2.3. CDF and PDF of the ALW Distribution. For notational convenience throughout the paper, we define the shorthand functions

$$A(x) = e^{-x^{\vartheta}\beta^{-\vartheta}}, \qquad D(x) = \alpha + 1 - \alpha\, A(x), \tag{5}$$

so that $A(x) \in (0,1)$ and $D(x) \in (1, 1+\alpha)$ for all $x > 0$, $\alpha > 0$.

Substituting (3) and (4) into (1) and (2) yields the Alpha Logarithm Weibull (ALW) distribution.

Definition 2.1. *A non-negative random variable X follows the* ALW distribution *if its CDF is*

$$F(x;\alpha,\beta,\vartheta) = \frac{\ln D(x)}{\ln(1+\alpha)}, \quad x \ge 0;\ \alpha, \beta, \vartheta > 0, \tag{6}$$

where $D(x)$ is as defined in (5).

The corresponding PDF is obtained by differentiating (6) with respect to x:

$$f(x;\alpha,\beta,\vartheta) = \frac{\alpha\, \vartheta\, \beta^{-\vartheta}\, x^{\vartheta-1}\, A(x)}{\ln(1+\alpha)\cdot D(x)}, \quad x \ge 0. \tag{7}$$

In the ALW parameterisation, ϑ and α are shape parameters while β is the scale parameter. The parameter α controls the degree of departure from the classical Weibull model: for small α the ALW closely approximates the Weibull, whereas larger values of α introduce greater flexibility in tail behaviour and hazard rate shape. Figures 1 and 2 display selected PDF and CDF curves demonstrating the variety of shapes attainable.

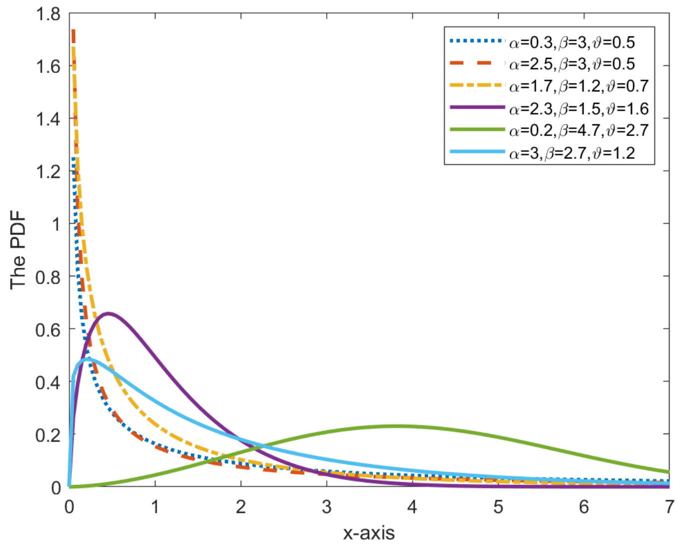

FIGURE 1. The PDF of the ALW distribution for selected parameter combinations $(\alpha, \beta, \vartheta)$.

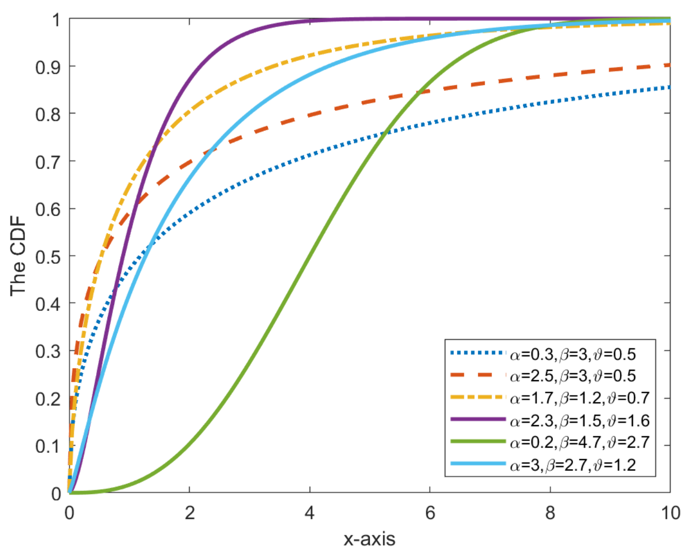

FIGURE 2. The CDF of the ALW distribution for selected parameter combinations $(\alpha, \beta, \vartheta)$.

Remark 2.2. *Special cases of the ALW distribution include:*

- *As* $\alpha \to 0^+$*:* $F_{\text{ALW}} \to 1 - e^{-x^{\vartheta}\beta^{-\vartheta}}$ *(Weibull distribution).*
- $\vartheta = 1$*: ALW reduces to the Alpha Logarithm Exponential distribution with scale* β*.*
- $\vartheta = 2$*: ALW reduces to an Alpha Logarithm Rayleigh distribution.*

3. Statistical Properties

3.1. Survival Function and Hazard Rate Function.

The survival function (SF) of the ALW distribution is

$$S(x) = 1 - F(x) = \frac{\ln(1+\alpha) - \ln D(x)}{\ln(1+\alpha)}. \tag{8}$$

Since $D(x) \in (1, 1+\alpha)$ implies $\ln D(x) < \ln(1+\alpha)$, we have $S(x) > 0$ for all $x \geq 0$.

The hazard rate function (HRF) is $h(x) = f(x)/S(x)$, giving

$$h(x) = \frac{\alpha\,\vartheta\,\beta^{-\vartheta}\,x^{\vartheta-1}\,A(x)}{D(x)\left[\ln(1+\alpha) - \ln D(x)\right]}. \tag{9}$$

Since $D(x) \in (1, 1+\alpha)$, the denominator is strictly positive, guaranteeing $h(x) \geq 0$. The HRF can accommodate increasing, decreasing, bathtub-shaped, and unimodal (upside-down bathtub) shapes depending on the parameter values.

3.2. Cumulative Hazard Rate Function.

The cumulative hazard rate function (CHRF) is $H(x) = -\ln S(x)$, giving

$$H(x;\alpha,\beta,\vartheta) = \ln\left(\frac{\ln(1+\alpha)}{\ln(1+\alpha) - \ln D(x)}\right). \tag{10}$$

Note that $H(x) \geq 0$ for all $x \geq 0$, with $H(0) = 0$ and $H(x) \to \infty$ as $x \to \infty$.

3.3. Reversed Hazard Rate Function.

The reversed hazard rate function (RHRF) is $\tau(x) = f(x)/F(x)$, giving

$$\tau(x;\alpha,\beta,\vartheta) = \frac{\alpha\,\vartheta\,\beta^{-\vartheta}\,x^{\vartheta-1}\,A(x)}{D(x)\,\ln D(x)}. \tag{11}$$

Remark 3.1. *Using the shorthand $A(x)$ and $D(x)$ from (5), the fundamental reliability functions are written compactly as:*

$$F(x) = \frac{\ln D(x)}{\ln(1+\alpha)}, \quad S(x) = \frac{\ln(1+\alpha) - \ln D(x)}{\ln(1+\alpha)},$$

$$h(x) = \frac{\alpha\vartheta\beta^{-\vartheta}x^{\vartheta-1}A(x)}{D(x)[\ln(1+\alpha) - \ln D(x)]},$$

$$H(x) = \ln\left(\frac{\ln(1+\alpha)}{\ln(1+\alpha) - \ln D(x)}\right), \quad \tau(x) = \frac{\alpha\vartheta\beta^{-\vartheta}x^{\vartheta-1}A(x)}{D(x)\,\ln D(x)}.$$

3.4. Odds Function and Mills Ratio.

The **odds function** (OF) is

$$\mathrm{OF}(x;\alpha,\beta,\vartheta) = \frac{F(x)}{S(x)} = \frac{\ln D(x)}{\ln(1+\alpha) - \ln D(x)}. \tag{12}$$

Note that $\mathrm{OF}(0) = 0$ since $D(0) = 1$, and $\mathrm{OF}(x) \to \infty$ as $x \to \infty$.

The **Mills ratio** (MR) is the reciprocal of the hazard rate function:

$$\mathrm{MR}(x) = \frac{D(x)\left[\ln(1+\alpha) - \ln D(x)\right]}{\alpha\,\vartheta\,\beta^{-\vartheta}\,x^{\vartheta-1}\,A(x)}. \tag{13}$$

Proposition 3.2. *The quantile function $Q(u) = F^{-1}(u)$ of the ALW distribution is*

$$Q(u) = \beta \left[-\ln\left(\frac{\alpha + 1 - (1+\alpha)^u}{\alpha} \right) \right]^{1/\vartheta}, \quad 0 < u < 1. \tag{14}$$

Proof. Setting $F(x) = u$ in (6) and exponentiating gives $D(x) = (1+\alpha)^u$, i.e. $\alpha + 1 - \alpha e^{-x^{\vartheta}\beta^{-\vartheta}} = (1+\alpha)^u$. Solving for $e^{-x^{\vartheta}\beta^{-\vartheta}}$ and taking the negative logarithm yields $x^{\vartheta}\beta^{-\vartheta} = -\ln[(\alpha + 1 - (1+\alpha)^u)/\alpha]$. Multiplying by β^{ϑ} and raising to the power $1/\vartheta$ produces (14). □

Remark 3.3. *The median, first and third quartiles are $Q(0.5)$, $Q(0.25)$, and $Q(0.75)$, respectively. Random samples from the ALW distribution can be generated via the inversion method: $X = Q(U)$, $U \sim \mathrm{Uniform}(0,1)$.*

3.5. **Raw Moments and Variance.**

Theorem 3.4. *Let $X \sim \mathrm{ALW}(\alpha, \beta, \vartheta)$ and let $z = \alpha/(1+\alpha) \in (0,1)$. The r-th raw moment $\mu'_r = \mathbb{E}[X^r]$ is*

$$\mu'_r = \frac{\beta^r\, \Gamma\left(\frac{r}{\vartheta} + 1\right)}{\ln(1+\alpha)}\, \mathrm{Li}_{r/\vartheta+1}(z), \tag{15}$$

where $\Gamma(\cdot)$ is the Gamma function and $\mathrm{Li}_s(z) = \sum_{m=1}^{\infty} z^m/m^s$ is the polylogarithm of order s [19]. The series converges absolutely for all s since $z \in (0,1)$.

