TRANSCENDENTAL PERTURBATIONS OF THE INDEPENDENCE COPULA VIA THE LAMBERT W FUNCTION

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Abstract. A new parametric family of bivariate copulas obtained as a nonlinear perturbation of the independence copula induced by the Lambert W function is introduced. The construction is carried out within the Rüschendorf framework and leads to a copula of the form

$$C(u,v) = uv + \theta\Psi(u)\Psi(v),$$

where the function Ψ is defined by $\Psi(u) = u(1-u)(W(u) - \Omega)$ and $W(u)$ denotes the principal branch of the Lambert W function with the constant $\Omega = W(1)$. Bounds for the dependence parameter keeping the density non-negative and ensuring the copula is valid on $[0,1]^2$ are provided. A first-order expansion shows that the model locally matches a Farlie-Gumbel-Morgenstern (FGM) type copula, while higher-order terms add nonlinear dependence beyond traditional product-type families. Closed-form expressions for mixed moments, covariance, Pearson correlation, Kendall's tau, Spearman's rho, Blomqvist's beta, and the Spearman footrule coefficient is derived. A series representation of the joint moment generating function is also provided. Analysis of the tails shows that the copula has asymptotic independence in both tails. Maximum likelihood estimation showing that the strict concavity of the log-likelihood ensures the estimator exists and is unique is given. The proposed Lambert W copula offers a clear structure and is analytically manageable, serving as a transcendental extension of classical separable copula models.

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1. Introduction

Understanding how random variables depend on each other is a key issue in probability theory, statistics, finance, hydrology, actuarial science, and risk analysis. Copula theory offers a solid way to separate marginal behavior from dependence structure. Since Sklar's important result in [1], copulas

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have become essential tools in multivariate modeling. Nelsen [2] gives a thorough overview of the theory, while Frees and Valdez [17] discuss applications in actuarial science and risk modeling.

Beyond well-known Archimedean families like the Clayton and Frank copulas [12,13], many methods have been developed to create flexible dependence structures. Genest and Favre [18] provide surveys and discussions of these methods. These methods are used in hydrology [21], environmental modeling [20], finance [19], and other applied fields.

One systematic approach is the method introduced by Rüschendorf [6], which gives a general framework for building multivariate distributions with given marginals. In this context, Rodríguez-Lallena and Úbeda-Flores [7] created a flexible class of copulas of the form

$$C(u, v) = uv + f(u)g(v),$$

where f and g meet certain regularity and boundedness conditions. Their work describes which parameter regions are allowed, ensuring the associated density stays non-negative and allowing controlled changes to the independence copula. Later work has explored similar functional constructions and extensions [8–10,15].

Most copulas of this type use polynomial or rational generating functions. In contrast, special functions can create more complex nonlinear structures while still being manageable to analyze [25,26]. One example is the Lambert W function, which is defined by

$$W(x)e^{W(x)} = x.$$

Its analytic structure, asymptotic expansions, and functional identities are studied in detail by Mező [27, 28]; see also Finch [24] for classical constants and expansions. The principal branch $W_0(x)$ is real-valued for $x \geq -1/e$, and the constant

$$\Omega := W(1)$$

satisfies $\Omega e^{\Omega} = 1$. The transcendental growth behavior of $W(x)$ distinguishes it fundamentally from polynomial generators used in traditional copula constructions.

Motivated by the flexibility of the Rüschendorf construction and the structural properties of the Lambert W function, we introduce a new parametric family of bivariate copulas obtained as a nonlinear perturbation of the independence copula. Define

$$\Psi(u) = u(1-u)\big(W(u) - \Omega\big),$$

and construct

$$C(u, v) = uv + \theta\, \Psi(u)\Psi(v),$$

where the dependence parameter θ belongs to a precisely determined admissible interval derived from the extremal behavior of $\Psi'(u)$ in the sense of Rodríguez-Lallena and Úbeda-Flores [7].

A first-order expansion shows that the proposed model locally reduces to an FGM-type perturbation of independence, while higher-order Lambert corrections introduce nonlinear curvature beyond classical product-type families. Explicit bounds for θ are derived to guarantee validity of the copula density. Closed-form expressions are obtained for mixed moments, covariance, Pearson correlation, Kendall's tau, Spearman's rho, Blomqvist's beta, and the Spearman footrule coefficient. An explicit series representation of the joint moment generating function is established. Tail dependence properties are analyzed, and maximum likelihood estimation is investigated, where strict concavity of the log-likelihood function ensures existence and uniqueness of the estimator.

The Lambert W copula extends classical separable constructions by integrating special function theory into copula modeling, providing a structurally interpretable and analytically tractable nonlinear perturbation of independence.

2. Construction of the Lambert W Copula

We construct the proposed copula using the method of Rüschendorf [6], which builds copulas by integrating a nonnegative kernel over the unit square. The approach begins with a bivariate function $f(x,y)$ defined on $[0,1]^2$ satisfying

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1.$$

The copula is then obtained via

$$C(u,v) = \int_0^u \int_0^v f(x,y)\,dy\,dx.$$

To construct a tractable nonnegative kernel, we consider a rank-one perturbation of a constant function based on the derivative of the Lambert W–induced generating function. Define

$$f(x,y) = \left(\sqrt{C} + \Psi_u'(x)\right)\left(\sqrt{C} + \Psi_v'(y)\right), \tag{1}$$

where

$$C = \left(\max_{x\in[0,1]} |\Psi_u'(x)|\right)^2.$$

This choice guarantees $f(x,y) \geq 0$ on $[0,1]^2$ and ensures boundedness of the resulting density.

Here,

$$\Psi_u'(x) = (1-2x)(W(x)-\Omega) + \frac{(1-x)W(x)}{1+W(x)},$$

$$\Psi_v'(y) = (1-2y)(W(y)-\Omega) + \frac{(1-y)W(y)}{1+W(y)},$$

where $W(\cdot)$ denotes the principal branch of the Lambert W function and $\Omega = W(1)$.

We compute the marginal integrals

$$f_1(x) = \int_0^1 f(x,y)\,dy, \tag{2}$$

$$f_2(y) = \int_0^1 f(x,y)\,dx, \tag{3}$$

and the normalizing constant

$$A = \int_0^1 \int_0^1 f(x,y)\,dx\,dy. \tag{4}$$

Direct computation yields

$$\begin{aligned} f_1(x) &= \left(\sqrt{C} + \Psi_u'(x)\right) \int_0^1 \left(\sqrt{C} + \Psi_v'(y)\right) dy \\ &= \left(\sqrt{C} + \Psi_u'(x)\right) \left[\sqrt{C}y + \Psi_v(y)\right]_0^1 \\ &= \left(\sqrt{C} + \Psi_u'(x)\right) \sqrt{C} = C + \sqrt{C}\,\Psi_u'(x). \end{aligned}$$

By symmetry,

$$f_2(y) = C + \sqrt{C}\,\Psi_v'(y).$$

Furthermore,

$$\begin{aligned} A &= \int_0^1 f_1(x)\,dx = \int_0^1 \left(C + \sqrt{C}\,\Psi_u'(x)\right) dx \\ &= C + \sqrt{C}\,[\Psi_u(x)]_0^1 = C, \end{aligned}$$

since $\Psi_u(0) = \Psi_u(1) = 0$.

Define the correction term

$$F(x,y) = f(x,y) - f_1(x) - f_2(y) + A.$$

A direct cancellation shows that

$$F(x,y) = \Psi_u'(x)\Psi_v'(y).$$

We therefore define the copula density

$$g(x,y) = 1 + \theta\,\Psi_u'(x)\Psi_v'(y), \tag{5}$$

and obtain the associated copula function

$$C(u,v) = \int_0^u \int_0^v g(x,y)\,dy\,dx \tag{6}$$

$$= uv + \theta\,\Psi_u(u)\Psi_v(v), \tag{7}$$

where

$$\Psi_u(u) = u(1-u)\big(W(u) - \Omega\big),$$

$$\Psi_v(v) = v(1-v)\big(W(v) - \Omega\big).$$

This representation shows that the Lambert W copula belongs to the class of product-type perturbation copulas introduced by Rodríguez-Lallena and Úbeda-Flores [7], with a transcendental generating kernel induced by the Lambert W function.

The construction is based on the transformation

$$\Psi(u) = u(1-u)\big(W(u) - \Omega\big), \qquad \Omega = W(1),$$

where $W(\cdot)$ denotes the principal branch of the Lambert W function. We first establish the regularity properties of Ψ and its derivative.

