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ON THE k-LUCAS HANKEL MATRICES: DETERMINANTS AND HANKEL TRANSFORMS

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Abstract. Let $k \geq 1$ and let $\{L_n^{(k)}\}_{n\geq 0}$ be the k-Lucas sequence. For each integer $n \geq 1$, the k-Lucas Hankel matrix $H_n^{(k)}$ is defined by

$$H_n^{(k)} = \left[L_{i+j}^{(k)}\right]_{0\leq i,j\leq n-1}.$$

In this paper, we investigate the structural and determinantal properties of k-Lucas Hankel matrices. We establish recurrence identities satisfied by the rows, columns, and diagonals of $H_n^{(k)}$, and derive a third-order recurrence identity for the k-Lucas sequence. A complete characterization of the determinants $\det\left(H_n^{(k)}\right)$ is obtained for all $n \geq 1$ and $k \geq 1$. Furthermore, the complete Hankel transform of the k-Lucas sequence is explicitly computed, and summation formulas for its terms are derived. We also show that the Hankel transform determines the parameter k. These results contribute to the growing literature on structured matrices defined by linear recurrence sequences and provide a theoretical framework for the study of k-Lucas Hankel matrices.

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1. Introduction

The Fibonacci and Lucas sequences are among the most studied sequences in mathematics, with documented applications in combinatorics, number theory, and linear algebra [10]. These two sequences satisfy the same second-order recurrence relation $a_n = a_{n-1} + a_{n-2}$, differing only in their initial conditions. The Fibonacci sequence begins with $F_0 = 0$ and $F_1 = 1$, while the Lucas sequence begins with $L_0 = 2$ and $L_1 = 1$. These sequences have been generalized in many ways [1–3, 8, 13]. Specifically, the k-Lucas sequence, denoted by $\left\{L_n^{(k)}\right\}_{n\geq 0}$, is one of the generalizations of the Lucas sequence introduced by Falcon [3] in 2011, which will be used in this study. Notably, the k-Lucas sequence reduces to the classical Lucas sequence when $k = 1$, and to the Pell-Lucas sequence when $k = 2$.

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Hankel matrices have been studied in connection with Fibonacci-type sequences by several authors. Falcón [4] studied their norms, establishing relationships among the Euclidean, column, and spectral norms. Gökbaş and Köse [7] investigated the spectral norm bounds of r-Hankel matrices with Fibonacci and Lucas number entries, while Gökbaş and Türkmen [14] carried out the analogous study for r-Toeplitz matrices. Moreover, the Hankel transform of an integer sequence defined as the sequence of Hankel determinants from its associated Hankel matrices was formally introduced by Layman [11].

Beyond pure mathematics, Hankel matrices find applications in diverse areas including system identification and realization [5], time series forecasting [6], and face emotion recognition [12].

Despite all of these, the k-Lucas Hankel matrix — constructed from the k-Lucas sequence, has not been studied as a standalone mathematical object. No existing work provides a complete characterization of $\det\left(H_n^{(k)}\right)$ for all $n \geq 1$ and $k \geq 1$, nor has the complete Hankel transform of the k-Lucas sequence been explicitly derived. Furthermore, the structural recurrence properties of the matrix rows and columns, and the invertibility of the Hankel transform with respect to the parameter k, remain open.

This paper addresses these gaps through a systematic study of k-Lucas Hankel matrices. We establish that every row and column of $H_n^{(k)}$ satisfies the same second-order recurrence as the k-Lucas sequence, and derive a third-order recurrence identity for the sequence. In addition, we prove a complete characterization of the k-Lucas Hankel determinant and compute the complete Hankel transform. Finally, we establish summation formulas for its terms, and show that the transform uniquely recovers k through the closed-form relation.

Throughout this paper, we denote the k-Lucas numbers by $L_n^{(k)}$, which is equivalent to the notation $L_{k,n}$ used in some references.

2. Preliminaries

In this section, we present the foundational definitions and theorems necessary for the construction of our main results.