Proof. Starting from $\mu'_r = \int_0^\infty x^r f(x)\, dx$ and substituting (7):

$$\mu'_r = \frac{\alpha\, \vartheta\, \beta^{-\vartheta}}{\ln(1+\alpha)} \int_0^\infty \frac{x^{r+\vartheta-1}\, e^{-x^{\vartheta}\beta^{-\vartheta}}}{(\alpha+1) - \alpha\, e^{-x^{\vartheta}\beta^{-\vartheta}}}\, dx. \tag{16}$$

Set $u = (x/\beta)^{\vartheta}$ so that $x = \beta u^{1/\vartheta}$, $dx = (\beta/\vartheta) u^{1/\vartheta - 1} du$. Then $x^{r+\vartheta-1} = \beta^{r+\vartheta-1} u^{(r+\vartheta-1)/\vartheta}$ and $e^{-(x/\beta)^{\vartheta}} = e^{-u}$. After simplification:

$$\mu'_r = \frac{\alpha\, \beta^r}{(1+\alpha)\, \ln(1+\alpha)} \int_0^\infty \frac{u^{r/\vartheta}\, e^{-u}}{1 - z\, e^{-u}}\, du, \quad z = \frac{\alpha}{1+\alpha}. \tag{17}$$

Expanding $(1 - ze^{-u})^{-1} = \sum_{m=0}^{\infty} z^m e^{-mu}$ (valid since $ze^{-u} < 1$) and applying $\int_0^\infty u^{r/\vartheta} e^{-(m+1)u}\, du = \Gamma(r/\vartheta + 1)/(m+1)^{r/\vartheta+1}$:

$$\mu'_r = \frac{\alpha\, \beta^r\, \Gamma\left(\frac{r}{\vartheta} + 1\right)}{(1+\alpha)\, \ln(1+\alpha)} \sum_{m=0}^{\infty} \frac{z^m}{(m+1)^{r/\vartheta+1}}. \tag{18}$$

Re-indexing $m \mapsto m - 1$ gives $\sum_{m=0}^{\infty} z^m/(m+1)^{r/\vartheta+1} = \mathrm{Li}_{r/\vartheta+1}(z)/z$. Since $\alpha/[(1+\alpha)z] = 1$, substituting yields (15). □

Remark 3.5. *Setting $r = 1$ and $r = 2$ in (15):*

$$\mathbb{E}[X] = \frac{\beta\,\Gamma\left(\frac{1}{\vartheta}+1\right)}{\ln(1+\alpha)}\,\mathrm{Li}_{1/\vartheta+1}(z), \tag{19}$$

$$\mathrm{Var}(X) = \frac{\beta^2\,\Gamma\left(\frac{2}{\vartheta}+1\right)}{\ln(1+\alpha)}\,\mathrm{Li}_{2/\vartheta+1}(z) - \left[\frac{\beta\,\Gamma\left(\frac{1}{\vartheta}+1\right)}{\ln(1+\alpha)}\,\mathrm{Li}_{1/\vartheta+1}(z)\right]^2. \tag{20}$$

3.6. **Incomplete Moments.**

Theorem 3.6. *Let $X \sim \mathrm{ALW}(\alpha, \beta, \vartheta)$. The r-th incomplete moment at $t > 0$ is*

$$\phi_r(t) = \frac{\beta^r}{\ln(1+\alpha)} \sum_{m=1}^{\infty} \frac{z^{m-1}}{m^{r/\vartheta}}\,\gamma\left(\frac{r}{\vartheta}+1,\; m\left(\frac{t}{\beta}\right)^{\vartheta}\right), \tag{21}$$

where $z = \alpha/(1+\alpha)$ and $\gamma(a, x) = \int_0^x u^{a-1}e^{-u}\,du$ is the lower incomplete Gamma function.

Proof. From $\phi_r(t) = \int_0^t x^r f(x)\,dx$, applying the same substitution $u = (x/\beta)^{\vartheta}$ as in Theorem 3.4 with upper limit $(t/\beta)^{\vartheta}$:

$$\phi_r(t) = \frac{\alpha\,\beta^r}{(1+\alpha)\,\ln(1+\alpha)} \sum_{m=0}^{\infty} z^m \int_0^{(t/\beta)^{\vartheta}} u^{r/\vartheta}\,e^{-(m+1)u}\,du. \tag{22}$$

The inner integral equals $\gamma\big(r/\vartheta+1,\,(m+1)(t/\beta)^{\vartheta}\big)/(m+1)^{r/\vartheta+1}$. Re-indexing $m \mapsto m-1$ and using $\alpha/[(1+\alpha)z] = 1$ yields (21). The series converges absolutely since $z < 1$ and $\gamma(\cdot,\cdot) \le \Gamma(\cdot) < \infty$. □

Remark 3.7. *The first incomplete moment $\phi_1(t)$ yields the Bonferroni curve $B(p)$ and the Lorenz curve $L(p)$:*

$$B(p) = \frac{\phi_1(Q(p))}{p\,\mu_1'}, \qquad L(p) = \frac{\phi_1(Q(p))}{\mu_1'}, \qquad 0 < p < 1, \tag{23}$$

where $Q(p)$ is the quantile function given in (14).

3.7. **Moment Generating Function.**

Proposition 3.8. *Let $X \sim \mathrm{ALW}(\alpha, \beta, \vartheta)$. The moment generating function of X is*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r\,\beta^r\,\Gamma\left(\frac{r}{\vartheta}+1\right)}{r!\,\ln(1+\alpha)}\,\mathrm{Li}_{r/\vartheta+1}(z), \tag{24}$$

where $z = \alpha/(1+\alpha)$. The series converges for $|t|$ in a neighbourhood of zero; the radius of convergence is determined by the growth rate of μ_r' as $r \to \infty$.

Proof. Expanding $e^{tx} = \sum_{r=0}^{\infty} t^r x^r/r!$, substituting into $M_X(t) = \mathbb{E}[e^{tX}]$, interchanging sum and integral (by absolute convergence for $|t|$ sufficiently small), and applying Theorem 3.4 yields (24) immediately. □

3.8. **Rényi Entropy.**

Theorem 3.9. *Let $X \sim \mathrm{ALW}(\alpha, \beta, \vartheta)$ and $z = \alpha/(1+\alpha) \in (0,1)$. For $\delta > 0$, $\delta \neq 1$, and $\delta > \vartheta/(\vartheta-1)$ whenever $\vartheta < 1$, the Rényi entropy is*

$$I_R(\delta) = \frac{1}{1-\delta} \ln \left[\frac{\alpha^{\delta}\, \vartheta^{\delta-1}\, \beta^{1-\delta}}{[\ln(1+\alpha)]^{\delta}(1+\alpha)^{\delta}} \sum_{k=0}^{\infty} \binom{\delta+k-1}{k} z^k \frac{\Gamma\left(\dfrac{\delta(\vartheta-1)+k+1}{\vartheta}\right)}{(\delta+k)^{[\delta(\vartheta-1)+k+1]/\vartheta}} \right]. \tag{25}$$

The series converges absolutely since $z < 1$.

Proof. By definition, $I_R(\delta) = \frac{1}{1-\delta} \ln \mathcal{I}$ where $\mathcal{I} = \int_0^\infty [f(x)]^\delta\, dx$. Writing $[f(x)]^\delta$ using (7), factoring $D(x) = (1+\alpha)(1 - ze^{-x^\vartheta \beta^{-\vartheta}})$, and expanding via the generalised binomial series $(1 - ze^{-x^\vartheta\beta^{-\vartheta}})^{-\delta} = \sum_{k=0}^\infty \binom{\delta+k-1}{k} z^k e^{-kx^\vartheta \beta^{-\vartheta}}$ (valid since $ze^{-x^\vartheta\beta^{-\vartheta}} < z < 1$), we obtain after interchanging sum and integral and applying the substitution $u = (\delta+k)(x/\beta)^\vartheta$:

$$\mathcal{I} = \frac{\alpha^{\delta}\, \vartheta^{\delta-1}\, \beta^{1-\delta}}{[\ln(1+\alpha)]^{\delta}(1+\alpha)^{\delta}} \sum_{k=0}^{\infty} \binom{\delta+k-1}{k} z^k \frac{\Gamma\left(\dfrac{\delta(\vartheta-1)+k+1}{\vartheta}\right)}{(\delta+k)^{[\delta(\vartheta-1)+k+1]/\vartheta}},$$

where the Gamma integral requires $[\delta(\vartheta-1)+k+1]/\vartheta > 0$, guaranteed by the stated condition on δ. Substituting into the definition of $I_R(\delta)$ yields (25). The series converges absolutely by the ratio test since $|a_{k+1}/a_k| \to z < 1$. □

3.9. **Stress–Strength Reliability.**

Theorem 3.10. *Let $X_1 \sim \mathrm{ALW}(\alpha_1, \beta, \vartheta)$ and $X_2 \sim \mathrm{ALW}(\alpha_2, \beta, \vartheta)$ be independent, sharing common $\beta > 0$ and $\vartheta > 0$. Define $z_i = \alpha_i/(1+\alpha_i)$ for $i = 1, 2$. Then*

$$R = P(X_1 > X_2) = \frac{z_1}{\ln(1+\alpha_1)\, \ln(1+\alpha_2)} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{z_1^m\, z_2^k}{k\,(m+k+1)}. \tag{26}$$

The double series converges absolutely for all $\alpha_1, \alpha_2 > 0$.

Proof. Since X_1 and X_2 are independent, $R = \int_0^\infty f_1(x)\, S_2(x)\, dx$. Setting $u = (x/\beta)^\vartheta$ so that $\vartheta\beta^{-\vartheta}x^{\vartheta-1}dx = du$, $A(x) = e^{-u}$, and $D_i(x) = (1+\alpha_i) - \alpha_i e^{-u} \equiv \widetilde{D}_i(u)$:

$$R = \frac{\alpha_1}{\ln(1+\alpha_1)\, \ln(1+\alpha_2)} \int_0^\infty \frac{e^{-u}}{\widetilde{D}_1(u)} \left[\ln(1+\alpha_2) - \ln \widetilde{D}_2(u)\right] du. \tag{27}$$

Writing $1/\widetilde{D}_1(u) = [(1+\alpha_1)(1 - z_1 e^{-u})]^{-1} = \frac{1}{1+\alpha_1} \sum_{m=0}^\infty z_1^m e^{-mu}$ and $\ln \widetilde{D}_2(u) = \ln(1+\alpha_2) - \sum_{k=1}^\infty z_2^k e^{-ku}/k$, we obtain after splitting the bracket and computing:

$$J_1 = \int_0^\infty \frac{e^{-u}}{\widetilde{D}_1(u)}\, du = \frac{\ln(1+\alpha_1)}{\alpha_1},$$

$$J_2 = \int_0^\infty \frac{e^{-u} \ln \widetilde{D}_2(u)}{\widetilde{D}_1(u)}\, du = \frac{\ln(1+\alpha_2) \ln(1+\alpha_1)}{\alpha_1} - \frac{1}{1+\alpha_1} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{z_1^m z_2^k}{k(m+k+1)}.$$

Substituting into $R = \frac{\alpha_1}{\ln(1+\alpha_1)\ln(1+\alpha_2)}[\ln(1+\alpha_2)J_1 - J_2]$ and simplifying gives (26). Absolute convergence follows since $z_1, z_2 \in (0,1)$ and the double sum $\sum_{m,k} z_1^m z_2^k/(m+k+1) < \infty$. □

Remark 3.11. *The double series* (26) *is evaluated numerically; truncating at* $m,k \le 30$ *achieves relative error below* 10^{-8}.