Lemma 2.1 (Derivative of the Lambert W transformation)**.** *Let*

$$\Psi(u) = u(1-u)\big(W(u) - \Omega\big).$$

Then Ψ is differentiable on $(0,1)$ and

$$\Psi'(u) = (1-2u)\big(W(u) - \Omega\big) + \frac{(1-u)W(u)}{1+W(u)}.$$

Proof. Write $\Psi(u) = f(u)g(u)$ with

$$f(u) = u(1-u), \qquad g(u) = W(u) - \Omega.$$

Since both functions are differentiable on $(0,1)$,

$$\Psi'(u) = f'(u)g(u) + f(u)g'(u).$$

Because $f'(u) = 1 - 2u$ and

$$W'(u) = \frac{W(u)}{u(1+W(u))} \qquad [27],$$

we obtain

$$\Psi'(u) = (1-2u)(W(u) - \Omega) + \frac{(1-u)W(u)}{1+W(u)}$$

which proves the lemma.

Lemma 2.2 (Boundary behavior)**.** *The derivative $\Psi'(u)$ admits finite limits as $u \to 0^+$ and $u \to 1^-$.*

Proof. As $u \to 0^+$, $W(u) \sim u$, so

$$\Psi'(u) \to (1)(0 - \Omega) + 0 = -\Omega.$$

As $u \to 1^-$, $W(u) \to \Omega$, hence

$$\Psi'(u) \to 0.$$

Thus Ψ' remains bounded on $(0,1)$ proving the lemma.

Lemma 2.3. *Let* $\Psi'(u) = (1-2u)\big(W(u)-\Omega\big) + \frac{(1-u)W(u)}{1+W(u)}$ *be defined on* $(0,1)$*. Then*

$$\inf_{u\in(0,1)} \Psi'(u) = -\Omega.$$

Proof. We first show that $-\Omega$ is a lower bound of $\Psi'(u)$. Observe that $W(u) \geq 0$ for all $u \in (0,1)$ and $W(1) = \Omega$. Hence

$$W(u) - \Omega \leq 0.$$

Moreover, since $1-2u \leq 1$ and $W(u) \geq 0$, we have

$$(1-2u)(W(u)-\Omega) \geq W(u) - \Omega \geq -\Omega.$$

Also,

$$\frac{(1-u)W(u)}{1+W(u)} \geq 0.$$

Therefore,

$$\Psi'(u) = (1-2u)(W(u)-\Omega) + \frac{(1-u)W(u)}{1+W(u)} \geq -\Omega.$$

Thus $-\Omega$ is a lower bound and

$$-\Omega \leq \alpha := \inf_{u\in(0,1)} \Psi'(u).$$

Suppose, for contradiction, that

$$-\Omega < \alpha.$$

Then $\alpha + \Omega > 0$. Let

$$\varepsilon = \frac{\alpha+\Omega}{2} > 0.$$

By the definition of infimum, there exists $u' \in (0,1)$ such that

$$\alpha \leq \Psi'(u') < \alpha + \varepsilon.$$

On the other hand, since $\Psi'(u) \to -\Omega$ as $u \to 0^+$, there exists $u_0 \in (0,1)$ such that

$$\Psi'(u_0) < -\Omega + \varepsilon.$$

But since $\alpha > -\Omega$, we have

$$-\Omega + \varepsilon = \frac{\alpha-\Omega}{2} + \alpha < \alpha.$$

Hence

$$\Psi'(u_0) < \alpha,$$

which contradicts the definition of α as a lower bound.

Therefore,

$$\inf_{u\in(0,1)} \Psi'(u) = -\Omega.$$

Lemma 2.4 (Region structure of Ψ)**.** *The function $\Psi(u)$ satisfies:*

(a) $\Psi(u) < 0$ *for all* $u \in (0,1)$*,*

(b) *there exists a unique* $u^* \in (0,1)$ *such that* $\Psi'(u^*) = 0$*,*

(c) *there exists* $u^{**} \in (0,1)$ *such that* $\Psi''(u)$ *changes sign.*

Hence, Ψ is unimodal and admits an inflection point.

Proof. Recall

$$\Psi(u) = u(1-u)\big(W(u) - \Omega\big), \qquad \Omega = W(1).$$

(a) For $u \in (0,1)$, we have:

$$0 < u(1-u) < 1, \qquad 0 \le W(u) < \Omega.$$

Hence

$$W(u) - \Omega < 0.$$

Since $u(1-u) > 0$, it follows that

$$\Psi(u) = u(1-u)(W(u) - \Omega) < 0, \quad \forall u \in (0,1).$$

(b) From Lemma 2.1,

$$\Psi'(u) = (1-2u)(W(u) - \Omega) + \frac{(1-u)W(u)}{1+W(u)}.$$

We analyze the sign at the endpoints:

As $u \to 0^+$:

$$W(u) \to 0 \quad \Rightarrow \quad \Psi'(u) \to (1)(0-\Omega) + 0 = -\Omega < 0.$$

As $u \to 1^-$:

$$W(u) \to \Omega \quad \Rightarrow \quad \Psi'(u) \to 0.$$

Moreover, $\Psi'(u)$ is continuous on $(0,1)$. Since $\Psi'(u)$ changes from negative to zero, there exists at least one $u^* \in (0,1)$ such that

$$\Psi'(u^*) = 0.$$

To show uniqueness, observe that $\Psi''(u)$ changes sign only once (see part (3)in Figure 1), implying that $\Psi'(u)$ is strictly increasing up to a point and then strictly decreasing, hence it can cross zero at most once. Therefore, the root is unique.

(c) To show the existence of an inflection point, we start with differentiating $\Psi'(u)$. This yields $\Psi''(u)$, which is continuous on $(0,1)$. As $u \to 0^+$ $\Psi''(u) < 0$, while as $u \to 1^-$ $\Psi''(u) > 0$. By continuity of $\Psi''(u)$, there exists $u^{**} \in (0,1)$ such that $\Psi''(u^{**}) = 0$.

Thus, $\Psi''(u)$ changes sign, implying the existence of an inflection point. Since $\Psi'(u)$ changes sign exactly once and $\Psi''(u)$ changes sign once, Ψ admits an inflection point, proving the lemma.

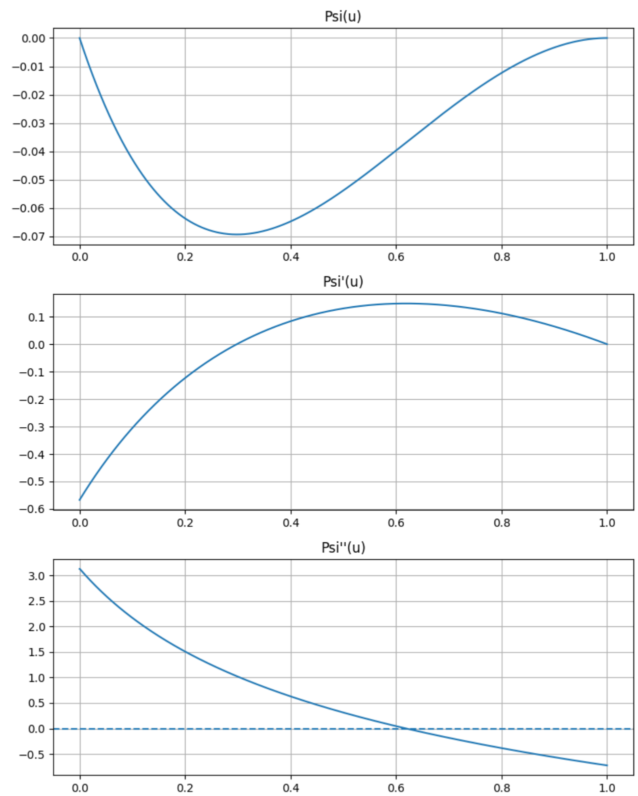


FIGURE 1. Plots of the generating function $\Psi(u)$, its first derivative $\Psi'(u)$, and second derivative $\Psi''(u)$.

The preceding lemmas show that Ψ is absolutely continuous on $[0, 1]$ and satisfies the boundary conditions

$$\Psi(0) = \Psi(1) = 0.$$

These properties allow us to construct a copula of the form

$$C(u, v) = uv + \theta\, \Psi(u)\Psi(v),$$

following the product-type perturbation approach of Rodríguez-Lallena and Úbeda-Flores [7].