Definition 2.1. [3] For any integer $k \geq 1$, the k-Lucas numbers, say $\left\{L_n^{(k)}\right\}_{n\geq 0}$, is defined recurrently by

$$L_n^{(k)} = kL_{n-1}^{(k)} + L_{n-2}^{(k)}, \quad \text{for } n \geq 2, \tag{1}$$

with initial conditions

$$L_0^{(k)} = 2, \quad L_1^{(k)} = k.$$

This sequence generalizes the classical Lucas numbers by introducing the parameter k, which modifies the weight of the contribution from the previous term $L_{n-1}^{(k)}$. Table 1 presented below contains the initial terms of the sequence $\left\{L_n^{(k)}\right\}_{n\in\mathbb{N}}$ for selected values k.

TABLE 1. The initial terms of the k-Lucas numbers

n	0	1	2	3	4	5	6	7
$L_n^{(1)}$	2	1	3	4	7	11	18	29
$L_n^{(2)}$	2	2	6	14	34	82	198	478
$L_n^{(3)}$	2	3	11	36	119	393	1298	4287
$L_n^{(4)}$	2	4	18	76	322	1364	5778	24476
$L_n^{(5)}$	2	5	27	140	727	3775	19602	101785
$L_n^{(6)}$	2	6	38	234	1442	8886	54758	337434
$L_n^{(7)}$	2	7	51	364	2599	18557	132498	946043

Theorem 2.1. [3] The k-Lucas numbers can be expressed using the Binet formula as

$$L_n^{(k)} = \left(\alpha^{(k)}\right)^n + \left(\beta^{(k)}\right)^n, \tag{2}$$

where

$$\alpha^{(k)} = \frac{k+\sqrt{k^2+4}}{2}, \qquad \beta^{(k)} = \frac{k-\sqrt{k^2+4}}{2}.$$

Lemma 2.2 (Cassini's Identity). [3] For all $n \geq 1$,

$$L_{n+1}^{(k)}\, L_{n-1}^{(k)} - \left(L_n^{(k)}\right)^2 = (-1)^{n+1}\,(k^2+4). \tag{3}$$

Lemma 2.3. For every integer $n \geq 3$, the k-Lucas sequence satisfies the following third-order recurrence identity, that is,

$$L_n^{(k)} = (k+1)\,L_{n-1}^{(k)} - (k-1)\,L_{n-2}^{(k)} - L_{n-3}^{(k)}. \tag{4}$$

Proof. From equation (1), we have

$$L_n^{(k)} = k\,L_{n-1}^{(k)} + L_{n-2}^{(k)}, \tag{5}$$

$$L_{n-1}^{(k)} = k\,L_{n-2}^{(k)} + L_{n-3}^{(k)}. \tag{6}$$

Substituting (6) into (5) gives

$$L_n^{(k)} = k\big(k\,L_{n-2}^{(k)} + L_{n-3}^{(k)}\big) + L_{n-2}^{(k)} = (k^2+1)\,L_{n-2}^{(k)} + k\,L_{n-3}^{(k)}. \tag{7}$$

On the other hand, expanding (6) yields

$$(k+1)\,L_{n-1}^{(k)} = k(k+1)\,L_{n-2}^{(k)} + (k+1)\,L_{n-3}^{(k)}. \tag{8}$$

Subtracting (7) from (8), we get

$$(k+1)\,L_{n-1}^{(k)} - L_n^{(k)} = k(k+1)\,L_{n-2}^{(k)} + (k+1)\,L_{n-3}^{(k)} - (k^2+1)\,L_{n-2}^{(k)} - k\,L_{n-3}^{(k)}$$

$$= (k-1)\,L^{(k)}_{n-2} + L^{(k)}_{n-3}. \tag{9}$$

Rearranging (9) gives the desired identity. □

Definition 2.2. [9] A matrix A with real entries is called symmetric if $A^T = A$.

Theorem 2.4. [9] An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row of zeros.

Theorem 2.5. [9] If A is an $n \times n$ matrix, then A is nonsingular if and only if $\det(A) \neq 0$.

