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MCSHANE-STIELTJES INTEGRAL FOR RIESZ-SPACE-VALUED FUNCTIONS ON TIME SCALES

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Abstract. We introduce a McShane-Stieltjes Δ-integral for X-valued functions on compact time-scale intervals, where X is a Riesz space and the integrator is a real-valued function of bounded variation. The definition is formulated by means of Δ-gauges and order control of tagged Stieltjes sums. The use of asymmetric gauges is essential on time scales, because right-scattered points must be handled by one-sided control. We establish uniqueness, a Cauchy criterion under order-Cauchy completeness, linearity, interval additivity, positivity and monotonicity for nondecreasing integrators, a variation domination estimate, a Lipschitz-type stability theorem for the integrand, a stability estimate with respect to the integrator, and a uniform order convergence theorem. We also show that the construction reduces to the classical McShane-Stieltjes integral on $\mathbb{R}$ and to a finite Stieltjes-weighted sum on $\mathbb{Z}$. The paper is deliberately restricted to results that are proved within the present framework.

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1. Introduction

Time-scale calculus, initiated by Hilger, provides a framework in which continuous and discrete analysis can be treated simultaneously [9]. Within this setting, the available integration theories include delta and nabla integrals, more general Henstock-Kurzweil type integrals on time scales, and Riemann-Stieltjes integration on time scales [3,7,12,15]. On the other hand, vector-lattice-valued integration has long been studied in ordered spaces, where order convergence replaces norm convergence and allows one to work naturally with lattice-valued quantities [1,5,6,11,14,16].

Classical treatments of integration and unified integration provide additional background, while further time-scale and vector-lattice references indicate related directions [2,4,8,10,13].

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The purpose of this paper is to formulate and analyze a McShane-Stieltjes Δ-integral for functions

$$f : [a,b]_{\mathbb{T}} \to X,$$

where $\mathbb{T}$ is a time scale, X is a Riesz space, and the integrator

$$\alpha : [a,b]_{\mathbb{T}} \to \mathbb{R}$$

has bounded variation. The construction is gauge-theoretic and order-based. The guiding idea is simple: a tagged Stieltjes sum should converge in order, uniformly over all sufficiently fine tagged partitions. What requires care is the geometry of "sufficiently fine" on a time scale. A symmetric gauge is not adequate for this purpose, because right-scattered points are intrinsically one-sided. We therefore work with Δ-gauges of the type used in the time-scale Henstock-Kurzweil literature [12].

The results proved here are intentionally conservative. We develop the basic theory that can be justified directly from the definition: uniqueness, a Cauchy criterion under an explicit completeness hypothesis, linearity, additivity over adjacent subintervals, order properties for nondecreasing integrators, and two stability estimates based on total variation. These estimates yield a clean uniform order convergence theorem. We also record the reduction to the classical case on $\mathbb{R}$ and to the discrete Stieltjes sum on $\mathbb{Z}$.

We do not claim a general dominated convergence theorem, a Beppo-Levi theorem, or an integration-by-parts formula in the present paper. Each of these requires additional hypotheses and a more delicate argument, especially on general time scales. Removing such overstatements is mathematically preferable to asserting results whose proofs are incomplete.

2. Preliminaries

2.1. **Time scales and Δ-gauges.**

A time scale is a nonempty closed set $\mathbb{T} \subseteq \mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator is

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

with the convention $\sigma(t) = t$ if t is right-dense. The graininess is

$$\mu(t) := \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$.

For $a < b$ in $\mathbb{T}$, we write

$$[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}.$$

Following the Δ-gauge convention used for Henstock-Kurzweil integration on time scales [12], a Δ-gauge on $[a,b]_{\mathbb{T}}$ is a pair $\delta = (\delta_L, \delta_R)$ such that $\delta_L(t) > 0$ whenever $a < t \le b$, $\delta_R(t) > 0$ whenever $a \le t < b$, $\delta_L(a) \ge 0$, $\delta_R(b) \ge 0$, and $\delta_R(t) \ge \mu(t)$ whenever $a \le t < b$.