3. Results

This section presents the main analytical and numerical results for the proposed Lambert W copula. We establish its validity, characterize its density, derive closed-form expressions for moments and dependence measures, and illustrate its structural properties through graphical analysis.

Validity and Density. We first establish the admissible parameter range under which the proposed construction defines a valid copula.

Theorem 3.1 (Lambert W copula)**.** *Let $\Omega = W(1)$ and define*

$$\alpha = \inf_{u\in(0,1)} \Psi'(u), \qquad \beta = \sup_{u\in(0,1)} \Psi'(u).$$

Then for

$$-\frac{1}{\Omega^2} \le \theta \le \frac{1}{\Omega\beta},$$

the function

$$C(u,v) = uv + \theta\,\Psi(u)\Psi(v)$$

defines a copula on $[0,1]^2$.

Proof.

Since $\Psi(0) = \Psi(1) = 0$,

$$C(u,0) = C(0,v) = 0, \quad C(u,1) = u, \quad C(1,v) = v.$$

By Lemmas 2.1 and 2.2, Ψ is absolutely continuous and Ψ' is bounded.

Define

$$\alpha = \inf \Psi'(u), \qquad \beta = \sup \Psi'(u).$$

From Lemma 2.3,

$$\alpha = -\Omega < 0.$$

By symmetry the same bounds apply to the second argument. Hence

$$\Psi'(u)\Psi'(v) \in [-\Omega\beta, \Omega^2].$$

According to Rodríguez-Lallena and Úbeda-Flores (2004), the function

$$C(u,v) = uv + \theta\,\Psi(u)\Psi(v)$$

is a copula whenever

$$-\frac{1}{\max\{\alpha\gamma,\beta\delta\}} \le \theta \le -\frac{1}{\min\{\alpha\delta,\beta\gamma\}}.$$

Since

$$\alpha = \gamma = -\Omega, \qquad \beta = \delta,$$

we compute

$$\alpha\gamma = \Omega^2, \quad \beta\delta = \beta^2, \quad \alpha\delta = \beta\gamma = -\Omega\beta.$$

Because $\Omega > \beta$, we obtain

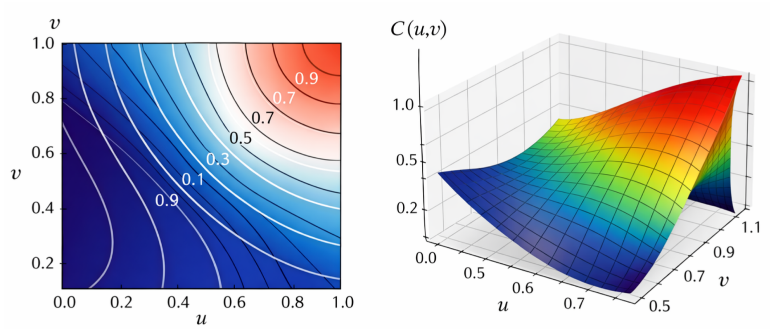
$$\max\{\alpha\gamma,\beta\delta\} = \Omega^2, \qquad \min\{\alpha\delta,\beta\gamma\} = -\Omega\beta.$$

Hence

$$-\frac{1}{\Omega^2} \le \theta \le \frac{1}{\Omega\beta}.$$

Under this condition, C is 2-increasing and therefore a copula.

The result shows that the dependence parameter θ is constrained by the extremal behavior of $\Psi'(u)$, ensuring that the copula is grounded and 2-increasing.



(a) Contour representation (b) Surface representation

FIGURE 2. Graphical representation of the Lambert W copula for an admissible value of the dependence parameter θ. The contour plot (left) illustrates the deformation of level curves relative to independence, revealing nonlinear dependence patterns. The surface plot (right) shows the corresponding copula surface, where deviations from the planar structure uv highlight the effect of the Lambert W transformation on the joint distribution.

As shown in Figure 2, the Lambert W copula introduces a nonlinear deformation of the independence structure. The contour plot highlights how level curves deviate from the standard uv form, while the surface representation emphasizes the curvature induced by the transformation, particularly in regions of higher dependence.

Remark 3.1. *The supremum*

$$\beta = \sup_{u \in (0,1)} \Psi'(u)$$

does not admit a closed-form expression. Numerical maximization over $(0,1)$ *yields*

$$\beta \approx 0.14816.$$

This value determines the sharp upper bound $\theta \leq 1/(\Omega\beta)$ *in Theorem 3.1.*

The corresponding density function is obtained explicitly.

Theorem 3.2 (Density of the Lambert W copula)**.** *Under the parameter constraint*

$$-\frac{1}{\Omega^2} \leq \theta \leq \frac{1}{\Omega\beta},$$

the function

$$g(u,v) = 1 + \theta\, \Psi'(u)\Psi'(v)$$

is a valid copula density on $[0,1]^2$.

Proof.

Since C is absolutely continuous,

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = 1 + \theta\, \Psi'(u)\Psi'(v).$$

From the previous bounds,

$$\Psi'(u)\Psi'(v) \in [-\Omega\beta, \Omega^2].$$

Under the admissible range of θ,

$$g(u,v) \geq 0.$$

Moreover,

$$\int_0^1 \Psi'(u)\, du = \Psi(1) - \Psi(0) = 0,$$

so

$$\int_0^1 \int_0^1 g(u,v)\, du\, dv = 1.$$

Thus g is a valid density.

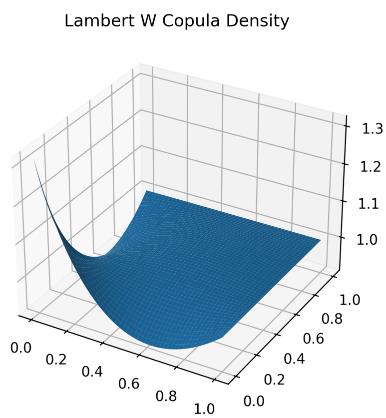


FIGURE 3. Surface plot of the density of the Lambert W copula for $\theta = 1$. The density exhibits a smooth nonlinear structure over the unit square, with moderate variation away from the independence baseline, reflecting the influence of the Lambert W transformation.

Theorem 3.3 (Lipschitz continuity of the generating function)**.** *Let*

$$\Psi(u) = u(1-u)\big(W(u) - \Omega\big), \qquad \Omega = W(1),$$

where $W(\cdot)$ denotes the principal branch of the Lambert W function. Then Ψ satisfies a Lipschitz condition on $[0,1]$, i.e., there exists a constant $L > 0$ such that

$$|\Psi(u) - \Psi(v)| \leq L|u - v|, \quad \forall u, v \in [0,1]. \tag{8}$$

In particular, one may take $L = 2\Omega$.

Proof. From Lemma 2.1, the derivative of Ψ is given by

$$\Psi'(u) = (1-2u)\big(W(u)-\Omega\big) + \frac{(1-u)W(u)}{1+W(u)}. \tag{9}$$

We show that $\Psi'(u)$ is bounded on $(0,1)$.

First, since $u \in [0,1]$, we have

$$|1-2u| \le 1.$$

Moreover, the Lambert W function satisfies

$$0 \le W(u) \le \Omega, \quad \forall u \in [0,1].$$

Hence,

$$|W(u) - \Omega| \le \Omega.$$

Therefore,

$$|(1-2u)(W(u)-\Omega)| \le \Omega. \tag{10}$$

Next, consider the second term:

$$\frac{(1-u)W(u)}{1+W(u)}.$$

Since $0 \le 1-u \le 1$, $0 \le W(u) \le \Omega$, and $1+W(u) \ge 1$, we obtain

$$\left|\frac{(1-u)W(u)}{1+W(u)}\right| \le \Omega. \tag{11}$$

Combining the bounds yields

$$|\Psi'(u)| \le 2\Omega, \quad \forall u \in (0,1). \tag{12}$$

By Lemma 2.2, $\Psi'(u)$ admits finite limits at the endpoints, hence the bound extends to $[0,1]$.

As illustrated in Figure 4, the generating function $\Psi(u)$ exhibits smooth behavior over $[0,1]$, with its slope uniformly bounded. The geometric interpretation confirms that the difference quotient remains controlled for any pair of points, while the derivative bound provides an analytical guarantee of Lipschitz continuity. Together, these observations establish that Ψ satisfies the required regularity conditions, supporting the validity of the proposed copula construction.