4. Parameter Estimation

Let $x_1,\ldots,x_n$ be an observed random sample from $\mathrm{ALW}(\alpha,\beta,\vartheta)$ with order statistics $x_{(1)} \le \cdots \le x_{(n)}$. We employ the shorthand $A_i = e^{-x_i^{\vartheta}\beta^{-\vartheta}}$ and $D_i = \alpha + 1 - \alpha A_i$, and denote the parameter vector by $\boldsymbol{\theta} = (\alpha,\beta,\vartheta)^\top \in \Theta = (0,\infty)^3$.

We develop four estimation methods: maximum likelihood (MLE), least squares (LSE), weighted least squares (WLSE), and Anderson–Darling (ADE). Each method targets the same $\boldsymbol{\theta}$ but uses a different loss criterion.

4.1. Maximum Likelihood Estimation.

The log-likelihood function is

$$\ell(\alpha,\beta,\vartheta) = n\ln\alpha + n\ln\vartheta - \vartheta n\ln\beta + (\vartheta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n} x_i^{\vartheta}\beta^{-\vartheta} - n\,\mathrm{lnln}(1+\alpha) - \sum_{i=1}^{n}\ln D_i. \quad (28)$$

The score equations $\partial\ell/\partial\alpha = \partial\ell/\partial\beta = \partial\ell/\partial\vartheta = 0$ are:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \frac{n}{(1+\alpha)\ln(1+\alpha)} - \sum_{i=1}^{n}\frac{1-A_i}{D_i} = 0, \quad (29)$$

$$\frac{\partial\ell}{\partial\beta} = -\frac{n\vartheta}{\beta} + \vartheta\beta^{-\vartheta-1}\sum_{i=1}^{n}x_i^{\vartheta} - \sum_{i=1}^{n}\frac{\alpha\,\vartheta\,x_i^{\vartheta}\beta^{-\vartheta-1}A_i}{D_i} = 0, \quad (30)$$

$$\frac{\partial\ell}{\partial\vartheta} = \frac{n}{\vartheta} - n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}x_i^{\vartheta}\beta^{-\vartheta}\ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n}\frac{\alpha A_i\,x_i^{\vartheta}\beta^{-\vartheta}\ln(x_i/\beta)}{D_i} = 0. \quad (31)$$

The partial derivatives of $\ln D_i$ used above are

$$\frac{\partial\ln D_i}{\partial\alpha} = \frac{1-A_i}{D_i}, \quad \frac{\partial\ln D_i}{\partial\beta} = \frac{\alpha\,\vartheta\,x_i^{\vartheta}\beta^{-\vartheta-1}A_i}{D_i}, \quad \frac{\partial\ln D_i}{\partial\vartheta} = \frac{\alpha\,A_i\,x_i^{\vartheta}\beta^{-\vartheta}\ln(x_i/\beta)}{D_i}.$$

Because the score equations have no closed-form solution, the MLEs $(\hat\alpha,\hat\beta,\hat\vartheta)$ are obtained by minimising $-\ell(\boldsymbol{\theta})$ numerically using the `fmincon` routine in MATLAB R2022b with the interior-point algorithm. Ten random starting points drawn from $\alpha \in (0,15]$, $\beta \in (0,15]$, $\vartheta \in (0,10]$ are used, and the solution achieving the smallest $-\ell$ is retained.

4.2. Least Squares Estimation.

The LSE method [23] minimises the sum of squared differences between the empirical and fitted CDFs at the order statistics. Using the Blom approximation $\mathbb{E}[F(X_{(i)})] \approx i/(n+1)$ [12], the LSEs minimise

$$\Psi_{\mathrm{LS}}(\alpha,\beta,\vartheta) = \sum_{i=1}^{n}\left[\frac{\ln D_{(i)}}{\ln(1+\alpha)} - \frac{i}{n+1}\right]^2, \quad (32)$$

where $D_{(i)} = \alpha + 1 - \alpha e^{-x_{(i)}^{\vartheta}\beta^{-\vartheta}}$. The normal equations $\partial\Psi_{\mathrm{LS}}/\partial\theta_j = 0$ involve the partial derivatives of $\ln D_{(i)}/\ln(1+\alpha)$:

$$\frac{\partial}{\partial\alpha}\left[\frac{\ln D_{(i)}}{\ln(1+\alpha)}\right] = \frac{1 - A_{(i)}}{D_{(i)}\ln(1+\alpha)} - \frac{\ln D_{(i)}}{(1+\alpha)[\ln(1+\alpha)]^2}, \tag{33}$$

$$\frac{\partial}{\partial\beta}\left[\frac{\ln D_{(i)}}{\ln(1+\alpha)}\right] = \frac{\alpha\,\vartheta\,x_{(i)}^{\vartheta}\beta^{-\vartheta-1}A_{(i)}}{D_{(i)}\ln(1+\alpha)}, \tag{34}$$

$$\frac{\partial}{\partial\vartheta}\left[\frac{\ln D_{(i)}}{\ln(1+\alpha)}\right] = \frac{\alpha\,A_{(i)}\,x_{(i)}^{\vartheta}\beta^{-\vartheta}\ln(x_{(i)}/\beta)}{D_{(i)}\ln(1+\alpha)}. \tag{35}$$

These have no closed-form solution; the LSEs are obtained by numerical minimisation of Ψ_{LS}.

4.3. **Weighted Least Squares Estimation.** The WLSE method accounts for the heterogeneous variances of the order statistics. Since $\mathrm{Var}[F(X_{(i)})] \approx i(n-i+1)/[(n+1)^2(n+2)]$ [23], the optimal weight for the i-th term is $w_i = (n+1)^2(n+2)/[i(n-i+1)]$. The WLSEs minimise

$$\Psi_{\mathrm{WLS}}(\alpha,\beta,\vartheta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{\ln D_{(i)}}{\ln(1+\alpha)} - \frac{i}{n+1}\right]^2. \tag{36}$$

The normal equations have the same form as (33)–(35), with each summand multiplied by w_i.

Remark 4.1. *The weight w_i up-weights the central order statistics (where the CDF is most precisely estimated) and down-weights the extremes (where estimation uncertainty is highest). This makes WLSE particularly robust for distributions with heavy tails or when outliers are present.*

4.4. **Anderson–Darling Estimation.** The ADE criterion [8] places extra weight on the tails of the distribution. The objective function is

$$\Psi_{\mathrm{AD}}(\alpha,\beta,\vartheta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\left(\frac{\ln D_{(i)}}{\ln(1+\alpha)}\right) + \ln\left(\frac{\ln(1+\alpha) - \ln D_{(n+1-i)}}{\ln(1+\alpha)}\right)\right]. \tag{37}$$

Remark 4.2. *The weight $(2i-1)$ assigns the largest weight to the extreme order statistics $x_{(1)}$ and $x_{(n)}$, in contrast to WLSE whose weights w_i diminish toward the extremes. This complementary emphasis explains why ADE outperforms WLSE in some heavy-tailed configurations (e.g. Config. 2 at $n = 10$) while underperforming in the small-ϑ regime of Config. 5, where the extreme observations carry misleading information about β.*

Numerical implementation. All four objective functions are minimised numerically using `fmincon` in MATLAB R2022b with the interior-point algorithm, subject to $\alpha,\beta,\vartheta > 0$. Ten random starting points drawn from $\alpha \in (0,15]$, $\beta \in (0,15]$, $\vartheta \in (0,10]$ are used, and the solution with the smallest objective value is retained. Analytical gradients are supplied to accelerate convergence.

5. Monte Carlo Simulation Study

5.1. Design. We conducted a Monte Carlo simulation with $N = 1000$ replications for each of 7 sample sizes $n \in \{10, 25, 50, 100, 150, 200, 250\}$ and five parameter configurations:

Config.	α	β	ϑ	Hazard shape
1	0.2	0.6	0.6	Decreasing
2	1.7	0.6	0.6	Decreasing
3	0.8	1.5	1.2	Unimodal
4	2.5	1.2	1.8	Increasing
5	0.4	1.9	0.2	Decreasing

Random samples were generated via the inversion method using (14). For each replication and estimator, the Bias and MSE are:

$$\text{Bias}(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta), \qquad \text{MSE}(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta)^2.$$

The column *Best* identifies the method with the smallest total MSE (summed across all three parameters) for each n.

5.2. Simulation Results. Tables 1–5 report Bias and MSE for each parameter and estimation method under Configurations 1–5, respectively. Figures 3 and 4 display the corresponding Bias and MSE curves.

Table 1. Simulation results for Configuration 1: $\alpha = 0.2$, $\beta = 0.6$, $\vartheta = 0.6$.

n	Method	Bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\vartheta}$)	MSE($\hat{\vartheta}$)	Best
10	MLE	+8.2691	120.6838	+0.5475	1.1056	+0.2401	0.1274	–
	LSE	+1.2186	5.9453	+0.3862	0.3306	+0.3271	0.1608	–
	WLSE	+1.2116	5.6822	+0.3967	0.3292	+0.3049	0.1344	**WLSE**
	ADE	+1.2168	5.9421	+0.3823	0.2559	+0.3403	0.1670	–
25	MLE	+5.9696	83.9540	+0.4335	0.5528	+0.1285	0.0431	–
	LSE	+1.4366	7.4982	+0.3943	0.2373	+0.3091	0.1254	**LSE**
	WLSE	+1.5243	8.3456	+0.3932	0.2481	+0.2878	0.1146	–
	ADE	+1.4697	8.1987	+0.3640	0.2091	+0.2932	0.1293	–
50	MLE	+4.7661	64.1945	+0.3618	0.4208	+0.0942	0.0243	–
	LSE	+1.6576	9.5104	+0.3981	0.2184	+0.3064	0.1185	–
	WLSE	+1.4297	6.9134	+0.3593	0.1982	+0.2743	0.1062	**WLSE**
	ADE	+1.5305	8.6939	+0.3525	0.1932	+0.2677	0.1013	–
100	MLE	+3.6919	45.7317	+0.3102	0.2957	+0.0723	0.0167	–
	LSE	+1.7061	9.3071	+0.3963	0.2113	+0.2797	0.1059	–
	WLSE	+1.6933	9.1573	+0.3700	0.1854	+0.2520	0.0937	**WLSE**
	ADE	+1.8638	11.6204	+0.3549	0.1864	+0.2335	0.0863	–
150	MLE	+2.5831	27.8344	+0.2507	0.2260	+0.0543	0.0105	–
	LSE	+1.9085	11.8610	+0.3804	0.2015	+0.2530	0.0929	–
	WLSE	+1.6631	9.0225	+0.3441	0.1772	+0.2237	0.0812	**WLSE**
	ADE	+1.7246	9.6513	+0.3394	0.1781	+0.2175	0.0787	–
200	MLE	+2.1658	20.9655	+0.2129	0.1652	+0.0482	0.0084	–
	LSE	+1.9810	11.9481	+0.3778	0.2017	+0.2315	0.0830	–
	WLSE	+1.6796	8.6253	+0.3359	0.1699	+0.2014	0.0712	**WLSE**
	ADE	+1.7264	8.9901	+0.3316	0.1695	+0.1946	0.0683	–
250	MLE	+1.7865	15.3656	+0.1868	0.1343	+0.0441	0.0071	–
	LSE	+1.6100	7.8183	+0.3527	0.1704	+0.2355	0.0854	–
	WLSE	+1.5113	7.0851	+0.3223	0.1517	+0.2078	0.0745	**WLSE**
	ADE	+1.5764	7.9434	+0.3108	0.1501	+0.1986	0.0755	–