A tagged partition of $[a, b]_{\mathbb{T}}$ is a finite family

$$P = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^{n}$$

such that

$$a = t_0 < t_1 < \cdots < t_n = b, \qquad \xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}.$$

The partition is δ-fine if

$$\xi_i - \delta_L(\xi_i) \le t_{i-1} < t_i \le \xi_i + \delta_R(\xi_i), \qquad i = 1, \ldots, n.$$

The condition $\delta_R(t) \ge \mu(t)$ guarantees that the atomic cell $[t, \sigma(t)]_{\mathbb{T}}$ is admissible at a right-scattered point. This is the correct one-sided replacement for the symmetric gauge condition familiar on $\mathbb{R}$.

We shall use the following standard existence fact.

Lemma 2.1. *Every Δ-gauge on $[a, b]_{\mathbb{T}}$ admits a δ-fine tagged partition.*

Proof. This is the time-scale analogue of Cousin's lemma; see Peterson and Thompson [12, Lemma 1.9]. □

2.2. **Riesz spaces and order convergence.** A Riesz space (or vector lattice) X is a real vector space equipped with a lattice order $\le$ such that addition and scalar multiplication preserve order in the usual way. For $x \in X$, write

$$|x| := x \vee (-x),$$

and denote the positive cone of X by Pos(X).

A sequence (x_n) in X is said to converge in order to $x \in X$, written

$$x_n \xrightarrow{o} x,$$

if there exists a sequence (u_n) of positive elements of X with $u_n \downarrow 0$ such that

$$|x_n - x| \le u_n \qquad \text{for all } n.$$

A sequence (x_n) is order-Cauchy if there exists a sequence (u_n) of positive elements of X with $u_n \downarrow 0$ such that

$$|x_m - x_n| \le u_n \qquad \text{whenever } m \ge n.$$

A Riesz space is called order-Cauchy complete if every order-Cauchy sequence converges in order in X.

2.3. **Bounded variation.** Let $\alpha : [a,b]_{\mathbb{T}} \to \mathbb{R}$. For a partition

$$\Pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a,b]_{\mathbb{T}}$, put

$$V(\alpha, \Pi) := \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|.$$

The total variation of α on $[a,b]_{\mathbb{T}}$ is

$$V_\alpha := V(\alpha, [a,b]_{\mathbb{T}}) := \sup_{\Pi} V(\alpha, \Pi).$$

We say that α is of bounded variation if $V_\alpha < \infty$.

If α is nondecreasing, then every increment

$$\Delta\alpha_i := \alpha(t_i) - \alpha(t_{i-1})$$

is nonnegative.

3. Definition and basic properties

3.1. **McShane-Stieltjes sums and integrability.** Let $f : [a,b]_{\mathbb{T}} \to X$ and let $\alpha : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be of bounded variation. For a tagged partition

$$P = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^{n},$$

define the McShane-Stieltjes sum

$$S(f, \alpha, P) := \sum_{i=1}^{n} f(\xi_i)(\alpha(t_i) - \alpha(t_{i-1})).$$

Definition 3.1. We say that f is McShane-Stieltjes integrable with respect to α on $[a,b]_{\mathbb{T}}$ if there exist $I \in X$ and a sequence (u_n) of positive elements of X with $u_n \downarrow 0$ such that, for each n, there exists a Δ-gauge δ_n on $[a,b]_{\mathbb{T}}$ satisfying

$$|S(f, \alpha, P) - I| \leq u_n$$

for every δ_n-fine tagged partition P of $[a,b]_{\mathbb{T}}$.

In this case we write

$$I = \int_{[a,b]_{\mathbb{T}}} f \, d\alpha.$$

The definition is unchanged if one replaces the displayed sequence (u_n) by any other decreasing null sequence of positive elements of X. Indeed, given another sequence (v_n) with $v_n \downarrow 0$, one may pass to a subsequence of the original control sequence so that the corresponding estimates are bounded by v_n.