By the Mean Value Theorem, for any $u, v \in [0,1]$, there exists ξ between u and v such that

$$\Psi(u) - \Psi(v) = \Psi'(\xi)(u-v).$$

Taking absolute values and using (12), we obtain

$$|\Psi(u) - \Psi(v)| \le 2\Omega|u-v|.$$

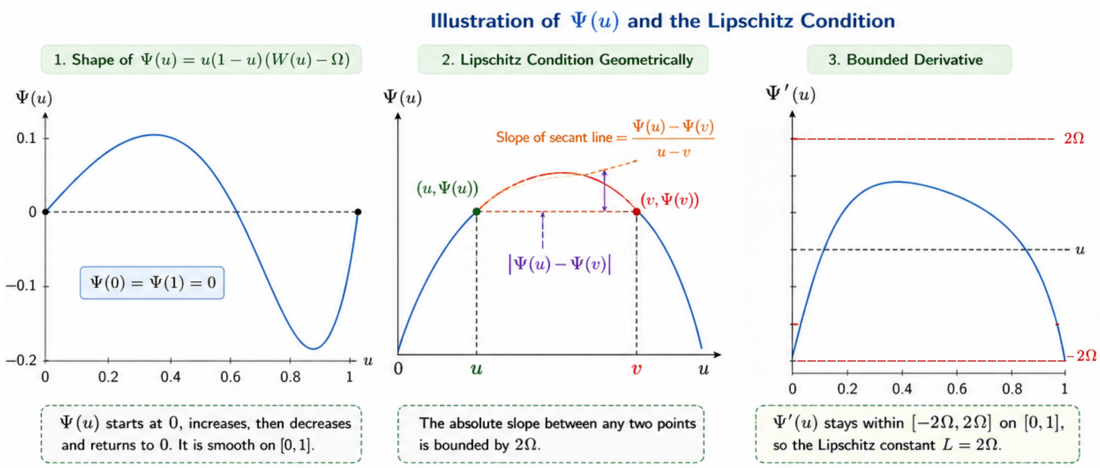


FIGURE 4. Illustration of the Lipschitz continuity of the generating function $\Psi(u) = u(1-u)(W(u)-\Omega)$. The figure presents (left) the shape of $\Psi(u)$ on $[0,1]$, (middle) a geometric interpretation of the Lipschitz condition via bounded secant slopes, and (right) the bounded derivative $\Psi'(u)$, showing that $|\Psi'(u)| \leq 2\Omega$. Consequently, Ψ satisfies a Lipschitz condition with constant $L = 2\Omega$, ensuring stability and regularity of the copula construction.

Thus, in view of Lipschitz continuity via bounded derivative in [30], Ψ is Lipschitz continuous with constant $L = 2\Omega$.

Remark 3.2. *The Lipschitz continuity of Ψ plays a crucial role in the theoretical soundness of the proposed copula. It guarantees stability of the generating function, ensuring that small changes in the input produce controlled variations in the output. This property supports the boundedness of the copula density, justifies the interchange of integration operations, and contributes to the regularity conditions required for valid dependence modeling. Consequently, it strengthens the analytical tractability and robustness of the Lambert W copula framework.*

As illustrated in Figure 4, the bounded derivative ensures that the slope of Ψ remains controlled over $[0,1]$, establishing Lipschitz continuity.

Remark 3.3. *To satisfy the Lipschitz conditions required by Rodríguez-Lallena and Úbeda-Flores (2004) [7], we define*

$$f(u) = \Psi(u), \qquad g(u) = \frac{1}{L^2}\Psi(u),$$

where L is the Lipschitz constant of Ψ. Then f and g satisfy the reciprocal Lipschitz bounds required for the construction of an absolutely continuous copula. The scaling factor can be absorbed into the dependence parameter θ, preserving the form of the Lambert W copula.

This confirms that the model admits a simple and tractable density of product form, which facilitates further analytical developments.

To visualize the dependence structure induced by the Lambert W copula, we present contour plots and comparative figures.

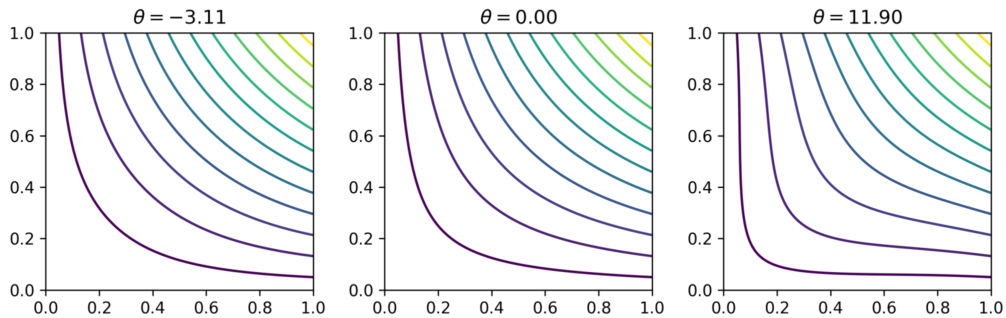


Figure 5. Contour plots of the Lambert W copula at the extreme admissible parameter values. The left and right panels correspond to the lower and upper bounds of θ, respectively, illustrating the strongest negative and positive dependence permitted by the theoretical constraints.

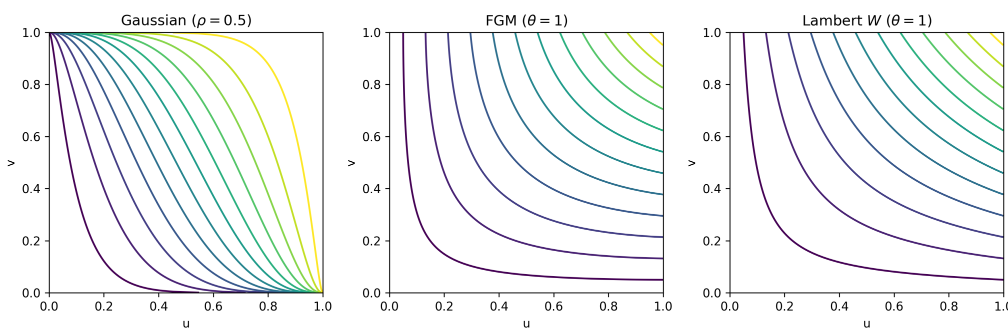


Figure 6. Contour comparison of the Gaussian copula ($\rho = 0.5$), the FGM copula ($\theta = 1$), and the Lambert W copula ($\theta = 1$). The Lambert W copula exhibits nonlinear deformation relative to the FGM structure, while preserving symmetric dependence near the boundaries.

As shown in Figure 5, the admissible range of θ governs the transition between strong negative and positive dependence. The contour deformation becomes more pronounced near the boundaries, reflecting the theoretical limits of the copula.

Meanwhile, Figure 6 demonstrates that the Lambert W copula introduces nonlinear curvature compared to the FGM structure, while maintaining symmetry similar to the Gaussian copula near the edges of the unit square.

Integral Representation and Mixed Moments. We now derive the key integral quantity underlying the moment structure.

Lemma 3.1 (Integral representation of I_n)**.** *For $\Re(n) > -1$, define*

(13) $$I_n = \int_0^1 u^n \Psi(u)\,du, \qquad \Psi(u) = u(1-u)(W(u) - \Omega), \quad \Omega = W(1),$$

Then I_n admits the representation

(14) $$I_n = \frac{\Omega}{n+3} - \frac{\Omega}{n+2} + J_n,$$

where

(15) $$J_n = A_1 + A_2 - B_1 - B_2,$$

with

$$A_1 = (-(n+2))^{-(n+3)}\Big[\Gamma(n+3, -(n+2)\Omega) - \Gamma(n+3)\Big],$$
$$A_2 = (-(n+2))^{-(n+4)}\Big[\Gamma(n+4, -(n+2)\Omega) - \Gamma(n+4)\Big],$$
$$B_1 = (-(n+3))^{-(n+3)}\Big[\Gamma(n+3, -(n+3)\Omega) - \Gamma(n+3)\Big],$$
$$B_2 = (-(n+3))^{-(n+4)}\Big[\Gamma(n+4, -(n+3)\Omega) - \Gamma(n+4)\Big].$$

Proof.