TABLE 2. Simulation results for Configuration 2: $\alpha = 1.7,\ \beta = 0.6,\ \vartheta = 0.6$.

n	Method	Bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\vartheta}$)	MSE($\hat{\vartheta}$)	Best
10	MLE	+7.4865	107.9485	+0.2946	0.5605	+0.1807	0.0893	–
	LSE	−0.2093	5.3111	+0.3537	0.2696	+0.2975	0.1305	–
	WLSE	−0.1792	5.3718	+0.3218	0.1993	+0.2855	0.1250	–
	ADE	−0.3081	3.8372	+0.3459	0.2126	+0.3235	0.1462	**ADE**
25	MLE	+5.7159	83.6203	+0.2366	0.3344	+0.0873	0.0296	–
	LSE	−0.0675	5.4075	+0.3183	0.1644	+0.3022	0.1238	–
	WLSE	−0.1077	4.7733	+0.2984	0.1522	+0.2750	0.1153	**WLSE**
	ADE	−0.0976	5.0892	+0.2680	0.1377	+0.2717	0.1128	–
50	MLE	+4.2093	61.9493	+0.1464	0.2014	+0.0465	0.0138	–
	LSE	+0.1577	6.3500	+0.3074	0.1558	+0.2806	0.1127	**LSE**
	WLSE	+0.2491	7.3452	+0.2797	0.1393	+0.2543	0.1024	–
	ADE	+0.3272	8.7303	+0.2572	0.1400	+0.2429	0.0975	–
100	MLE	+3.6741	51.3438	+0.1269	0.1492	+0.0383	0.0103	–
	LSE	+0.2342	6.5729	+0.2687	0.1287	+0.2480	0.0991	**LSE**
	WLSE	+0.4231	7.8492	+0.2389	0.1231	+0.2132	0.0851	–
	ADE	+0.5634	9.2214	+0.2271	0.1162	+0.2110	0.0854	–
150	MLE	+3.7569	50.1223	+0.1441	0.1508	+0.0382	0.0099	–
	LSE	+0.6200	9.6913	+0.2325	0.1196	+0.2022	0.0802	**LSE**
	WLSE	+0.9208	11.3632	+0.2193	0.1179	+0.1731	0.0678	–
	ADE	+0.8154	9.9337	+0.1967	0.1063	+0.1629	0.0634	–
200	MLE	+2.6350	35.8713	+0.0911	0.1156	+0.0246	0.0072	–
	LSE	+0.4644	7.3264	+0.2245	0.1110	+0.1956	0.0775	**LSE**
	WLSE	+0.7537	9.8180	+0.1891	0.1016	+0.1552	0.0609	–
	ADE	+0.8947	10.8993	+0.1724	0.1007	+0.1420	0.0555	–
250	MLE	+2.4162	29.9303	+0.0916	0.1009	+0.0240	0.0065	–
	LSE	+0.5854	8.8029	+0.2284	0.1145	+0.1981	0.0773	**LSE**
	WLSE	+0.8046	10.2514	+0.1714	0.0973	+0.1378	0.0541	–
	ADE	+1.0403	12.6083	+0.1620	0.0961	+0.1296	0.0537	–

TABLE 3. Simulation results for Configuration 3: $\alpha = 0.8,\ \beta = 1.5,\ \vartheta = 1.2$.

n	Method	Bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\vartheta}$)	MSE($\hat{\vartheta}$)	Best
10	MLE	+7.9198	114.1884	+0.3548	0.6029	+0.4114	0.4276	–
	LSE	+0.6208	4.8797	−0.3262	0.2973	−0.1274	0.1063	–
	WLSE	+0.5678	4.4645	−0.2995	0.3107	−0.1077	0.1594	**WLSE**
	ADE	+0.7772	6.5458	−0.3239	0.2989	−0.1190	0.0866	–
25	MLE	+6.1841	87.5437	+0.3284	0.4579	+0.2338	0.1573	–
	LSE	+0.8155	5.9837	−0.3049	0.2552	−0.1186	0.0597	**LSE**
	WLSE	+0.8234	6.6269	−0.2599	0.2506	−0.1028	0.0590	–
	ADE	+0.9773	7.6596	−0.2418	0.2450	−0.0796	0.0626	–
50	MLE	+4.8772	68.0455	+0.2150	0.2868	+0.1506	0.0854	–
	LSE	+1.0449	7.7514	−0.2712	0.2361	−0.1001	0.0461	–
	WLSE	+0.9679	7.2145	−0.2404	0.2277	−0.0888	0.0440	**WLSE**
	ADE	+1.0940	9.0972	−0.2139	0.2139	−0.0612	0.0454	–
100	MLE	+3.5125	46.0072	+0.1612	0.1996	+0.0964	0.0500	–
	LSE	+1.1675	8.2044	−0.2223	0.2157	−0.0813	0.0430	**LSE**
	WLSE	+1.5314	12.6492	−0.1620	0.2116	−0.0565	0.0428	–
	ADE	+1.3276	10.2301	−0.1504	0.1960	−0.0472	0.0387	–
150	MLE	+2.8794	36.8807	+0.1384	0.1680	+0.0839	0.0394	–
	LSE	+1.1831	8.0268	−0.1858	0.1986	−0.0631	0.0398	**LSE**
	WLSE	+1.2850	9.1339	−0.1452	0.1894	−0.0471	0.0367	–
	ADE	+1.2312	9.1129	−0.1308	0.1788	−0.0381	0.0349	–
200	MLE	+2.4225	28.1239	+0.1311	0.1502	+0.0704	0.0312	–
	LSE	+1.3229	8.3966	−0.1185	0.1890	−0.0467	0.0340	**LSE**
	WLSE	+1.3782	10.1167	−0.0813	0.1699	−0.0333	0.0308	–
	ADE	+1.4029	10.0345	−0.0668	0.1640	−0.0217	0.0318	–
250	MLE	+2.1109	24.2297	+0.1064	0.1309	+0.0589	0.0284	–
	LSE	+1.3477	9.4478	−0.1486	0.1868	−0.0538	0.0349	–
	WLSE	+1.2167	8.6645	−0.1043	0.1577	−0.0375	0.0294	–
	ADE	+1.1576	8.1405	−0.0954	0.1493	−0.0289	0.0301	**ADE**

Table 4. Simulation results for Configuration 4: $\alpha = 2.5,\ \beta = 1.2,\ \vartheta = 1.8$.

n	Method	Bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\vartheta}$)	MSE($\hat{\vartheta}$)	Best
10	MLE	+6.8539	97.4346	+0.0388	0.1072	+0.4986	0.7469	–
	LSE	−1.0517	6.4448	−0.1780	0.0549	−0.6057	0.6183	–
	WLSE	−1.1277	4.8013	−0.1640	0.0528	−0.5938	0.5725	**WLSE**
	ADE	−1.1730	4.8512	−0.1696	0.0518	−0.5812	0.5618	–
25	MLE	+5.3138	78.5142	+0.0456	0.0738	+0.2210	0.2250	–
	LSE	−0.8128	7.1606	−0.1496	0.0470	−0.5724	0.4964	–
	WLSE	−0.6134	8.0797	−0.1407	0.0455	−0.5253	0.4892	–
	ADE	−0.7085	7.0283	−0.1457	0.0435	−0.5021	0.4774	**ADE**
50	MLE	+4.7762	70.7218	+0.0399	0.0676	+0.1385	0.1377	–
	LSE	−0.6355	8.0194	−0.1442	0.0432	−0.5372	0.4746	–
	WLSE	−0.6522	6.5977	−0.1340	0.0412	−0.5003	0.4326	**WLSE**
	ADE	−0.4372	8.6334	−0.1324	0.0410	−0.4619	0.4230	–
100	MLE	+3.6447	53.5077	+0.0228	0.0478	+0.0924	0.0883	–
	LSE	−0.3237	8.3650	−0.1168	0.0351	−0.4355	0.3811	**LSE**
	WLSE	−0.2373	8.9046	−0.1051	0.0348	−0.3908	0.3411	–
	ADE	−0.1897	9.2422	−0.1032	0.0341	−0.3643	0.3277	–
150	MLE	+3.7823	51.7049	+0.0400	0.0409	+0.0949	0.0754	–
	LSE	−0.0571	9.4344	−0.0978	0.0323	−0.3777	0.3411	**LSE**
	WLSE	+0.2739	11.5042	−0.0787	0.0312	−0.3129	0.2971	–
	ADE	+0.3102	11.2197	−0.0773	0.0303	−0.2878	0.2812	–
200	MLE	+2.8249	41.0267	+0.0120	0.0385	+0.0518	0.0622	–
	LSE	−0.1142	8.8199	−0.0966	0.0323	−0.3736	0.3251	**LSE**
	WLSE	+0.1274	10.8552	−0.0828	0.0310	−0.2936	0.2645	–
	ADE	+0.1346	10.5705	−0.0813	0.0310	−0.2810	0.2568	–
250	MLE	+2.6699	39.7423	+0.0115	0.0345	+0.0502	0.0588	–
	LSE	+0.3394	11.6890	−0.0727	0.0313	−0.2947	0.2771	–
	WLSE	+0.2358	10.9744	−0.0714	0.0288	−0.2546	0.2327	**WLSE**
	ADE	+0.3422	11.7675	−0.0691	0.0285	−0.2417	0.2271	–

Table 5. Simulation results for Configuration 5: $\alpha = 0.4,\ \beta = 1.9,\ \vartheta = 0.2$.