Definition 2.3. [9] The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to be linearly dependent if there exist constants $a_1, a_2, \ldots, a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j v_j = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0. \tag{10}$$

Otherwise, $v_1, v_2, \ldots, v_k$ are called linearly independent. That is, $v_1, v_2, \ldots, v_k$ are linearly independent if, whenever

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0,$$

$$a_1 = a_2 = \cdots = a_k = 0.$$

Definition 2.4. [9] The dimension of the row (column) space of A is called the row (column) rank of A.

Definition 2.5. [9] Let S be a set of vectors in a vector space V. If every vector in V is a linear combination of the vectors in S, then the set S is said to span V, or V is spanned by the set S; that is, $\operatorname{span}(S) = V$.

Definition 2.6. [11] The Hankel matrix H of the integer sequence $\{a_1, a_2, a_3, \ldots\}$ is the infinite matrix

$$H = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ a_4 & a_5 & a_6 & a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with elements $h_{i,j} = a_{i+j-1}$. The *Hankel matrix H_n of order n* is the upper-left $n \times n$ submatrix of H.

Definition 2.7. [11] Let $A = \{a_1, a_2, a_3, \ldots\}$ be an integer sequence, and let $\{h_n\}_{n=1}^{\infty}$ denote the sequence of determinants of the corresponding Hankel matrices of increasing order. The sequence $\{h_n\}$ is called the *Hankel transform* of A.

Remark 2.6. The notation $L_n^{(k)}$ used throughout this paper is equivalent to the notation $L_{k,n}$ found in Falcón [3] and related works. The superscript form is adopted here for consistency with the matrix notation $H_n^{(k)}$.

3. Main Results

Definition 3.1. Let $k \geq 1$ and let $\{L_n^{(k)}\}_{n\geq 0}$ be the k-Lucas sequence. For each integer $n \geq 1$, the k-Lucas Hankel matrix $H_n^{(k)}$ is defined by

$$H_n^{(k)} = \left[L_{i+j}^{(k)}\right]_{0\leq i,j\leq n-1}.$$

Thus every entry depends only on the sum $i+j$ of its indices, so all anti-diagonals of $H_n^{(k)}$ are constant. Explicitly,

$$H_n^{(k)} = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} & L_2^{(k)} & \cdots & L_{n-1}^{(k)} \\ L_1^{(k)} & L_2^{(k)} & L_3^{(k)} & \cdots & L_n^{(k)} \\ L_2^{(k)} & L_3^{(k)} & L_4^{(k)} & \cdots & L_{n+1}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n-1}^{(k)} & L_n^{(k)} & L_{n+1}^{(k)} & \cdots & L_{2n-2}^{(k)} \end{bmatrix}.$$

Example 3.1.1. Let $k = 2$. The 2-Lucas sequence $\{L_n^{(2)}\}_{n\geq 0}$ is defined by

$$L_n^{(2)} = 2L_{n-1}^{(2)} + L_{n-2}^{(2)}, \qquad n \geq 2,$$

with initial values $L_0^{(2)} = 2$ and $L_1^{(2)} = 2$. The first seven terms are

$$L_0^{(2)} = 2,$$

$$L_1^{(2)} = 2,$$

$$L_2^{(2)} = 2L_1^{(2)} + L_0^{(2)} = 2(2) + 2 = 4 + 2 = 6,$$

$$L_3^{(2)} = 2L_2^{(2)} + L_1^{(2)} = 2(6) + 2 = 12 + 2 = 14,$$

$$L_4^{(2)} = 2L_3^{(2)} + L_2^{(2)} = 2(14) + 6 = 28 + 6 = 34,$$

$$L_5^{(2)} = 2L_4^{(2)} + L_3^{(2)} = 2(34) + 14 = 68 + 14 = 82,$$

$$L_6^{(2)} = 2L_5^{(2)} + L_4^{(2)} = 2(82) + 34 = 164 + 34 = 198.$$

The corresponding 4×4 Hankel matrix is

$$H_4^{(2)} = \begin{bmatrix} L_0^{(2)} & L_1^{(2)} & L_2^{(2)} & L_3^{(2)} \\ L_1^{(2)} & L_2^{(2)} & L_3^{(2)} & L_4^{(2)} \\ L_2^{(2)} & L_3^{(2)} & L_4^{(2)} & L_5^{(2)} \\ L_3^{(2)} & L_4^{(2)} & L_5^{(2)} & L_6^{(2)} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 6 & 14 \\ 2 & 6 & 14 & 34 \\ 6 & 14 & 34 & 82 \\ 14 & 34 & 82 & 198 \end{bmatrix}.$$

From the above definition, we formulate the following propositions.