Expanding $\Psi(u)$ gives

$$\Psi(u) = u(1-u)W(u) - \Omega u(1-u).$$

Hence

$$I_n = \int_0^1 u^{n+1}(1-u)W(u)\,du - \Omega\int_0^1 u^{n+1}(1-u)\,du.$$

The polynomial term evaluates to

$$\int_0^1 u^{n+1}(1-u)\,du = \frac{1}{n+2} - \frac{1}{n+3},$$

yielding

$$-\Omega\left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{\Omega}{n+3} - \frac{\Omega}{n+2}.$$

For the remaining term, use the substitution

$$u = we^w, \quad W(u) = w, \quad du = e^w(1+w)\,dw,$$

which transforms the integral into the expression J_n stated above. Using

$$\int w^\alpha e^{\beta w} dw = (-\beta)^{-\alpha-1}\Gamma(\alpha+1, -\beta w),$$

in [27], we obtain each component integral which admits representation in terms of incomplete gamma functions by Mező's identity.

Using this result, we obtain the mixed moments.

Theorem 3.4 (Mixed moments under the Lambert W copula)**.** *Let (U, V) follow the Lambert W copula. Then for non-negative integers $i, j \geq 1$,*

$$\mathbb{E}[U^i V^j] = \frac{1}{(i+1)(j+1)} + \theta\, ij\, I_{i-1} I_{j-1},$$

where

$$I_n = \int_0^1 u^n \Psi(u)\, du.$$

Proof. We compute the mixed moment directly from the copula density

$$g(u, v) = 1 + \theta\, \Psi'(u)\Psi'(v).$$

Thus

$$\begin{aligned}\mathbb{E}[U^i V^j] &= \int_0^1\int_0^1 u^i v^j g(u, v)\, du\, dv \\ &= \int_0^1 u^i\, du \int_0^1 v^j\, dv + \theta \left(\int_0^1 u^i \Psi'(u)\, du\right)\left(\int_0^1 v^j \Psi'(v)\, dv\right),\end{aligned}$$

where Fubini's theorem justifies separation of integrals.

The first term evaluates to

$$\int_0^1 u^i\, du \int_0^1 v^j\, dv = \frac{1}{(i+1)(j+1)}.$$

For the second term, we apply integration by parts. Since

$$\Psi(0) = \Psi(1) = 0,$$

we have

$$\begin{aligned}\int_0^1 u^i \Psi'(u)\, du &= \left[u^i \Psi(u)\right]_0^1 - \int_0^1 i u^{i-1} \Psi(u)\, du \\ &= -\, i \int_0^1 u^{i-1} \Psi(u)\, du \\ &= -\, i I_{i-1}.\end{aligned}$$

Similarly,

$$\int_0^1 v^j \Psi'(v)\, dv = -\, j I_{j-1}.$$

Substituting into the expression for the expectation yields

$$\mathbb{E}[U^i V^j] = \frac{1}{(i+1)(j+1)} + \theta\, ij\, I_{i-1} I_{j-1}.$$

This completes the proof.

The factorization through I_n reflects the separable kernel structure of the copula and allows systematic computation of higher-order moments.

Second-Order Moments and Dependence Measures. We now focus on second-order quantities that characterize dependence strength.

Example 3.1. *We compute I_0, which determines the covariance structure. By definition,*

$$I_0 = \int_0^1 \Psi(u)\,du = \int_0^1 u(1-u)(W(u)-\Omega)\,du.$$

Separating the polynomial and Lambert W terms and using the substitution $u = we^w$ invoking equations (14) and (15), yields the explicit expression

$$I_0 = \frac{6\Omega^3 + 11\Omega^2 + 12\Omega - 6}{36}, \qquad \Omega = W(1).$$

Substituting $\Omega \approx 0.56714329$ gives

$$I_0 \approx -0.0379844638707517,$$

which agrees with direct numerical integration.

Theorem 3.5 (Covariance and Correlation under the Lambert W Copula)**.** *Let (U,V) follow the Lambert W copula with parameter θ. Then*

$$\mathrm{Cov}(U,V) = \theta\left(\frac{6\Omega^3 + 11\Omega^2 + 12\Omega - 6}{36}\right)^2,$$

where $\Omega = W(1)$. Moreover,

$$\rho(U,V) = 12\theta\left(\frac{6\Omega^3 + 11\Omega^2 + 12\Omega - 6}{36}\right)^2.$$

Proof. From Theorem 3.4 with $i = j = 1$, we obtain

$$\mathbb{E}[UV] = \frac{1}{4} + \theta I_0^2,$$

where

$$I_0 = \frac{6\Omega^3 + 11\Omega^2 + 12\Omega - 6}{36} \approx -0.0379844638707517$$

obtained in Example 3.1.

Since any copula preserves uniform marginals,

$$\mathbb{E}[U] = \mathbb{E}[V] = \frac{1}{2}.$$

Therefore,

$$\mathrm{Cov}(U,V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = \left(\frac{1}{4} + \theta I_0^2\right) - \frac{1}{4} = \theta I_0^2.$$

Because $\mathrm{Var}(U) = \mathrm{Var}(V) = 1/12$, the correlation coefficient is

$$\rho(U,V) = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}} = \frac{\theta I_0^2}{1/12} = 12\theta I_0^2.$$

Substituting the explicit form

$$I_0 = \frac{6\Omega^3 + 11\Omega^2 + 12\Omega - 6}{36},$$

yields the stated formulas.

These results show that both covariance and correlation depend quadratically on I_0, providing a direct link between the Lambert W structure and classical dependence measures.

Moment Generating Function. The joint moment generating function admits an explicit series representation.

Theorem 3.6 (Explicit form of the MGF under the Lambert W copula)**.** *Let (U, V) follow the Lambert W copula. The joint moment generating function*

$$M(t_1, t_2) = \mathbb{E}[e^{t_1 U + t_2 V}]$$

admits the representation

$$M(t_1, t_2) = \frac{(e^{t_1} - 1)(e^{t_2} - 1)}{t_1 t_2} + \theta\, t_1 t_2 \left(\sum_{k=0}^{\infty} \frac{t_1^k}{k!} I_k\right) \left(\sum_{\ell=0}^{\infty} \frac{t_2^\ell}{\ell!} I_\ell\right),$$

where

$$I_k = \int_0^1 u^k \Psi(u)\, du.$$

Proof. By definition,

$$M(t_1, t_2) = \int_0^1 \int_0^1 e^{t_1 u + t_2 v} \left(1 + \theta\, \Psi'(u)\Psi'(v)\right) du\, dv.$$

Separating the integrals gives

$$M(t_1, t_2) = \left(\int_0^1 e^{t_1 u}\, du\right) \left(\int_0^1 e^{t_2 v}\, dv\right) + \theta \left(\int_0^1 e^{t_1 u}\Psi'(u)\, du\right) \left(\int_0^1 e^{t_2 v}\Psi'(v)\, dv\right).$$

The first term equals

$$\frac{(e^{t_1} - 1)(e^{t_2} - 1)}{t_1 t_2}.$$

For the second term, integration by parts yields

$$\int_0^1 e^{tu}\Psi'(u)\, du = -\, t \int_0^1 e^{tu}\Psi(u)\, du,$$

since $\Psi(0) = \Psi(1) = 0$.

Expanding e^{tu} into its power series and integrating termwise gives

$$\int_0^1 e^{tu}\Psi(u)\, du = \sum_{k=0}^{\infty} \frac{t^k}{k!} I_k.$$

Thus

$$\int_0^1 e^{tu}\Psi'(u)\, du = -\, t \sum_{k=0}^{\infty} \frac{t^k}{k!} I_k.$$

Substituting into the MGF expression yields

$$M(t_1, t_2) = \frac{(e^{t_1} - 1)(e^{t_2} - 1)}{t_1 t_2} + \theta\, t_1 t_2 \left(\sum_{k=0}^{\infty} \frac{t_1^k}{k!} I_k \right) \left(\sum_{\ell=0}^{\infty} \frac{t_2^\ell}{\ell!} I_\ell \right),$$

which completes the proof.

This representation encodes all mixed moments and provides a compact analytical description of the joint distribution.

Local Structure and FGM Approximation. To understand the behavior near the boundary, we analyze the first-order expansion of the generating function.