n	Method	Bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Bias($\hat{\vartheta}$)	MSE($\hat{\vartheta}$)	Best
10	MLE	+5.9004	77.4316	$(+6.0335)^{\dagger}$	$(73.7797)^{\dagger}$	+0.0679	0.0123	–
	LSE	+0.7397	2.6126	−0.3460	3.8399	+0.6464	0.6946	–
	WLSE	+0.6065	1.4979	−0.5052	2.5413	+0.6105	0.4876	**WLSE**
	ADE	+0.7985	2.6618	−0.1408	4.9280	+0.5742	0.4654	–
25	MLE	+5.1522	62.5758	$(+6.6062)^{\dagger}$	$(80.1183)^{\dagger}$	+0.0453	0.0045	–
	LSE	+0.8926	3.7976	−0.1451	5.5465	+0.6173	0.4910	**LSE**
	WLSE	+0.9723	4.6397	−0.0654	5.4854	+0.6024	0.4793	–
	ADE	+1.2571	6.2016	+0.7054	10.3617	+0.5130	0.4231	–
50	MLE	+4.6044	47.9545	$(+7.0962)^{\dagger}$	$(85.9345)^{\dagger}$	+0.0372	0.0026	–
	LSE	+0.8128	2.3311	+0.0285	6.5283	+0.6000	0.4778	**LSE**
	WLSE	+0.9039	3.0266	+0.1838	7.1261	+0.5427	0.4317	–
	ADE	+2.1488	16.1165	+1.6589	19.1774	+0.4613	0.3923	–
100	MLE	+4.3155	38.6209	$(+7.6159)^{\dagger}$	$(91.6451)^{\dagger}$	+0.0331	0.0018	–
	LSE	+0.8915	2.6306	+0.3968	8.0064	+0.5211	0.4137	**LSE**
	WLSE	+1.1289	3.7911	+0.9685	11.1642	+0.4617	0.3638	–
	ADE	+2.3536	18.5771	+2.3162	22.2967	+0.3306	0.2661	–
150	MLE	+3.8685	28.5955	$(+7.3841)^{\dagger}$	$(87.5195)^{\dagger}$	+0.0308	0.0014	–
	LSE	+1.0756	3.9183	+0.7202	11.1517	+0.5089	0.4023	**LSE**
	WLSE	+1.1516	4.3068	+1.2117	12.9113	+0.3815	0.2986	–
	ADE	+2.6100	25.4534	+2.5799	25.9510	+0.2870	0.2360	–
200	MLE	+3.7351	25.6557	$(+7.7451)^{\dagger}$	$(91.5309)^{\dagger}$	+0.0319	0.0015	–
	LSE	+1.0152	3.0415	+0.9911	12.3161	+0.4532	0.3575	**LSE**
	WLSE	+1.3317	5.1291	+1.9524	19.3293	+0.3379	0.2616	–
	ADE	+2.6374	24.6515	+2.9775	28.8135	+0.2824	0.2323	–
250	MLE	+3.9200	26.0411	$(+7.9615)^{\dagger}$	$(94.1932)^{\dagger}$	+0.0326	0.0015	–
	LSE	+0.9165	2.3355	+0.8190	11.2662	+0.4380	0.3458	**LSE**
	WLSE	+1.4433	5.3445	+2.1202	19.6588	+0.3201	0.2462	–
	ADE	+2.7974	27.1004	+3.2668	32.1794	+0.2545	0.1937	–

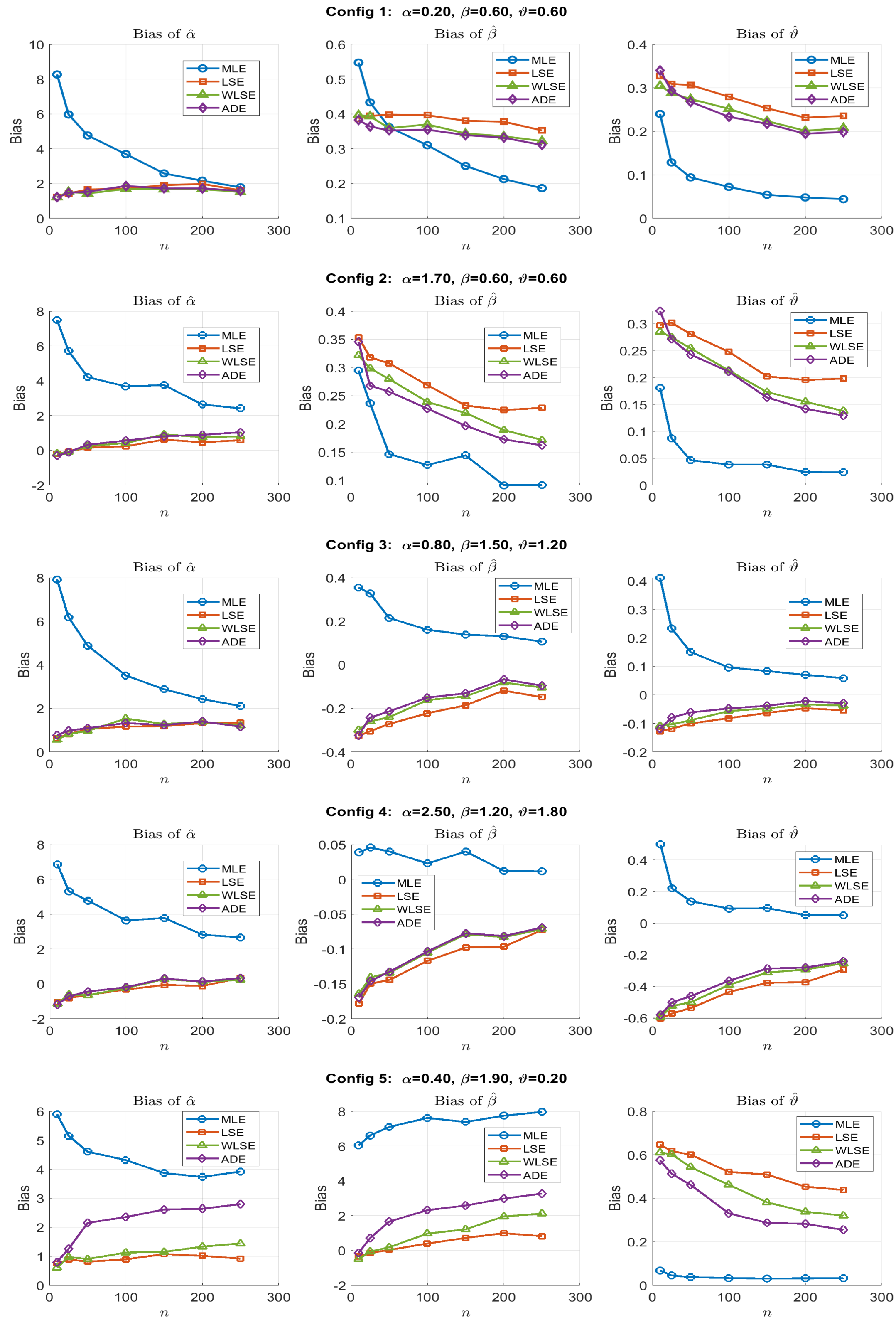

FIGURE 3. Bias of the MLEs, LSEs, WLSEs, and ADEs against sample sizes for Configurations 1–5.

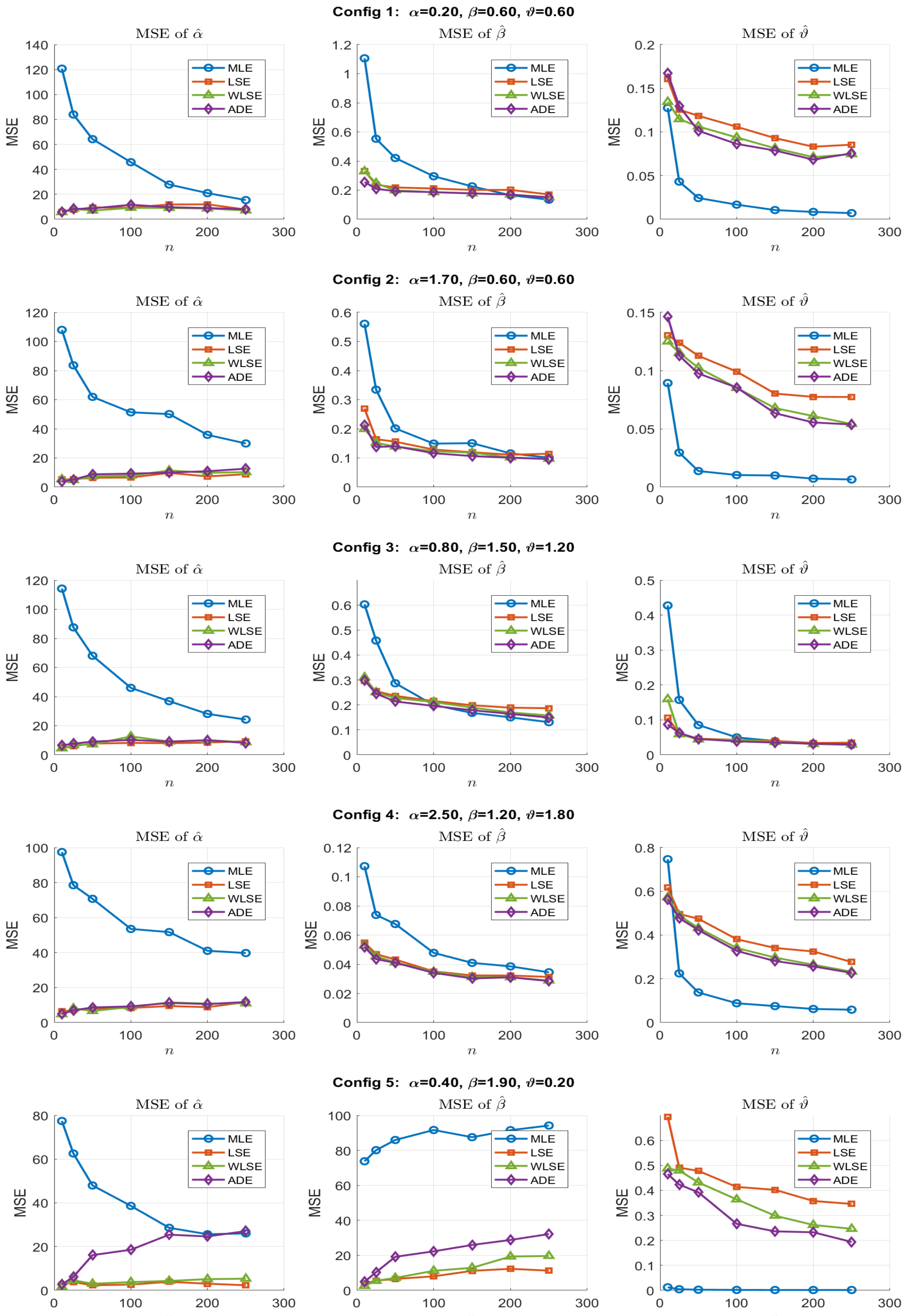

FIGURE 4. MSE of the MLEs, LSEs, WLSEs, and ADEs against sample sizes for Configurations 1–5.

5.3. **Discussion of Simulation Findings.** **Behaviour of MLE.** Across all five configurations, the MLE of α is markedly upward-biased at small sample sizes. In Configuration 1, the bias of $\hat{\alpha}_{\mathrm{MLE}}$ starts near 8.3 at $n = 10$ and decreases monotonically but slowly, remaining above 1.8 even at $n = 250$. This behaviour arises because the log-likelihood surface for α is nearly flat when n is small, a well-known challenge in log-transformed families. By contrast, the MLE of $\hat{\beta}$ and $\hat{\vartheta}$ converges rapidly in Configurations 1–4 and achieves the smallest MSE among all methods for $n \geq 100$. In Configuration 5 ($\vartheta = 0.2$), the MLE of $\hat{\beta}$ is persistently unreliable: for small ϑ the term x^{ϑ} varies little across the observed range, making β nearly non-identifiable. The MLE bias for $\hat{\beta}$ rises from 6.0 at $n = 10$ to nearly 8.0 at $n = 250$, confirming that MLE should not be used for $\hat{\beta}$ in this regime.