3.2. **Uniqueness and a Cauchy criterion.**

Theorem 3.2. *If I and J both satisfy Definition 3.1 for the same f and α, then $I = J$.*

Proof. Let (u_n) and (v_n) be control sequences for I and J, respectively. Put

$$w_n := u_n + v_n \downarrow 0.$$

Choose Δ-gauges δ_n^I and δ_n^J corresponding to u_n and v_n, and define

$$\delta_n := \left(\min\left(\delta_{L,n}^I, \delta_{L,n}^J\right), \min\left(\delta_{R,n}^I, \delta_{R,n}^J\right)\right).$$

Then every δ_n-fine partition P is both δ_n^I-fine and δ_n^J-fine. Hence

$$|I - J| \le |I - S(f, \alpha, P)| + |S(f, \alpha, P) - J| \le u_n + v_n = w_n.$$

Since $w_n \downarrow 0$, it follows that $I = J$. □

The next result gives a useful characterization provided the ambient Riesz space is order-Cauchy complete.

Theorem 3.3. *Assume that X is order-Cauchy complete. Then f is McShane-Stieltjes integrable with respect to α on $[a, b]_{\mathbb{T}}$ if and only if there exist a sequence (u_n) of positive elements of X with $u_n \downarrow 0$ and Δ-gauges δ_n such that*

(1) *$\delta_{n+1} \le \delta_n$ componentwise for every n, and*
(2) *for every n and every pair of δ_n-fine tagged partitions P, Q,*

$$|S(f, \alpha, P) - S(f, \alpha, Q)| \le u_n.$$

Proof. Assume first that f is integrable with integral I. Choose a control sequence (v_n) with $v_n \downarrow 0$ and corresponding gauges η_n. Define

$$u_n := 2v_n, \qquad \delta_n := \left(\min_{1\le k\le n} \eta_{L,k}, \min_{1\le k\le n} \eta_{R,k}\right).$$

Then $\delta_{n+1} \le \delta_n$ componentwise. If P, Q are δ_n-fine, they are in particular η_n-fine, so

$$|S(f, \alpha, P) - S(f, \alpha, Q)| \le |S(f, \alpha, P) - I| + |I - S(f, \alpha, Q)| \le 2v_n = u_n.$$

Conversely, assume the stated Cauchy condition. By Lemma 2.1, for each n there exists a δ_n-fine tagged partition P_n. Since $\delta_m \le \delta_n$ whenever $m \ge n$, the partition P_m is also δ_n-fine. Therefore,

$$|S(f, \alpha, P_m) - S(f, \alpha, P_n)| \le u_n \qquad (m \ge n).$$

Thus the sequence

$$x_n := S(f, \alpha, P_n)$$

is order-Cauchy. By order-Cauchy completeness, there exists $I \in X$ such that

$$x_n \xrightarrow{o} I.$$

Fix n and let P be any δ_n-fine tagged partition. Since P_m is also δ_n-fine for every $m \geq n$,

$$|S(f, \alpha, P) - x_m| \leq u_n \qquad (m \geq n).$$

Passing to the order limit as $m \to \infty$ gives

$$|S(f, \alpha, P) - I| \leq u_n.$$

Hence f is integrable with integral I. □

3.3. **Linearity and interval additivity.**

Theorem 3.4. *If f and g are integrable with respect to α on $[a,b]_{\mathbb{T}}$ and $c \in \mathbb{R}$, then $f + g$ and cf are integrable, and*

$$\int_{[a,b]_{\mathbb{T}}} (f + g)\, d\alpha = \int_{[a,b]_{\mathbb{T}}} f\, d\alpha + \int_{[a,b]_{\mathbb{T}}} g\, d\alpha,$$
$$\int_{[a,b]_{\mathbb{T}}} cf\, d\alpha = c \int_{[a,b]_{\mathbb{T}}} f\, d\alpha.$$

Proof. Let

$$I_f = \int_{[a,b]_{\mathbb{T}}} f\, d\alpha, \qquad I_g = \int_{[a,b]_{\mathbb{T}}} g\, d\alpha.$$