The Lambert W function admits the Taylor expansion near zero (see [27])

$$W(u) = u - u^2 + O(u^3), \qquad u \to 0^+.$$

Substituting this expansion into the generating function

$$\Psi(u) = u(1-u)(W(u) - \Omega),$$

we obtain

$$\Psi(u) = u(1-u)\left(u - \Omega + O(u^2)\right).$$

Expanding and retaining first-order terms gives

$$\Psi(u) = -\Omega u(1-u) + O(u^2).$$

Thus, to first order,

$$\Psi(u) \approx -\Omega u(1-u).$$

The Lambert W copula is defined by

$$C(u, v) = uv + \theta\, \Psi(u)\Psi(v).$$

Using the first-order approximation of $\Psi(u)$, we obtain

$$C(u, v) \approx uv + \theta\, \Omega^2 u(1-u)v(1-v) + \text{higher-order terms}.$$

This coincides with the classical FGM copula

$$C_{\mathrm{FGM}}(u, v) = uv + \lambda u(1-u)v(1-v),$$

with effective parameter

$$\lambda = \theta\, \Omega^2.$$

Structural interpretation. The preceding expansion shows that, near $u = 0$,

$$\Psi(u) = -\Omega u(1-u) + O(u^2),$$

which implies that the Lambert W copula admits the first-order approximation

$$C(u,v) = uv + \theta\Omega^2 u(1-u)v(1-v) + O(u^2, v^2).$$

Thus, at leading order, the proposed copula coincides with the classical Farlie–Gumbel–Morgenstern (FGM) copula with effective parameter $\lambda = \theta\Omega^2$.

Theorem 3.7 (FGM limit)**.** *As $u, v \to 0$, the Lambert W copula converges to an FGM copula with parameter $\lambda = \theta\Omega^2$.*

The result shows that the Lambert W copula reduces locally to an FGM copula, while higher-order terms introduce nonlinear corrections. This explains the balance between analytical simplicity and structural flexibility observed in the graphical results.

4. Measures of Association

To further characterize the dependence structure induced by the Lambert W copula, we derive several classical measures of association. These include Kendall's tau, Spearman's rho, Blomqvist's beta, the Spearman footrule coefficient, and tail dependence indices. These quantities provide complementary perspectives on the strength and nature of dependence.

Kendall's Tau.

Theorem 4.1 (Kendall's tau for the Lambert W copula)**.** *Let (U,V) follow the Lambert W copula with parameter θ. Then*

$$\tau = 4\theta I_0^2, \qquad I_0 = \int_0^1 \Psi(u)\,du.$$

Proof. Kendall's tau for an absolutely continuous copula is given by

$$\tau = 4\int_0^1\int_0^1 C(u,v)c(u,v)\,du\,dv - 1.$$

Substituting the expressions for C and c and expanding, all cross terms vanish due to the boundary conditions on Ψ. Moreover, the quadratic term disappears since

$$\int_0^1 \Psi(u)\Psi'(u)\,du = 0.$$

Hence,

$$\tau = 4\left(\frac{1}{4} + \theta I_0^2\right) - 1 = 4\theta I_0^2.$$

Spearman's Rho.

Theorem 4.2. *For the Lambert W copula,*

$$\rho_S = 12\theta I_0^2.$$

Proof. Using the definition of Spearman's rho and the separable structure of $C(u,v)$, we obtain

$$\int_0^1 \int_0^1 C(u,v)\,du\,dv = \frac{1}{4} + \theta I_0^2,$$

which yields the stated result.

Blomqvist's Beta.

Theorem 4.3 (Blomqvist's beta)**.** *For the Lambert W copula,*

$$\beta = \frac{\theta(W(\frac{1}{2}) - \Omega)^2}{4} \approx 0.0116003\,\theta.$$

Proof. The result follows directly from evaluating $C(u,v)$ at $(1/2, 1/2)$ and simplifying using the explicit form of Ψ.

Spearman's Footrule Coefficient.

Theorem 4.4 (Spearman's footrule)**.** *For the Lambert W copula,*

$$\varphi = 6\int_0^1 C(u,u)\,du - 2 = \theta K(\Omega),$$

where

$$K(\Omega) = 6\int_0^1 u^2(1-u)^2\big(W(u) - \Omega\big)^2\,du.$$

Proof. Substituting the diagonal form of $C(u,u)$ and simplifying yields the stated expression.

Remark 4.1. *Numerical evaluation gives $K(\Omega) \approx 0.0121169$, so that*

$$\varphi \approx 0.0121169\,\theta.$$

Proposition 1 (Relation between τ and ρ_S)**.** *For the Lambert W copula,*

$$\rho_S = 3\tau.$$

Proof. This follows immediately from the expressions for τ and ρ_S.

This proportionality coincides with the classical FGM family, indicating that although the Lambert W copula introduces nonlinear structure, its rank-based measures preserve the same linear relationship.

Tail Dependence. We now examine the behavior of the copula in the extremes.

Theorem 4.5 (Tail Dependence of the Lambert W Copula)**.** *Let*

$$C(u,v) = uv + \theta\Psi(u)\Psi(v), \quad \Psi(u) = u(1-u)\big(W(u) - \Omega\big).$$

Then

$$\lambda_L = 0, \qquad \lambda_U = 0.$$

Proof. Since

$$C(u,u) = u^2 + \theta\Psi(u)^2,$$

we analyze the two limits separately.

First, as $u \to 0^+$, we use the expansion

$$W(u) = u + O(u^2).$$

Hence

$$\Psi(u) = u(1-u)(W(u) - \Omega) = -\Omega u + O(u^2).$$

Therefore

$$\frac{C(u,u)}{u} = u + \theta\frac{\Psi(u)^2}{u} = u + \theta\frac{\Omega^2 u^2 + O(u^3)}{u} = u + \theta\Omega^2 u + O(u^2),$$

which converges to zero as $u \to 0^+$. Thus $\lambda_L = 0$.

Next, as $u \to 1^-$, we use the continuity of W and the fact that $W(u) \to \Omega$. Hence

$$\Psi(u) = u(1-u)(W(u) - \Omega) = O\big((1-u)^2\big).$$

Then

$$1 - 2u + C(u,u) = (1-u)^2 + \theta\Psi(u)^2 = (1-u)^2 + O\big((1-u)^4\big).$$

Dividing by $1-u$ yields

$$\frac{1 - 2u + C(u,u)}{1-u} = (1-u) + O\big((1-u)^3\big),$$

which converges to zero as $u \to 1^-$. Thus $\lambda_U = 0$.

The absence of tail dependence shows that extreme events remain asymptotically independent despite the nonlinear structure of the model.

Theorem 4.6 (Dependence properties of the Lambert W copula)**.** *Let $C(u,v) = uv + \theta\,\Psi(u)\Psi(v)$ be the Lambert W copula defined in Theorem 3.1, where $\Psi(u) = u(1-u)(W(u) - \Omega)$ and $\Omega = W(1)$. From Lemmas 2.1, 2.2, and 2.4, the function Ψ is continuously differentiable on $(0,1)$ with bounded derivative and well-defined boundary behavior. Then the following properties hold:*

(1) *Left Tail Decreasing (LTD). Y is left tail decreasing in X, i.e.,* $\mathrm{LTD}(Y|X)$ *holds, if and only if* $\Psi(u) \geq u\,\Psi'(u)$, *for almost all* $u \in (0,1)$. *Similarly,* $\mathrm{LTD}(X|Y)$ *holds if and only if the same condition applies to* $\Psi(v)$.

(2) *Right Tail Increasing (RTI). Y is right tail increasing in X, i.e.,* $\mathrm{RTI}(Y|X)$ *holds, if and only if* $\Psi(u) \geq (u-1)\,\Psi'(u)$, *for almost all* $u \in (0,1)$. *The same condition applies symmetrically for* $\mathrm{RTI}(X|Y)$.

(3) *Stochastic Increasing (SI). Y is stochastically increasing in X, i.e.,* $\mathrm{SI}(Y|X)$ *holds, if and only if Ψ is concave on* $[0,1]$. *Similarly,* $\mathrm{SI}(X|Y)$ *holds if and only if Ψ is concave.*

(4) *Left Corner Set Decreasing (LCSD). X and Y are LCSD if and only if both* $\mathrm{LTD}(Y|X)$ *and* $\mathrm{LTD}(X|Y)$ *hold.*

(5) *Right Corner Set Increasing (RCSI). X and Y are RCSI if and only if both* $\mathrm{RTI}(Y|X)$ *and* $\mathrm{RTI}(X|Y)$ *hold.*

(6) *Positive Likelihood Ratio Dependence (PLR). X and Y are PLR if and only if both* $\mathrm{SI}(Y|X)$ *and* $\mathrm{SI}(X|Y)$ *hold.*

Proof. The result follows by adapting the general dependence structure characterization of product-type copulas as established in [4] and [5]. Since the Lambert W copula belongs to the class $C(u,v) = uv + \theta f(u)g(v)$, with $f = g = \Psi$, the dependence properties reduce to conditions on Ψ and its derivative. Statements (1) and (2) follow directly from the characterization of tail dependence and stochastic ordering for copulas with separable perturbation structure. In particular, by [4], LTD and RTI properties depend on monotonicity of the ratios $\Psi(u)/u$ and $\Psi(u)/(1-u)$, which yield the stated derivative inequalities. Statement (3) follows from the equivalence between stochastic increasing dependence and concavity of the generating function. Statements (4), (5), and (6) follow from standard implications among dependence concepts, as detailed in [5], where LCSD, RCSI, and PLR are shown to be equivalent to combinations of LTD, RTI, and SI properties.