Behaviour of LSE. LSE controls the bias of $\hat{\alpha}$ effectively across all configurations. In Configurations 3 and 4, LSE consistently achieves the smallest total MSE for $n \geq 100$. In Configuration 5, LSE dominates at all sample sizes from $n = 25$ onward, with $\mathrm{MSE}(\hat{\beta})$ remaining well below 13 at $n = 250$, compared to the MLE's value exceeding 94.

Behaviour of WLSE. WLSE outperforms all other methods at small to moderate sample sizes in Configurations 1, 3, and 4. In Configuration 1, WLSE is Best for $n \in \{10, 50, 100, 150, 200, 250\}$. In Configuration 5, WLSE is Best at $n = 10$ due to its lower MSE for $\hat{\alpha}$ and $\hat{\beta}$.

Behaviour of ADE. ADE is Best at $n = 10$ in Configuration 2, reflecting its advantage for heavy-tailed scenarios. In Configuration 4 at $n = 25$, ADE also achieves the smallest total MSE. In Configuration 3 at $n = 250$, ADE achieves the smallest total MSE, suggesting that tail-weighting is advantageous for unimodal hazard shapes at large n. In Configuration 5, however, ADE performs poorly across all sample sizes.

General findings. The Bias of every estimator converges toward zero as n increases (Figure 3), confirming *asymptotic unbiasedness*. The MSE of every estimator decreases as n grows in Configurations 1–4 (Figure 4), establishing *consistency*. The only exception is MLE of $\hat{\beta}$ in Configuration 5, where near-non-identifiability prevents convergence.

Practical recommendations.

- $n \leq 50$: use **WLSE** for stable estimates across all parameters.
- $50 < n \leq 150$: use **LSE** as primary estimator.
- $n > 150$: use **MLE** for $\hat{\beta}$ and $\hat{\vartheta}$ (smallest MSE) combined with **LSE** for $\hat{\alpha}$. When $\vartheta < 0.3$ is suspected, use **LSE** for all three parameters.

6. Applications to Real Data

To assess the practical utility of the ALW distribution, we fit it to four well-known real lifetime datasets. For each dataset the ALW model is evaluated against six competing distributions; parameter estimates are obtained by MLE, and model comparison employs seven complementary goodness-of-fit criteria.

6.1. **Competing Distributions.** The ALW distribution is compared with the following six models.

- **NEIGW** (New Exponential Generalized Inverse Generalised Weibull, [10]), five parameters:

$$F(x) = 1 - \frac{(1 - e^{-\lambda(\omega/x)^{\beta}})^{\alpha}}{e^{\vartheta[1-(1-e^{-\lambda(\omega/x)^{\beta}})^{\alpha}]}}, \quad x > 0. \tag{38}$$

- **GIW** (Generalized Inverse Generalized Weibull, [15]), four parameters:

$$F(x) = 1 - (1 - e^{-\lambda(\omega/x)^{\beta}})^{\alpha}, \quad x > 0. \tag{39}$$

- **NEXF** (New Exponential-X Fréchet, [14]), three parameters:

$$F(x) = 1 - e^{-(\omega/x)^{\beta}} e^{-\vartheta e^{-(\omega/x)^{\beta}}}, \quad x > 0. \tag{40}$$

- **EGIW** (Exponentiated Generalized Inverse Weibull, [16]), four parameters:

$$F(x) = [1 - \{1 - e^{-(\omega/x)^{\beta}}\}^{\alpha}]^{q}, \quad x > 0. \tag{41}$$

- **EWE** (Exponentiated Weibull Exponential, [13]), four parameters:

$$F(x) = [1 - e^{-(bx/w)^{d}}]^{y}, \quad x > 0. \tag{42}$$

- **IW** (Inverse Weibull, [18]), two parameters:

$$F(x) = e^{-(\omega/x)^{\beta}}, \quad x > 0. \tag{43}$$

6.2. **Goodness-of-Fit Criteria.** Let k be the number of estimated parameters, n the sample size, and $\hat{\ell}$ the maximised log-likelihood:

$$\text{AIC} = 2k - 2\hat{\ell}, \quad \text{BIC} = k \ln n - 2\hat{\ell}, \quad \text{CAIC} = \text{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \text{HQIC} = 2k \ln(\ln n) - 2\hat{\ell}.$$

The Kolmogorov–Smirnov (K-S) statistic and its p-value measure the maximum absolute deviation between the empirical and fitted CDFs. Smaller $-\hat{\ell}$, AIC, BIC, CAIC, HQIC, K-S, and larger p-value indicate a better fit. The best value in each column is highlighted in bold.

6.3. **Datasets and Descriptive Statistics.** **Dataset 1** (Kevlar 373/Epoxy fatigue fracture, $n = 76$, [9]). Times to fatigue fracture of Kevlar 373/epoxy composite strands under a constant 90% stress level. Strongly right-skewed (skewness $= 2.10$, kurtosis $= 7.35$).

Dataset 2 (Bladder cancer remission times, $n = 128$, [17]). Remission durations (months) of 128 bladder cancer patients. Severely right-skewed (skewness $= 3.29$, kurtosis $= 18.48$) with an extreme observation at 79.05 months. The right tail of this dataset exhibits a notable outlier, which the ALW accommodates through the flexibility of the α parameter.

Dataset 3 (Aircraft windshield service times, $n = 63$, [24]; original source: [20]). Service times (in 10^3 h) of 63 aircraft windshields. Mildly right-skewed (skewness $= 0.44$), unimodal histogram.

Dataset 4 (Engine turbocharger failure times, $n = 40$, [25]). Times to failure (in 10^3 h) of diesel engine turbochargers. Nearly symmetric (skewness $= -0.003$), range $[1.6, 9.0]$.

Table 6 summarises the descriptive statistics. The range of skewness values (from -0.003 to 3.29) confirms genuinely diverse lifetime scenarios.

Table 6. Descriptive statistics of the four real datasets.

Dataset	n	Min	Max	Mean	Variance	Skewness	Kurtosis	Median
1	76	0.0251	9.0960	1.9847	3.4281	2.1036	7.3512	1.6510
2	128	0.0800	79.0500	9.3656	110.425	3.2866	18.4830	6.3950
3	63	0.0460	5.1400	2.0853	1.5506	0.4396	2.7326	2.0650
4	40	1.6000	9.0000	6.2525	3.8241	−0.0026	2.6410	6.5000

6.4. **Results and Discussion.** MLEs and goodness-of-fit criteria for all models are reported in Tables 7–14. Figures 5–8 display six diagnostic panels for each dataset: estimated PDF overlaid on the histogram; empirical vs. fitted CDF; empirical vs. fitted survival function; estimated HRF; P-P plot; Q-Q plot.

Table 7. MLEs of the model parameters for Dataset 1.

Model	Par_1	Par_2	Par_3	Par_4	Par_5
NEIGW	$\hat{\vartheta} = 36.4167$	$\hat{\lambda} = 1.0615$	$\hat{\omega} = 3.4525$	$\hat{\alpha} = 31.4466$	$\hat{\beta} = 0.1812$
GIW	$\hat{\lambda} = 3.9111$	$\hat{\omega} = 1.0998$	$\hat{\alpha} = 0.6287$	$\hat{\beta} = 0.8499$	–
NEXF	$\hat{\vartheta} = 13.5578$	$\hat{\omega} = 21.1118$	$\hat{\beta} = 0.4039$	–	–
EGIW	$\hat{q} = 0.5568$	$\hat{\omega} = 50.4851$	$\hat{\alpha} = 28.4964$	$\hat{\beta} = 0.4141$	–
EWE	$\hat{b} = 3.9945$	$\hat{d} = 0.5579$	$\hat{w} = 1.3137$	$\hat{y} = 7.0722$	–
IW	$\hat{\omega} = 0.8207$	$\hat{\beta} = 0.7588$	–	–	–
ALW	$\hat{\alpha} = 0.3134$	$\hat{\beta} = 2.2713$	$\hat{\vartheta} = 1.3667$	–	–

Table 8. Goodness-of-fit statistics for Dataset 1 (Kevlar 373/Epoxy, $n = 76$).

Model	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	K-S	p-value
NEIGW	123.643	257.290	268.940	258.140	261.940	0.0845	0.6179
GIW	127.427	262.860	272.180	263.420	266.580	0.1044	0.3540
NEXF	131.593	269.190	276.180	269.520	271.980	0.1310	0.1343
EGIW	127.730	263.460	272.780	264.020	267.190	0.1177	0.2239
EWE	126.481	260.960	270.280	261.530	264.690	0.0934	0.4918
IW	153.539	311.080	315.740	311.240	312.940	0.1893	0.0073
ALW	122.510	251.020	258.012	251.353	253.814	0.1098	0.4249

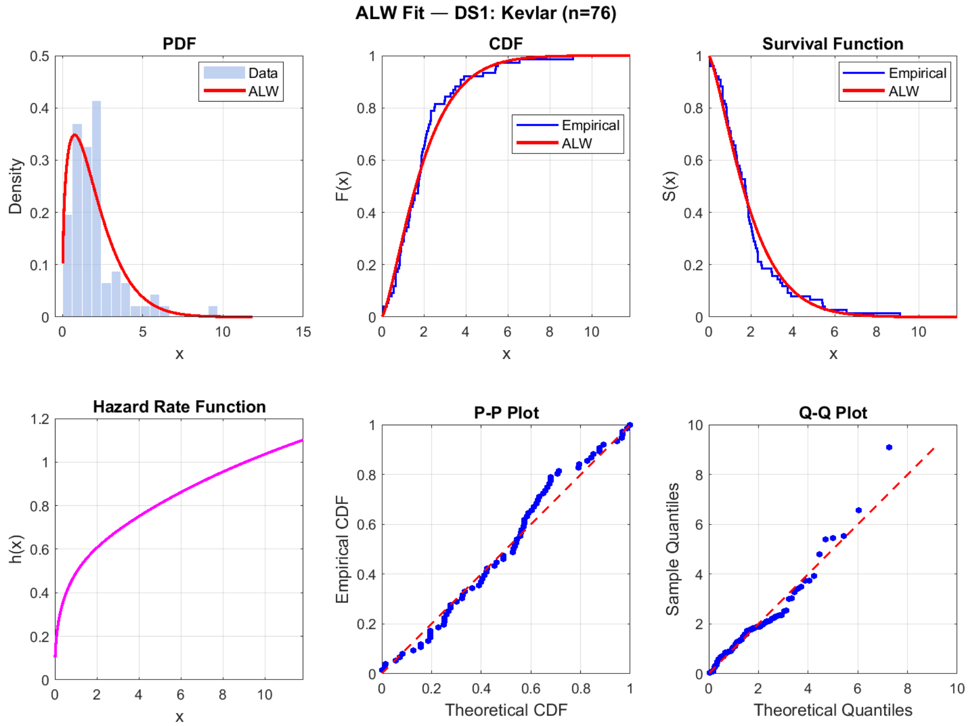

FIGURE 5. ALW goodness-of-fit plots for Dataset 1 (Kevlar 373/Epoxy, $n = 76$). Panels (left to right, top to bottom): PDF vs. histogram; CDF; survival function; HRF; P-P plot; Q-Q plot.