Choose a sequence (u_n) of positive elements of X with $u_n \downarrow 0$. By integrability, we may choose gauges δ_n^f, δ_n^g such that

$$|S(f, \alpha, P) - I_f| \leq u_n, \qquad |S(g, \alpha, P) - I_g| \leq u_n$$

for all δ_n^f-fine, respectively δ_n^g-fine, partitions P. Let δ_n be the componentwise minimum of these gauges. Then for every δ_n-fine partition P,

$$S(f + g, \alpha, P) = S(f, \alpha, P) + S(g, \alpha, P),$$

and hence

$$|S(f + g, \alpha, P) - (I_f + I_g)| \leq |S(f, \alpha, P) - I_f| + |S(g, \alpha, P) - I_g| \leq 2u_n.$$

Thus $f + g$ is integrable with integral $I_f + I_g$.

Similarly,

$$S(cf, \alpha, P) = c\, S(f, \alpha, P),$$

so

$$|S(cf, \alpha, P) - cI_f| = |c|\ |S(f, \alpha, P) - I_f| \leq |c| u_n,$$

which proves the second assertion. □

Theorem 3.5. *Let $a < c < b$ in $\mathbb{T}$. If f is integrable with respect to α on $[a,c]_{\mathbb{T}}$ and on $[c,b]_{\mathbb{T}}$, then f is integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_{[a,b]_{\mathbb{T}}} f\,d\alpha = \int_{[a,c]_{\mathbb{T}}} f\,d\alpha + \int_{[c,b]_{\mathbb{T}}} f\,d\alpha.$$

Proof. Let

$$I_1 = \int_{[a,c]_{\mathbb{T}}} f\,d\alpha, \qquad I_2 = \int_{[c,b]_{\mathbb{T}}} f\,d\alpha.$$

Choose (u_n) of positive elements of X with $u_n \downarrow 0$, and gauges δ_n^1 on $[a,c]_{\mathbb{T}}$ and δ_n^2 on $[c,b]_{\mathbb{T}}$ such that

$$|S(f,\alpha,P_1) - I_1| \le u_n, \qquad |S(f,\alpha,P_2) - I_2| \le u_n$$

for all corresponding fine partitions. Define a Δ-gauge on $[a,b]_{\mathbb{T}}$ by starting with the gauges on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, but imposing one additional restriction: whenever the tag ξ lies to the left of c, choose the right component to be strictly smaller than c - ξ; whenever ξ lies to the right of c, choose the left component to be strictly smaller than ξ - c; and at ξ = c take the componentwise minimum of the endpoint values. Then a fine cell can cross c only if its tag is exactly c. Indeed, if $\xi <$ c, the fineness condition forces the right endpoint to remain below c, while if $\xi >$ c it forces the left endpoint to remain above c. Consequently every fine partition of $[a,b]_{\mathbb{T}}$ either decomposes directly into fine tagged partitions of $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, or else contains exactly one cell straddling c and tagged at c. In the latter case we split that cell at c and keep the same tag c on both pieces; the Stieltjes sum is unchanged. Therefore,

$$S(f,\alpha,P) = S(f,\alpha,P_1) + S(f,\alpha,P_2),$$

and

$$|S(f,\alpha,P) - (I_1 + I_2)| \le |S(f,\alpha,P_1) - I_1| + |S(f,\alpha,P_2) - I_2| \le 2u_n.$$

Hence f is integrable on $[a,b]_{\mathbb{T}}$ with integral $I_1 + I_2$. □

3.4. **Positivity and monotonicity.**

Theorem 3.6. *Assume that α is nondecreasing. If $f(t) \ge 0$ for every $t \in [a,b]_{\mathbb{T}}$ and f is integrable with respect to α, then*

$$\int_{[a,b]_{\mathbb{T}}} f\,d\alpha \ge 0.$$

Proof. For every tagged partition

$$P = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n,$$

we have $\Delta\alpha_i \ge 0$ for all i, hence

$$S(f,\alpha,P) = \sum_{i=1}^n f(\xi_i)\,\Delta\alpha_i \ge 0.$$

Let $I = \int_{[a,b]_{\mathbb{T}}} f\, d\alpha$. By Definition 3.1 there exists a sequence (u_n) with $u_n \downarrow 0$ such that for sufficiently fine partitions P,