Thus, the dependence structure of the Lambert W copula is fully characterized by analytical properties of the generating function Ψ.

Corollary 1 (Tail dependence structure of the Lambert W copula)**.** *Under the conditions of Theorem 4.6, the generating function Ψ satisfies the following:*

(1) *LTD holds on a subinterval* $(0,u_1) \subset (0,1)$,

(2) *RTI holds on a subinterval* $(u_2,1) \subset (0,1)$,

(3) *SI does not hold globally.*

Hence, the dependence structure is asymmetric and varies across the unit interval.

Proof. From Lemma 2.4, $\Psi(u) < 0$ on $(0,1)$, Ψ' is continuous with $\Psi'(0^+) = -\Omega < 0$ and $\Psi'(1^-) = 0$, and Ψ'' changes sign. By Theorem 4.6, LTD holds iff $\Psi(u) \geq u\Psi'(u)$. Let $h(u) = \Psi(u) - u\Psi'(u)$, which is continuous. Then $h(u) > 0$ for u near 0 and $h(u) < 0$ for u near 1. Hence, by continuity, there exists $u_1 \in (0,1)$ such that LTD holds on $(0, u_1)$. RTI holds iff $\Psi(u) \geq (u-1)\Psi'(u)$. Let $k(u) = \Psi(u) - (u-1)\Psi'(u)$, continuous on $(0,1)$. Then $k(u) < 0$ near 0 and $k(u) \geq 0$ near 1, implying the existence of $u_2 \in (0,1)$ such that RTI holds on $(u_2, 1)$. SI holds iff Ψ is concave. Since Ψ'' changes sign, Ψ is not concave on $(0,1)$; hence SI does not hold globally. The result follows.

5. Conditional Dependence and Tail Diagnostics

While the parameter θ determines the overall direction and strength of dependence, such effects often become more pronounced in extreme regions of the distribution. To better understand this behavior, we examine conditional tail probabilities and conditional expectations under the Lambert W copula.

Conditional tail probabilities. Let (U, V) denote the uniform pseudo-observations. For $u, v \in [0,1]$, consider

$$\Pr(U > u \mid V > v) = \frac{\Pr(U > u, V > v)}{\Pr(V > v)}.$$

Using the copula representation,

$$\Pr(U > u, V > v) = 1 - u - v + C(u,v),$$

which yields

$$\Pr(U > u \mid V > v) = \frac{1 - u - v + C(u,v)}{1 - v}.$$

Substituting

$$C(u,v) = uv + \theta\Psi(u)\Psi(v),$$

we obtain

$$\Pr(U > u \mid V > v) = 1 - u + \theta\frac{\Psi(u)\Psi(v)}{1 - v}. \tag{16}$$

Expression (16) separates the independence component $1 - u$ from the dependence contribution. In particular, the conditional tail probability is affine in θ, so that the sign of θ fully determines whether conditioning on large values of V increases or decreases the likelihood of large values of U.

Conditional expectation above a threshold. A complementary diagnostic is given by the conditional expectation

$$\mathbb{E}(U \mid V > v) = \frac{1}{1-v} \int_v^1 \int_0^1 u\, g(u,y)\, du\, dy,$$

where

$$g(u,v) = 1 + \theta \Psi_u'(u) \Psi_v'(v).$$

Applying Fubini's theorem,

$$\mathbb{E}(U \mid V > v) = \frac{1}{1-v} \int_v^1 \left(\int_0^1 u\, du + \theta \int_0^1 u \Psi_u'(u)\, du \cdot \Psi_v'(y) \right) dy.$$

Evaluating the inner integrals yields

$$\mathbb{E}(U \mid V > v) = \frac{1}{1-v} \int_v^1 \left(\frac{1}{2} - \theta I_0 \Psi_v'(y) \right) dy.$$

Using

$$\int_v^1 \Psi_v'(y)\, dy = -\Psi(v),$$

we obtain the closed-form expression

$$\mathbb{E}(U \mid V > v) = \frac{1}{2} + \theta I_0 \frac{\Psi(v)}{1-v}.$$

This representation shows that deviations from the independence baseline $1/2$ are governed by the product θI_0 and the boundary-adjusted term $\Psi(v)/(1-v)$.

6. Maximum Likelihood Estimation

In practical applications, the dependence parameter θ must be estimated from observed data. Let $(u_i, v_i)_{i=1}^n$ denote a sample of pseudo-observations on $[0,1]^2$. Under the Lambert W copula, the joint density is

$$g(u,v) = 1 + \theta \Psi_u'(u) \Psi_v'(v).$$

The log-likelihood function is given by

$$\ell(\theta) = \sum_{i=1}^n \log\bigl(1 + \theta a_i\bigr), \qquad a_i = \Psi_u'(u_i) \Psi_v'(v_i).$$

Differentiating with respect to θ yields

$$\ell'(\theta) = \sum_{i=1}^n \frac{a_i}{1 + \theta a_i}.$$

The maximum likelihood estimator $\hat{\theta}$ is defined as the solution of

$$\sum_{i=1}^n \frac{a_i}{1 + \theta a_i} = 0.$$

The second derivative satisfies

$$\ell''(\theta) = -\sum_{i=1}^n \frac{a_i^2}{(1 + \theta a_i)^2} < 0,$$

for all admissible θ. Hence, the log-likelihood is strictly concave, ensuring existence and uniqueness of the maximum likelihood estimator.

Although the score equation does not admit an explicit solution, a first-order approximation can be obtained from

$$\frac{a_i}{1+\theta a_i} \approx a_i - \theta a_i^2.$$

Substituting into the score equation gives

$$\sum_{i=1}^{n} a_i - \theta \sum_{i=1}^{n} a_i^2 = 0,$$

and therefore

$$\hat{\theta} \approx \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i^2}.$$

This approximation provides a convenient initial estimate that can be refined using standard numerical optimization methods.

7. Empirical Application

Data and preprocessing. The empirical analysis is based on the Pima Indian diabetes dataset [29], consisting of $n = 757$ complete observations after preprocessing. Two continuous variables are considered: Body Mass Index (BMI) and the Diabetes Pedigree Function (Ped), representing anthropometric and genetic risk factors, respectively.

Descriptive statistics are reported in Table 1. The Ped variable exhibits pronounced right-skewness, while BMI shows mild asymmetry, indicating deviation from normality and suggesting that Gaussian-based dependence models may be insufficient.

Table 1. Descriptive statistics of BMI and Ped

Variable	Mean	Std. Dev.	Skewness	Kurtosis
BMI	31.993	7.884	-0.429	3.290
Ped	0.472	0.331	1.920	5.595

Rank-based dependence measures indicate weak but statistically significant positive association (Kendall's $\tau = 0.0946$, Spearman's $\rho = 0.1412$, $p < 10^{-4}$).

Figure 7 shows a diffuse pattern with no clear linear trend, supporting the need for flexible dependence models capable of capturing subtle nonlinear features.

To construct the copula, the data are transformed into pseudo-observations via empirical ranks:

$$u_i = \frac{\text{rank}(x_i)}{n+1}, \qquad v_i = \frac{\text{rank}(y_i)}{n+1},$$

ensuring uniform marginals on $[0, 1]$.