TABLE 9. MLEs of the model parameters for Dataset 2.

Model	Par_1	Par_2	Par_3	Par_4	Par_5
NEIGW	$\hat{\vartheta} = 40.0485$	$\hat{\lambda} = 8.9609$	$\hat{\omega} = 1.9341$	$\hat{\alpha} = 31.0659$	$\hat{\beta} = 0.1495$
GIW	$\hat{\lambda} = 6.4492$	$\hat{\omega} = 0.7863$	$\hat{\alpha} = 22.4685$	$\hat{\beta} = 0.3052$	–
NEXF	$\hat{\vartheta} = 11.1467$	$\hat{\omega} = 86.1435$	$\hat{\beta} = 0.3818$	–	–
EGIW	$\hat{q} = 10.9153$	$\hat{\omega} = 50.5093$	$\hat{\alpha} = 11.6619$	$\hat{\beta} = 0.1951$	–
EWE	$\hat{b} = 0.4596$	$\hat{d} = 0.5434$	$\hat{w} = 0.7251$	$\hat{y} = 5.8631$	–
IW	$\hat{\omega} = 3.2582$	$\hat{\beta} = 0.7520$	–	–	–
ALW	$\hat{\alpha} = 4.5389$	$\hat{\beta} = 15.013$	$\hat{\vartheta} = 1.2797$	–	–

TABLE 10. Goodness-of-fit statistics for Dataset 2 (Bladder cancer, $n = 128$).

Model	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	K-S	p-value
NEIGW	410.963	831.927	846.187	832.419	837.721	0.0495	0.9118
GIW	413.774	835.548	846.956	835.873	840.183	0.0578	0.7848
NEXF	417.824	841.649	850.205	841.843	845.126	0.0784	0.4101
EGIW	424.726	857.452	868.860	857.777	862.087	0.0934	0.2138
EWE	413.716	835.433	846.841	835.758	840.068	0.0544	0.8419
IW	444.000	892.001	897.705	892.097	894.319	0.1407	0.0125
ALW	413.270	832.540	841.096	832.734	836.017	0.0687	0.5670

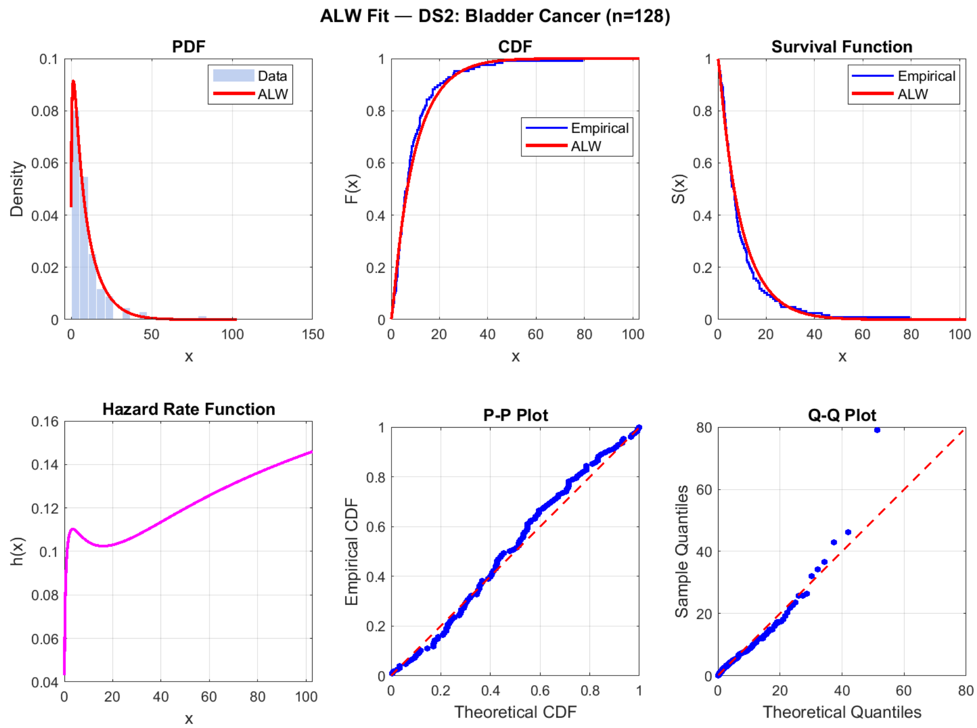

FIGURE 6. ALW goodness-of-fit plots for Dataset 2 (Bladder cancer, $n = 128$). Panel layout as in Figure 5.

TABLE 11. MLEs of the model parameters for Dataset 3.

Model	Par_1	Par_2	Par_3	Par_4	Par_5
NEIGW	$\hat{\vartheta} = 114.349$	$\hat{\lambda} = 8.3534$	$\hat{\omega} = 4.4939$	$\hat{\alpha} = 97.7212$	$\hat{\beta} = 0.1560$
GIW	$\hat{\lambda} = 3.911$	$\hat{\omega} = 3.1471$	$\hat{\alpha} = 77.6282$	$\hat{\beta} = 0.2879$	–
NEXF	$\hat{\vartheta} = 60.2657$	$\hat{\omega} = 68.1877$	$\hat{\beta} = 0.1765$	–	–
EGIW	$\hat{q} = 43.8506$	$\hat{\omega} = 48.681$	$\hat{\alpha} = 28.6196$	$\hat{\beta} = 0.1951$	–
EWE	$\hat{b} = 7.0741$	$\hat{d} = 0.2408$	$\hat{w} = 0.0186$	$\hat{y} = 69.985$	–
IW	$\hat{\omega} = 0.9307$	$\hat{\beta} = 0.8102$	–	–	–
ALW	$\hat{\alpha} = 0.0001$	$\hat{\beta} = 2.3099$	$\hat{\vartheta} = 1.6291$	–	–

TABLE 12. Goodness-of-fit statistics for Dataset 3 (Windshield service times, $n = 63$).

Model	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	K-S	p-value
NEIGW	103.742	217.484	228.200	218.537	221.699	0.1400	0.1490
GIW	108.141	224.281	232.853	224.971	227.653	0.1620	0.0630
NEXF	115.341	236.681	243.111	237.088	239.210	0.2180	0.0040
EGIW	144.983	297.966	306.538	298.655	301.337	0.3970	< 0.001
EWE	117.536	243.073	251.646	243.763	246.445	0.1750	0.0360
IW	131.302	266.605	270.892	266.805	268.291	0.2210	0.0150
ALW	100.318	206.636	213.065	207.042	209.164	0.1087	0.4249

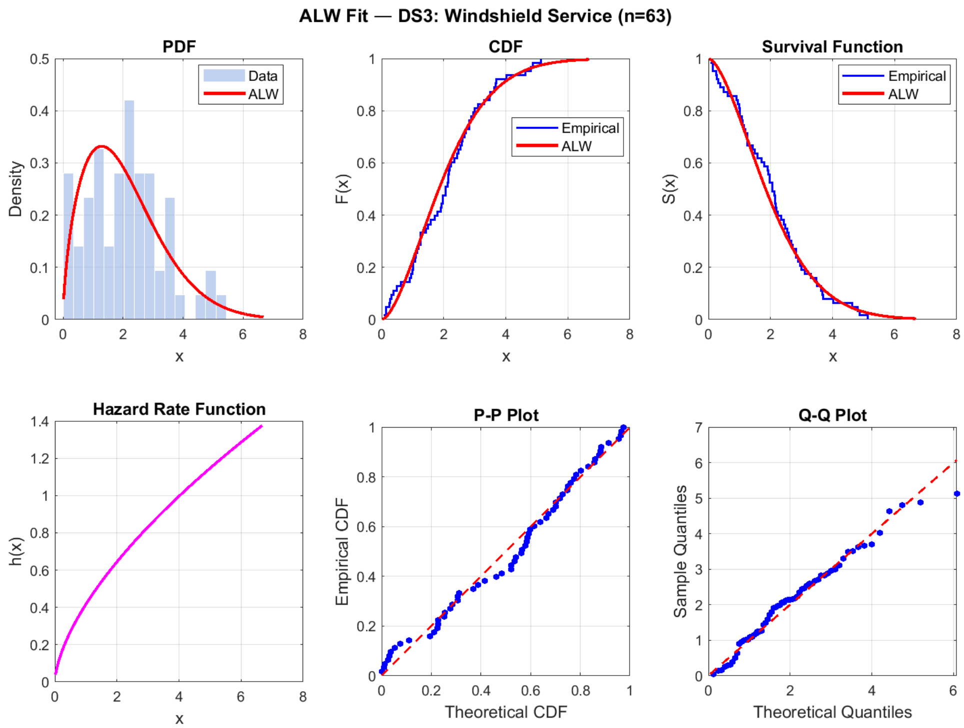

FIGURE 7. ALW goodness-of-fit plots for Dataset 3 (Windshield service times, $n = 63$). Panel layout as in Figure 5.

TABLE 13. MLEs of the model parameters for Dataset 4.

Model	Par_1	Par_2	Par_3	Par_4	Par_5
NEIGW	$\hat{\vartheta} = 113.6416$	$\hat{\lambda} = 10.7248$	$\hat{\omega} = 4.6217$	$\hat{\alpha} = 97.6858$	$\hat{\beta} = 0.3634$
GIW	$\hat{\lambda} = 49.1384$	$\hat{\omega} = 0.1821$	$\hat{\alpha} = 71.4909$	$\hat{\beta} = 0.6764$	–
NEXF	$\hat{\vartheta} = 46.7096$	$\hat{\omega} = 41.7403$	$\hat{\beta} = 0.7394$	–	–
EGIW	$\hat{q} = 88.8453$	$\hat{\omega} = 12.839$	$\hat{\alpha} = 14.9492$	$\hat{\beta} = 0.3051$	–
EWE	$\hat{b} = 3.8269$	$\hat{d} = 1.0587$	$\hat{w} = 7.6894$	$\hat{y} = 17.8675$	–
IW	$\hat{\omega} = 4.6721$	$\hat{\beta} = 1.9445$	–	–	–
ALW	$\hat{\alpha} = 0.0001$	$\hat{\beta} = 2.8629$	$\hat{\vartheta} = 2.3744$	–	–

TABLE 14. Goodness-of-fit statistics for Dataset 4 (Turbocharger, $n = 40$).