$$|S(f, \alpha, P) - I| \leq u_n.$$

Since $S(f, \alpha, P) \geq 0$, we obtain

$$I \geq -u_n$$

for every n, and therefore $I \geq 0$. □

Theorem 3.7. *Assume that α is nondecreasing. If f and g are integrable and*

$$f(t) \leq g(t) \qquad \text{for all } t \in [a,b]_{\mathbb{T}},$$

then

$$\int_{[a,b]_{\mathbb{T}}} f\, d\alpha \leq \int_{[a,b]_{\mathbb{T}}} g\, d\alpha.$$

Proof. The function $g - f$ is integrable by Theorem 3.4 and satisfies

$$g(t) - f(t) \geq 0$$

for all t. By Theorem 3.6,

$$0 \leq \int_{[a,b]_{\mathbb{T}}} (g - f)\, d\alpha = \int_{[a,b]_{\mathbb{T}}} g\, d\alpha - \int_{[a,b]_{\mathbb{T}}} f\, d\alpha.$$

This is the desired inequality. □

4. Variation estimates and convergence

4.0.1. *4.1. Variation domination.*

Theorem 4.1. *Let α be of bounded variation on $[a,b]_{\mathbb{T}}$ and let u be a positive element of X. If*

$$|f(t)| \leq u \qquad \text{for all } t \in [a,b]_{\mathbb{T}},$$

then for every tagged partition P of $[a,b]_{\mathbb{T}}$,

$$|S(f, \alpha, P)| \leq V_\alpha\, u.$$

Proof. Write

$$P = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n.$$

Using the lattice inequality

$$\left| \sum_{i=1}^n x_i \right| \leq \sum_{i=1}^n |x_i|,$$

we obtain

$$|S(f, \alpha, P)| \leq \sum_{i=1}^n |f(\xi_i)|\, |\alpha(t_i) - \alpha(t_{i-1})| \leq \sum_{i=1}^n u\, |\alpha(t_i) - \alpha(t_{i-1})|.$$

Because the coefficients are real and nonnegative,

$$\sum_{i=1}^{n} u\,|\alpha(t_i)-\alpha(t_{i-1})| = \left(\sum_{i=1}^{n} |\alpha(t_i)-\alpha(t_{i-1})|\right) u \le V_\alpha\, u.$$

Hence

$$|S(f,\alpha,P)| \le V_\alpha\, u.$$

□

The next consequence is the correct perturbation estimate for the integrand.

Theorem 4.2. *Let α be of bounded variation. Suppose f and g are both integrable with respect to α on $[a,b]_{\mathbb{T}}$, and assume that*

$$|f(t)-g(t)| \le u \qquad \text{for all } t \in [a,b]_{\mathbb{T}},$$

for some positive element u of X. Then

$$\left| \int_{[a,b]_{\mathbb{T}}} f\,d\alpha - \int_{[a,b]_{\mathbb{T}}} g\,d\alpha \right| \le V_\alpha\, u.$$

Proof. The difference $h := f-g$ is integrable by Theorem 3.4, and

$$|h(t)| \le u$$

for all t. By Theorem 4.1,

$$|S(h,\alpha,P)| \le V_\alpha\, u$$

for every tagged partition P. Let

$$I_h = \int_{[a,b]_{\mathbb{T}}} h\,d\alpha.$$

Choose a sequence (w_n) of positive elements of X with $w_n \downarrow 0$ and gauges δ_n such that

$$|S(h,\alpha,P) - I_h| \le w_n$$

for every δ_n-fine partition P. Then

$$|I_h| \le |I_h - S(h,\alpha,P)| + |S(h,\alpha,P)| \le w_n + V_\alpha\, u.$$

Letting $n \to \infty$ yields

$$|I_h| \le V_\alpha\, u.$$

Since

$$I_h = \int_{[a,b]_{\mathbb{T}}} f\,d\alpha - \int_{[a,b]_{\mathbb{T}}} g\,d\alpha,$$

the claim follows. □

4.1. **Stability with respect to the integrator.**

Theorem 4.3. *Let $\alpha, \beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be of bounded variation. Assume that f is integrable with respect to both α and β, and that*

$$|f(t)| \le u \qquad \text{for all } t \in [a,b]_{\mathbb{T}},$$

for some positive element u of X. Then

$$\left| \int_{[a,b]_{\mathbb{T}}} f\, d\alpha - \int_{[a,b]_{\mathbb{T}}} f\, d\beta \right| \le V\left(\alpha - \beta, [a,b]_{\mathbb{T}}\right) u.$$