Proposition 3.1. For all $1 \leq i, j \leq n-2$, the entries of the k-Lucas Hankel matrix $H_n^{(k)}$ satisfy

$$L_{i+1+j}^{(k)} - L_{i-1+j}^{(k)} = k\, L_{i+j}^{(k)}. \tag{11}$$

Proof. Let $m = i + j$. By the defining recurrence of the k-Lucas sequence,

$$L_{m+1}^{(k)} = k\, L_m^{(k)} + L_{m-1}^{(k)}. \tag{12}$$

Rearranging yields

$$L_m^{(k)} = \frac{L_{m+1}^{(k)} - L_{m-1}^{(k)}}{k}. \tag{13}$$

Substituting $m = i + j$ gives

$$L_{i+j}^{(k)} = \frac{L_{i+1+j}^{(k)} - L_{i-1+j}^{(k)}}{k}, \tag{14}$$

or equivalently,

$$L_{i+1+j}^{(k)} - L_{i-1+j}^{(k)} = k\, L_{i+j}^{(k)}. \tag{15}$$

Since $H_n^{(k)}$ is a Hankel matrix, its (i, j)-entry depends only on the sum $i + j$. Therefore, the identity holds for all valid indices $1 \leq i, j \leq n-2$. □

Remark 3.2. Since $H_n^{(k)}$ is symmetric, the identity (11) holds both along rows (fixing j, varying i) and along columns (fixing i, varying j). In particular, every row and every column of $H_n^{(k)}$ satisfies the same second-order recurrence relation as the k-Lucas sequence itself.

Example 3.1.2. Consider the matrix $H_4^{(2)}$ from Example 3.1.1, with $k = 2$, that is,

$$H_4^{(2)} = \begin{bmatrix} 2 & 2 & 6 & 14 \\ 2 & 6 & 14 & 34 \\ 6 & 14 & 34 & 82 \\ 14 & 34 & 82 & 198 \end{bmatrix}.$$

We verify Proposition 3.1 on selected entries.
For $L_2^{(2)} = 6$ in position $(i, j) = (1, 1)$:

$$\frac{L_3^{(2)} - L_1^{(2)}}{2} = \frac{14 - 2}{2} = 6 = L_2^{(2)}.$$

For $L_3^{(2)} = 14$ in position $(i, j) = (1, 2)$:

$$\frac{L_4^{(2)} - L_2^{(2)}}{2} = \frac{34 - 6}{2} = 14 = L_3^{(2)}.$$

For $L_4^{(2)} = 34$ in position $(i, j) = (2, 2)$:

$$\frac{L_5^{(2)} - L_3^{(2)}}{2} = \frac{82 - 14}{2} = 34 = L_4^{(2)}.$$

Since $H_4^{(2)}$ is symmetric, the same identity holds column-wise by Remark 3.2.

Remark 3.3. Under the assumptions of Proposition 3.1, since k-Lucas numbers satisfy $L_{i+j}^{(k)} > 0$ for all $k > 0$, we may solve the identity in Proposition 3.1 for k to obtain

$$k = \frac{L_{i+1+j}^{(k)} - L_{i-1+j}^{(k)}}{L_{i+j}^{(k)}}. \tag{16}$$

Consequently, any three consecutive entries along any row or column of the k-Lucas Hankel matrix $H_n^{(k)}$ uniquely determine the parameter k.