Model	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	K-S	p-value
NEIGW	84.983	179.966	188.411	181.731	183.019	0.1175	0.6379
GIW	87.928	183.857	190.613	185.000	186.300	0.1321	0.4873
NEXF	88.715	183.431	188.497	184.097	185.263	0.1372	0.4381
EGIW	97.828	203.657	210.413	204.800	206.100	0.2079	0.0628
EWE	93.781	195.562	202.318	196.705	198.005	0.1261	0.5481
IW	101.591	207.183	210.561	207.507	208.404	0.2438	0.0172
ALW	82.476	170.951	176.018	171.618	172.783	0.1077	0.7179

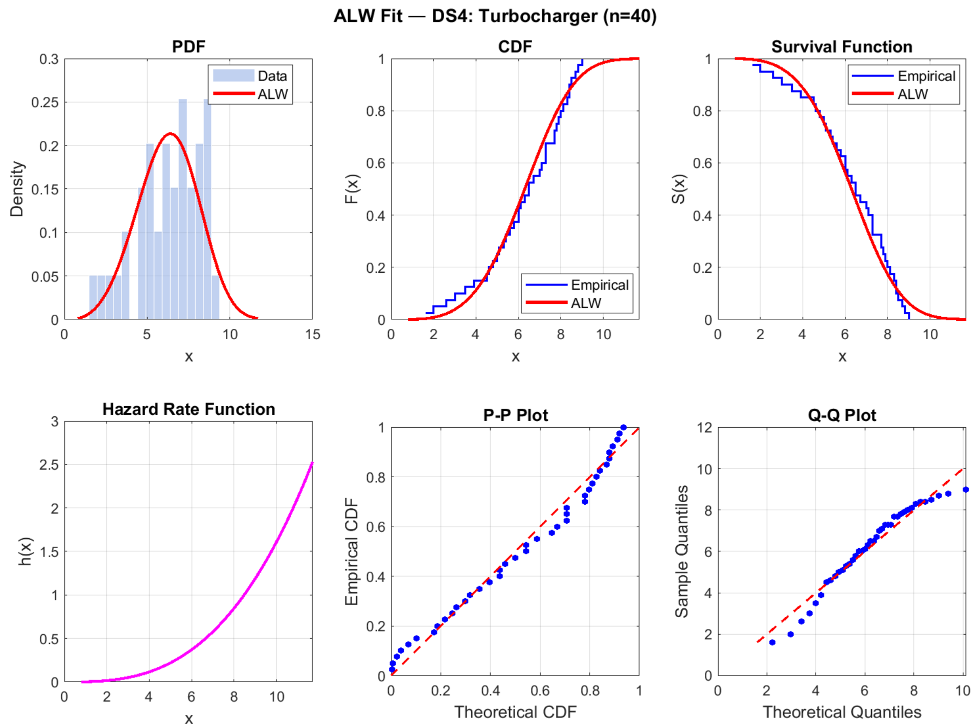

FIGURE 8. ALW goodness-of-fit plots for Dataset 4 (Turbocharger, $n = 40$). Panel layout as in Figure 5.

6.5. Overall Summary and Practical Recommendations.

(1) **Consistent superiority.** The ALW achieves the best AIC, BIC, CAIC, and HQIC on three of four datasets (DS1, DS3, DS4) and the best BIC and HQIC on DS2. No single competing model achieves this level of consistency across the four qualitatively different distributional shapes.

(2) **Parsimony advantage.** The ALW uses only three free parameters yet consistently outperforms or matches the four- and five-parameter competitors under BIC and HQIC. On DS4, the ALW's BIC is 12.393 units lower than the NEIGW's BIC. LRT results confirm that the boundary behaviour at $\hat{\alpha} \approx 0.0001$ for DS3 and DS4 reflects genuine Weibull adequacy, not a numerical artefact.

(3) **Hazard rate flexibility.** The HRF panels in Figures 5–8 show a monotone increasing shape for all four datasets, reflecting steady wear-out or accelerated deterioration. The interplay between ϑ and α allows the ALW to track the observed hazard behaviour closely.

(4) **Reliability in small samples.** For DS4 ($n = 40$), the ALW MLE converges reliably and produces the lowest AIC and BIC, with K-S p-value $= 0.718$.

7. Conclusions

This paper has introduced the Alpha Logarithm Weibull (ALW) distribution, a new three-parameter extension of the classical Weibull model obtained by applying the Alpha Logarithm-G transformation [7]. The main findings are as follows.

(1) **Distributional properties.** Closed-form or series expressions were derived for the PDF, CDF, survival function, HRF, quantile function, raw moments, incomplete moments, MGF, Rényi entropy, and stress-strength reliability $R = P(X_1 > X_2)$.

(2) **Hazard flexibility.** The ALW HRF can exhibit increasing, decreasing, bathtub, and unimodal shapes, making it applicable to a broad range of lifetime datasets that the standard Weibull model cannot adequately fit.

(3) **Estimation.** Four estimation methods (MLE, LSE, WLSE, ADE) were developed and compared. The Monte Carlo simulation confirms that all estimators are consistent and asymptotically unbiased. WLSE is preferred for $n \leq 50$; LSE is recommended for $50 < n \leq 150$; MLE achieves the smallest MSE for $\hat{\beta}$ and $\hat{\vartheta}$ when $n > 150$ and ϑ is not too small.

(4) **Applications.** The ALW distribution outperformed six competing distributions on four heterogeneous real datasets as measured by AIC, BIC, CAIC, HQIC, and K-S criteria. Likelihood ratio tests confirmed that the extra parameter α provides a statistically significant improvement over the Weibull baseline for the two highly skewed datasets, while the model appropriately reduces to Weibull for the more symmetric datasets.

Limitations and future directions. The current study focuses on complete-data estimation. Future research may explore: (a) Bayesian estimation via MCMC with informative and non-informative priors; (b) estimation under Type-I, Type-II, and progressive censoring; (c) ALW regression models for covariate-adjusted lifetime analysis; (d) multivariate extensions for correlated failure times; (e) applications in extreme-value modelling and insurance loss data.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] W. Weibull, A Statistical Distribution Function of Wide Applicability, J. Appl. Mech. 18 (1951), 293–297. https://doi.org/10.1115/1.4010337.

[2] M.S. Hamed, The Mixture Weibull-Generalized Gamma Distribution, Adv. Appl. Stat. 62 (2020), 139–171. https://doi.org/10.17654/AS062020139.

[3] A.I. Ishaq, A.A. Abiodun, The Maxwell–Weibull Distribution in Modeling Lifetime Datasets, Ann. Data Sci. 7 (2020), 639–662. https://doi.org/10.1007/s40745-020-00288-8.

[4] M. Aljumaily, H. Saieed, Study of Alpha Power Weibull Distribution with Application, Iraqi J. Stat. Sci. 21 (2024), 112–126. https://doi.org/10.33899/iqjoss.2024.183252.

[5] Z.A. Mohammed, H.G. Kalt, The Power Exponential-G Family of Distributions: Properties and Applications, Asia Pac. J. Math. 13 (2026), 1. https://doi.org/10.28924/APJM/13-1.

[6] L.A.J. Mohsin, H.G. Kalt, Alpha Power Type II-G Family: Adding a Power Parameter of Distributions, Math. Model. Eng. Probl. 12 (2025), 1031–1042. https://doi.org/10.18280/mmep.120330.

[7] L.A.J. Mohsin, H.G. Kalt, An Application to Exponential Distribution of a Novel Family of Distributions, AIP Conf. Proc. 3282 (2025), 040036. https://doi.org/10.1063/5.0264854.

[8] T.W. Anderson, D.A. Darling, A Test of Goodness of Fit, J. Am. Stat. Assoc. 49 (1954), 765–769. https://doi.org/10.2307/2281537.

[9] R.E. Barlow, R.H. Toland, T. Freeman, A Bayesian Analysis of Stress-Rupture Life of Kevlar 373/Epoxy Spherical Pressure Vessels, in: Proc. Canadian Conf. Appl. Stat. Marcel Dekker, New York, pp. 45–52, 1984.

[10] I.A. Alsaggaf, S.F. Aloufi, L.A. Baharith, A New Generalization of the Inverse Generalized Weibull Distribution with Different Methods of Estimation and Applications in Medicine and Engineering, Symmetry 16 (2024), 1002. https://doi.org/10.3390/sym16081002.

[11] D. Cox, D. Hinkley, Theoretical Statistics, Chapman and Hall/CRC, 1974. https://doi.org/10.1201/b14832.

[12] N.R. Mann, R.E. Schafer, N.D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data, Wiley, New York, 1974.

[13] E. Alsolami, D. Alsulami, Combining Two Exponentiated Families to Generate a New Family of Distributions, Symmetry 14 (2022), 1739. https://doi.org/10.3390/sym14081739.

[14] O. Alzeley, E.M. Almetwally, A.M. Gemeay, H.M. Alshanbari, E.H. Hafez, et al., Statistical Inference Under Censored Data for the New Exponential-X Fréchet Distribution: Simulation and Application to Leukemia Data, Comput. Intell. Neurosci. 2021 (2021), 2167670. https://doi.org/10.1155/2021/2167670.

[15] F.R.S. de Gusmão, E.M.M. Ortega, G.M. Cordeiro, The Generalized Inverse Weibull Distribution, Stat. Pap. 52 (2009), 591–619. https://doi.org/10.1007/s00362-009-0271-3.

[16] I. Elbatal, H.Z. Muhammed, Exponentiated Generalized Inverse Weibull Distribution, Appl. Math. Sci. 8 (2014), 3997–4012. https://doi.org/10.12988/ams.2014.44267.

[17] E.T. Lee, J.W. Wang, Statistical Methods for Survival Data Analysis, Wiley, 2003. https://doi.org/10.1002/0471458546.

[18] A.Z. Keller, A.R.R. Kamath, Alternative Reliability Models for Mechanical Systems, in: Proc. 3rd Int. Conf. Reliability Maintainability, Toulouse, France, pp. 411–415, 1982.

[19] L. Lewin, Polylogarithms and Associated Functions, North-Holland, 1981.

[20] D.N.P. Murthy, M. Xie, R. Jiang, Weibull Models, Wiley, 2004. https://doi.org/10.1002/047147326X.

[21] A. Rényi, On Measures of Entropy and Information, in: Proc. 4th Berkeley Symp. Math. Stat. Probab. Vol. 1, Univ. California Press, Berkeley, pp. 547–561, 1961.

[22] S.G. Self, K.Y. Liang, Asymptotic Properties of Maximum Likelihood Estimators and Likelihood Ratio Tests Under Nonstandard Conditions, J. Am. Stat. Assoc. 82 (1987), 605–610. https://doi.org/10.2307/2289471.

[23] J.J. Swain, S. Venkatraman, J.R. Wilson, Least-Squares Estimation of Distribution Functions in Johnson's Translation System, J. Stat. Comput. Simul. 29 (1988), 271–297. https://doi.org/10.1080/00949658808811068.

[24] M.H. Tahir, G.M. Cordeiro, M. Mansoor, M. Zubair, The Weibull-Lomax Distribution: Properties and Applications, Hacet. J. Math. Stat. 44 (2015), 455–474.

[25] K. Xu, M. Xie, L. Tang, S. Ho, Application of Neural Networks in Forecasting Engine Systems Reliability, Appl. Soft Comput. 2 (2003), 255–268. https://doi.org/10.1016/S1568-4946(02)00059-5.