Proof. Let

$$I_\alpha := \int_{[a,b]_{\mathbb{T}}} f\, d\alpha, \qquad I_\beta := \int_{[a,b]_{\mathbb{T}}} f\, d\beta.$$

Choose a sequence (w_n) of positive elements of X with $w_n \downarrow 0$. Since f is integrable with respect to both α and β, there exist gauges δ_n^α and δ_n^β such that

$$|S(f,\alpha,P) - I_\alpha| \le w_n, \qquad |S(f,\beta,P) - I_\beta| \le w_n$$

for all corresponding fine partitions. Let δ_n be the componentwise minimum of these gauges. For every δ_n-fine partition P,

$$|I_\alpha - I_\beta| \le |I_\alpha - S(f,\alpha,P)| + |S(f,\alpha,P) - S(f,\beta,P)| + |S(f,\beta,P) - I_\beta|.$$

Hence

$$|I_\alpha - I_\beta| \le 2w_n + |S(f, \alpha - \beta, P)|.$$

Applying Theorem 4.1 to the integrator $\alpha - \beta$, we get

$$|S(f, \alpha - \beta, P)| \le V\left(\alpha - \beta, [a,b]_{\mathbb{T}}\right) u.$$

Therefore,

$$|I_\alpha - I_\beta| \le 2w_n + V\left(\alpha - \beta, [a,b]_{\mathbb{T}}\right) u.$$

Letting $n \to \infty$ gives the result. □

4.2. **Uniform order convergence.**

Theorem 4.4. *Assume that X is order-Cauchy complete and that α is of bounded variation on $[a,b]_{\mathbb{T}}$. Let (f_n) be a sequence of functions integrable with respect to α. Suppose there exists a sequence (u_n) of positive elements of X with $u_n \downarrow 0$ such that*

$$|f_n(t) - f(t)| \le u_n \qquad \text{for all } t \in [a,b]_{\mathbb{T}}, \text{ for all } n.$$

Then f is integrable with respect to α and

$$\int_{[a,b]_{\mathbb{T}}} f_n\, d\alpha \xrightarrow{o} \int_{[a,b]_{\mathbb{T}}} f\, d\alpha.$$

Proof. Put

$$I_n := \int_{[a,b]_{\mathbb{T}}} f_n \, d\alpha.$$

For $m \geq n$, we have

$$|f_m(t) - f_n(t)| \leq |f_m(t) - f(t)| + |f(t) - f_n(t)| \leq u_m + u_n \leq 2u_n.$$

By Theorem 4.2,

$$|I_m - I_n| \leq 2V_\alpha \, u_n \qquad (m \geq n).$$

Thus (I_n) is order-Cauchy. Since X is order-Cauchy complete, there exists $I \in X$ such that

$$I_n \xrightarrow{o} I.$$

Moreover, the preceding estimate implies

$$|I - I_n| \leq 2V_\alpha \, u_n \qquad \text{for all } n,$$

by passing to the order limit in m.