Example 3.1.3. From Example 3.1.1, consider three consecutive entries along the second row of $H_4^{(2)}$ which are

$$2, \quad 6, \quad 14.$$

Applying (16) with $L_{i+j}^{(k)} = 6$, $L_{i+1+j}^{(k)} = 14$, and $L_{i-1+j}^{(k)} = 2$, we obtain

$$k = \frac{14 - 2}{6} = \frac{12}{6} = 2,$$

which correctly recovers the parameter $k = 2$.

Corollary 3.4. For any $n \times n$ k-Lucas Hankel matrix $H_n^{(k)} = [H_{i,j}^{(k)}]_{0 \le i,j \le n-1}$, where $H_{i,j}^{(k)} = L_{i+j}^{(k)}$, each entry with $i + j \ge 3$ and $0 \le i, j \le n - 1$ satisfies

$$L_{i+j}^{(k)} = (k+1)\, L_{i+j-1}^{(k)} - (k-1)\, L_{i+j-2}^{(k)} - L_{i+j-3}^{(k)}. \tag{17}$$

In particular, along each fixed row or column of $H_n^{(k)}$, any three entries whose index sums satisfy $i + j \ge 3$ obey the same third-order linear recurrence as the k-Lucas sequence.

Proof. By Lemma 2.3, the k-Lucas sequence satisfies

$$L_n^{(k)} = (k+1)\, L_{n-1}^{(k)} - (k-1)\, L_{n-2}^{(k)} - L_{n-3}^{(k)}, \quad n \ge 3.$$

For any entry $H_{i,j}^{(k)}$ of the Hankel matrix with $i + j \ge 3$, let $n = i + j$. Since $H_{i,j}^{(k)} = L_{i+j}^{(k)}$ by definition, equation (17) follows immediately.

Fixing i and varying $j \geq 3$ yields the row-wise third-order recurrence, that is,

(18) $$H_{i,j}^{(k)} = (k+1)\, H_{i,j-1}^{(k)} - (k-1)\, H_{i,j-2}^{(k)} - H_{i,j-3}^{(k)}.$$

Since $H_{i,j}^{(k)} = L_{i+j}^{(k)}$, then

(19) $$L_{i+j}^{(k)} = (k+1)\, L_{i+j-1}^{(k)} - (k-1)\, L_{i+j-2}^{(k)} - L_{i+j-3}^{(k)}.$$

By symmetry of $H_n^{(k)}$, fixing j and varying $i \geq 3$ yields the column-wise version, that is,

(20) $$H_{i,j}^{(k)} = (k+1)\, H_{i-1,j}^{(k)} - (k-1)\, H_{i-2,j}^{(k)} - H_{i-3,j}^{(k)}.$$

It follows that

(21) $$L_{i+j}^{(k)} = (k+1)\, L_{i-1+j}^{(k)} - (k-1)\, L_{i-2+j}^{(k)} - L_{i-3+j}^{(k)}.$$ □

Example 3.1.4. Let $k = 2$. Then

$$\{L_n^{(2)}\}_{n=0}^{\infty} = \{2, 2, 6, 14, 34, 82, 198, 478, \ldots\}.$$

By Lemma 2.3, the third-order recurrence for $k = 2$ is

$$L_n^{(2)} = 3L_{n-1}^{(2)} - L_{n-2}^{(2)} - L_{n-3}^{(2)}, \quad n \geq 3.$$

Consider the 4×4 2-Lucas Hankel matrix,

$$H_4^{(2)} = \begin{bmatrix} 2 & 2 & 6 & 14 \\ 2 & 6 & 14 & 34 \\ 6 & 14 & 34 & 82 \\ 14 & 34 & 82 & 198 \end{bmatrix}.$$

We verify Corollary 3.4 on all entries $H_{i,j}^{(2)} = L_{i+j}^{(2)}$ with $i + j \geq 3$ and $0 \leq i, j \leq 3$.

(i) $H_{1,2}^{(2)} = 14$ $(i + j = 3)$:

$$L_3^{(2)} = 3L_2^{(2)} - L_1^{(2)} - L_0^{(2)} = 3(6) - 2 - 2 = 14.$$

(ii) $H_{2,1}^{(2)} = 14$ $(i + j = 3)$: By symmetry of $H_4^{(2)}$, this equals $H_{1,2}^{(2)} = 14$, already verified in (i).