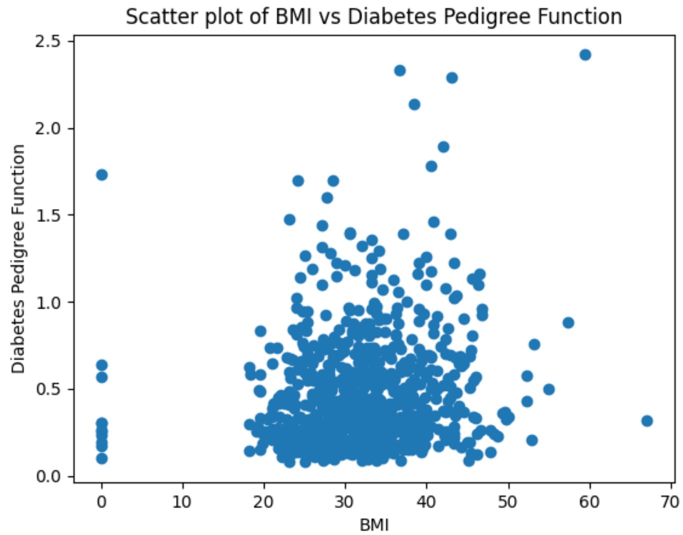


FIGURE 7. Scatter plot of BMI and Diabetes Pedigree Function. The distribution exhibits weak dependence with substantial dispersion and skewness, indicating the absence of strong linear structure and suggesting nonlinear dependence.

Model and estimation. The Lambert W copula is defined as

$$C(u,v) = uv + \theta\Psi(u)\Psi(v), \quad \Psi(u) = u(1-u)(W(u) - \Omega).$$

The dependence parameter θ is estimated via maximum likelihood:

$$\ell(\theta) = \sum_{i=1}^{n} \log\left(1 + \theta\,\Psi'(u_i)\Psi'(v_i)\right),$$

subject to admissibility constraints ensuring non-negativity of the density.

For comparison, Gaussian and FGM copulas are also fitted using maximum likelihood. Model performance is evaluated using likelihood-based criteria (AIC and BIC) and error-based metrics (RMSE, MAE, and MSE).

Results and graphical comparison. Table 2 summarizes the goodness-of-fit results.

TABLE 2. Goodness-of-fit comparison of copula models

Model	AIC	BIC	τ	RMSE/MAE/MSE
Gaussian	-15.609	-10.965	0.0946	0.1706/0.1019/0.0291
FGM	-13.177	-8.534	0.0946	0.1411/0.1059/0.0199
Lambert W	-5.276	-0.633	0.0946	0.1073/0.0627/0.0115

The Gaussian copula attains the lowest AIC and BIC values, indicating the best fit in terms of likelihood-based criteria. This is consistent with the relatively weak dependence observed in the data, where simple symmetric structures adequately capture global dependence patterns.

However, the Lambert W copula achieves the lowest RMSE, MAE, and MSE, indicating superior local approximation of the dependence structure. This suggests that the nonlinear generator $\Psi(u)$ enables the model to capture subtle variations in the joint distribution that are not represented by classical copulas.

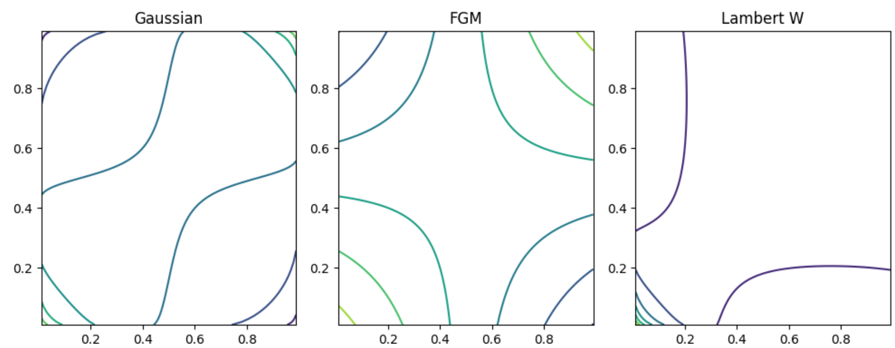


FIGURE 8. Contour comparison of Gaussian, FGM, and Lambert W copulas. The Gaussian copula exhibits elliptical symmetry, while the Lambert W copula introduces nonlinear deformation, capturing more complex dependence patterns.

Figure 8 shows that the Lambert W copula departs from classical symmetric structures, allowing greater flexibility in modeling dependence.

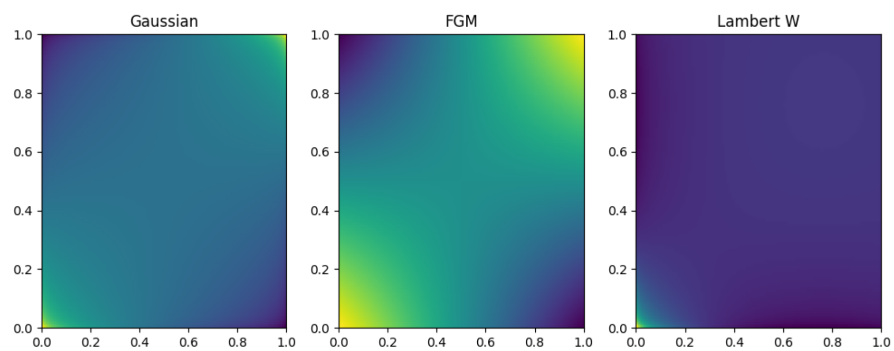


FIGURE 9. Density comparison of Gaussian, FGM, and Lambert W copulas. The Gaussian copula exhibits smooth symmetric variation, while the FGM copula shows limited dependence structure. In contrast, the Lambert W copula displays asymmetric and nonlinear concentration of probability mass, particularly near boundary regions, reflecting its ability to capture localized dependence features.

The density plots further highlight differences in mass allocation, with the Lambert W copula adapting more effectively to local dependence features.

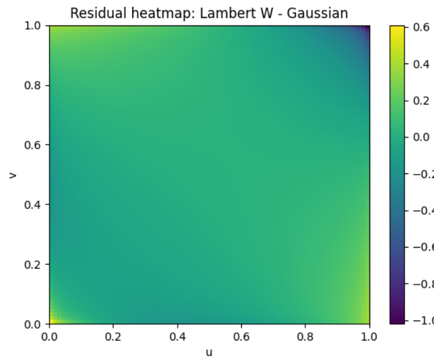


FIGURE 10. Residual heatmap between the Lambert W copula and the Gaussian copula. Positive regions indicate where the Lambert W copula assigns higher density, revealing areas where nonlinear dependence is better captured.

The residual heatmap provides localized evidence of model improvement, showing that the Lambert W copula captures dependence features not represented by the Gaussian model, particularly away from the central region.

Discussion. The results highlight a trade-off between likelihood optimality and structural flexibility. While the Gaussian copula provides the best global fit, it imposes a rigid dependence structure. In contrast, the Lambert W copula offers enhanced flexibility through its nonlinear generator, resulting in improved local accuracy.

Taken together, the numerical and graphical evidence demonstrates that the proposed Lambert W copula provides a meaningful extension of classical copula models, particularly in settings characterized by weak but nontrivial dependence.

Remark. The proposed framework admits natural extensions to conditional dependence modeling through partial derivatives of the copula function. Such extensions may support predictive and risk-based applications and are left for future work.

8. CONCLUSIONS

This paper introduced a new parametric family of bivariate copulas constructed as a nonlinear perturbation of the independence copula through the Lambert W function. The model

$$C(u,v) = uv + \theta\,\Psi(u)\Psi(v), \quad \Psi(u) = u(1-u)(W(u)-\Omega),$$

was developed within the Rüschendorf framework, providing a flexible yet tractable extension of classical product-type copulas.

Explicit admissible bounds for the dependence parameter were derived to ensure validity of the copula density. Closed-form expressions were obtained for mixed moments, covariance, correlation measures, and the joint moment generating function. A first-order expansion showed that the model locally reduces to an FGM-type copula, while higher-order terms introduce nonlinear corrections driven by the Lambert W function. Tail analysis confirmed asymptotic independence in both tails.

Empirical results demonstrated a clear distinction between global and local model performance. While the Gaussian copula achieved the best fit in terms of likelihood-based criteria, the Lambert W copula consistently provided lower approximation errors, indicating improved local representation of the dependence structure. This highlights the advantage of incorporating nonlinear generators when modeling weak but nontrivial dependence.

Overall, the proposed construction integrates special function theory into copula modeling, yielding a dependence structure that is both analytically transparent and structurally flexible. The Lambert W copula therefore offers a meaningful extension of classical perturbation-based families while preserving interpretability and computational feasibility.

Future work may consider multivariate extensions, alternative special-function generators, and broader empirical comparisons with established copula families. Further investigation of statistical efficiency, identifiability, and geometric properties may provide deeper insight into the role of transcendental structures in dependence modeling.

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