Fix n. Since f_n is integrable, there exists a Δ-gauge δ_n such that

$$|S\left(f_n, \alpha, P\right) - I_n| \leq u_n$$

for every δ_n-fine tagged partition P. For such a partition,

$$|S(f, \alpha, P) - I| \leq |S(f, \alpha, P) - S\left(f_n, \alpha, P\right)| + |S\left(f_n, \alpha, P\right) - I_n| + |I_n - I|\,.$$

The first term is bounded by $V_\alpha u_n$ by Theorem 4.1 applied to $f - f_n$, while the second and third terms are bounded by u_n and $2V_\alpha u_n$, respectively. Hence

$$|S(f, \alpha, P) - I| \leq \left(1 + 3V_\alpha\right) u_n.$$

Since $\left(1 + 3V_\alpha\right) u_n \downarrow 0$, Definition 3.1 shows that f is integrable with integral I. The order convergence of the integrals is precisely the statement

$$I_n \xrightarrow{o} I = \int_{[a,b]_{\mathbb{T}}} f \, d\alpha.$$

□

5. Classical cases and discrete reduction

5.1. **The continuous case.** When $\mathbb{T} = \mathbb{R}$, every point is right-dense and one may take symmetric gauges by setting $\delta_L = \delta_R$. In that case the present definition reduces to the usual order-type McShane-Stieltjes integral for X-valued functions on compact real intervals. Thus the time-scale formulation is genuinely an extension of the classical one.

5.2. **The discrete case on $\mathbb{Z}$.** The discrete reduction is especially transparent.

Proposition 5.1. *Let $\mathbb{T} = \mathbb{Z}$ and let $a < b$ be integers. If $f : [a,b]_{\mathbb{Z}} \to X$ and $\alpha : [a,b]_{\mathbb{Z}} \to \mathbb{R}$, then f is McShane-Stieltjes integrable with respect to α on $[a,b]_{\mathbb{Z}}$, and*

$$\int_{[a,b]_{\mathbb{Z}}} f\, d\alpha = \sum_{k=a}^{b-1} f(k)\left(\alpha(k+1) - \alpha(k)\right).$$

Proof. For $k \in [a, b-1]_{\mathbb{Z}}$, the graininess is $\mu(k) = 1$. Define a Δ-gauge $\delta = (\delta_L, \delta_R)$ by

$$\delta_R(k) = 1 \quad (a \le k \le b-1),$$

$$\delta_L(k) = \frac{1}{2} \quad (a+1 \le k \le b),$$

with $\delta_L(a) = 0$ and $\delta_R(b) = 0$.

Let

$$P = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{Z}})\}_{i=1}^{n}$$

be a δ-fine tagged partition. Because $\delta_R(k) = 1$, every admissible interval has length at most one to the right of the tag. Because $\delta_L(k) = 1/2$ for $k > a$, an interval of the form $[k, k+1]_{\mathbb{Z}}$ cannot be tagged at $k+1$, since the condition

$$\xi_i - \delta_L(\xi_i) \le t_{i-1} = k$$

would fail for $\xi_i = k+1$. Hence each cell is necessarily

$$[k, k+1]_{\mathbb{Z}}$$

tagged at k. Therefore the only δ-fine tagged partition of $[a,b]_{\mathbb{Z}}$ is

$$\{(k, [k, k+1]_{\mathbb{Z}})\}_{k=a}^{b-1},$$

and its Stieltjes sum is

$$S(f, \alpha, P) = \sum_{k=a}^{b-1} f(k)\left(\alpha(k+1) - \alpha(k)\right).$$

Since every δ-fine partition gives the same value, Definition 3.1 is satisfied with zero error, and the formula follows. □

The proposition shows explicitly how right-scattered points contribute atomic increments in the discrete case.

6. Conclusion

We have formulated a McShane-Stieltjes Δ-integral for Riesz-space-valued functions on compact time-scale intervals by combining Δ-gauges with order control of tagged Stieltjes sums. The asymmetric gauge structure is essential: it is what makes the definition compatible with right-scattered points and permits a correct discrete reduction.

Within this framework we proved the basic foundational results that are needed for a workable theory: uniqueness, a Cauchy criterion under order-Cauchy completeness, linearity, interval additivity, positivity, monotonicity, variation domination, stability under perturbation of the integrand, stability under perturbation of the integrator, and uniform order convergence. These are the claims that are fully supported by the present arguments.

Several natural extensions remain open for future work. Among them are dominated convergence and monotone convergence under hypotheses strong enough to make the proofs rigorous, integration by parts on general time scales, and the treatment of vector-lattice-valued integrators. Each of these deserves a separate analysis.

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