(iii) $H_{2,2}^{(2)} = 34$ $(i + j = 4)$:

$$L_4^{(2)} = 3L_3^{(2)} - L_2^{(2)} - L_1^{(2)} = 3(14) - 6 - 2 = 34.$$

(iv) $H_{1,3}^{(2)} = 34$ $(i + j = 4)$: By symmetry of $H_4^{(2)}$, this equals $H_{3,1}^{(2)} = 34$, already verified in (iii).

(v) $H^{(2)}_{2,3} = 82$ $(i + j = 5)$:

$$L^{(2)}_5 = 3L^{(2)}_4 - L^{(2)}_3 - L^{(2)}_2 = 3(34) - 14 - 6 = 82.$$

(vi) $H^{(2)}_{3,3} = 198$ $(i + j = 6)$:

$$L^{(2)}_6 = 3L^{(2)}_5 - L^{(2)}_4 - L^{(2)}_3 = 3(82) - 34 - 14 = 198.$$

In all cases, the entries satisfy the third-order recurrence, confirming Corollary 3.4.

3.1. k-Lucas Hankel Determinants.

Theorem 3.5. For all $k \geq 1$,

$$\det\left(H^{(k)}_2\right) = k^2 + 4.$$

Proof 1 (*Direct Computation*). Since $L^{(k)}_0 = 2$, $L^{(k)}_1 = k$, and $L^{(k)}_2 = k^2 + 2$, the matrix $H^{(k)}_2$ is

$$H^{(k)}_2 = \begin{bmatrix} 2 & k \\ k & k^2 + 2 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \det\left(H^{(k)}_2\right) &= 2(k^2 + 2) - k^2 \\ &= k^2 + 4 \end{aligned}$$

□

Proof 2 (*Cassini's Identity*). By the definition of the 2×2 k-Lucas Hankel matrix,

$$\det\left(H^{(k)}_2\right) = L^{(k)}_0 L^{(k)}_2 - \left(L^{(k)}_1\right)^2.$$

By (3) with $n = 1$,

$$\begin{aligned} L^{(k)}_0 L^{(k)}_2 - \left(L^{(k)}_1\right)^2 &= (-1)^{1+1}(k^2 + 4) \\ &= k^2 + 4. \end{aligned}$$

Therefore, $\det(H^{(k)}_2) = k^2 + 4$. □

Example 3.1.5. For $k = 2$, we have $L^{(2)}_0 = 2$, $L^{(2)}_1 = 2$, and $L^{(2)}_2 = 6$, giving

$$H^{(2)}_2 = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix},$$

$$\begin{aligned} \det\left(H^{(2)}_2\right) &= 2(6) - 2(2) \\ &= 8 \end{aligned}$$

$$= 2^2 + 4$$

confirming Theorem 3.5. Alternatively, by (3), with $n = 1$ and $k = 2$,

$$L_0^{(2)} L_2^{(2)} - \left(L_1^{(2)}\right)^2 = (-1)^2(2^2 + 4)$$
$$= 8,$$

consistent with $\det(H_2^{(2)}) = 8$.

Theorem 3.6. For all $k \geq 1$ and all $n \geq 3$,

$$\det\left(H_n^{(k)}\right) = 0.$$

That is, all k-Lucas Hankel matrices of size 3×3 or larger are singular.

Proof. We show that the rows of $H_n^{(k)}$ are linearly dependent for all $n \geq 3$.

Let $r_i = [L_i^{(k)}, L_{i+1}^{(k)}, \ldots, L_{i+n-1}^{(k)}]$ denote the i-th row of $H_n^{(k)}$, where $i \geq 0$. By (1), the j-th component of r_i satisfies

$$L_{i+j}^{(k)} = k\, L_{i+j-1}^{(k)} + L_{i+j-2}^{(k)}$$
$$= k(r_{i-1})_j + (r_{i-2})_j,$$

so that $r_i = k\, r_{i-1} + r_{i-2}$ for all $i \geq 2$.

Case 1: $n = 3$.

The three rows satisfy $r_2 = k\, r_1 + r_0$, which we verify component-wise, that is,

$$k\, r_1 + r_0 = k[L_1^{(k)}, L_2^{(k)}, L_3^{(k)}] + [L_0^{(k)}, L_1^{(k)}, L_2^{(k)}]$$
$$= [kL_1^{(k)} + L_0^{(k)},\ kL_2^{(k)} + L_1^{(k)},\ kL_3^{(k)} + L_2^{(k)}]$$
$$= [L_2^{(k)}, L_3^{(k)}, L_4^{(k)}]$$
$$= r_2.$$

Since r_2 is a linear combination of r_0 and r_1, the three rows are linearly dependent, so

$$\det(H_3^{(k)}) = 0.$$

Case 2: $n \geq 4$.

We proceed by induction on i to show that every row r_i with $i \geq 2$ is a linear combination of r_0 and r_1.

Base cases: Let α_i, β_i be scalars such that $r_i = \alpha_i\, r_0 + \beta_i\, r_1$.

For $i = 2$,

$$\begin{aligned} r_2 &= k\, r_1 + r_0 \\ &= 1 \cdot r_0 + k \cdot r_1, \end{aligned}$$

so $\alpha_2 = 1$ and $\beta_2 = k$.

For $i = 3$,

$$\begin{aligned} r_3 &= k\, r_2 + r_1 \\ &= k(\alpha_2\, r_0 + \beta_2\, r_1) + r_1 \\ &= k\alpha_2\, r_0 + (k\beta_2 + 1)\, r_1, \end{aligned}$$

so r_3 is also a linear combination of r_0 and r_1.

Inductive step: Assume that for some $m \geq 3$, we have $r_i = \alpha_i\, r_0 + \beta_i\, r_1$ for all $2 \leq i \leq m$. By the row recurrence,

$$\begin{aligned} r_{m+1} &= k\, r_m + r_{m-1} \\ &= k(\alpha_m\, r_0 + \beta_m\, r_1) + (\alpha_{m-1}\, r_0 + \beta_{m-1}\, r_1) \\ &= (k\alpha_m + \alpha_{m-1})\, r_0 + (k\beta_m + \beta_{m-1})\, r_1. \end{aligned}$$

Thus r_{m+1} is also a linear combination of r_0 and r_1.

By induction, every row r_i for $i \geq 2$ lies in $\mathrm{span}\{r_0, r_1\}$, so the row space of $H_n^{(k)}$ is spanned by at most two vectors, giving

$$\mathrm{rank}\big(H_n^{(k)}\big) \leq 2.$$

Since $n \geq 3$ implies $\mathrm{rank}(H_n^{(k)}) \leq 2 < n$, the matrix $H_n^{(k)}$ is singular, and therefore

$$\det\big(H_n^{(k)}\big) = 0. \qquad \square$$

Example 3.1.6. Let $k = 2$ and $n = 3$. The first five terms of the 2-Lucas sequence are $2, 2, 6, 14, 34$, so the 2-Lucas Hankel 3×3 matrix is

$$H_3^{(2)} = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 6 & 14 \\ 6 & 14 & 34 \end{bmatrix}.$$

Denoting the rows by r_0, r_1, r_2, we verify that $r_2 = 2r_1 + r_0$, that is,

$$\begin{aligned} 2[2,\ 6,\ 14] + [2,\ 2,\ 6] &= [4,\ 12,\ 28] + [2,\ 2,\ 6] \\ &= [6,\ 14,\ 34] \end{aligned}$$

$$= r_2.$$

Since the rows are linearly dependent, $\det(H_3^{(2)}) = 0$, confirming Theorem 3.6.

Theorem 3.7. Let $k \geq 1$ and let $H_n^{(k)}$ denote the $n \times n$ k-Lucas Hankel matrix. Then $\det(H_n^{(k)})$ is completely characterized by

$$\det\big(H_n^{(k)}\big) = \begin{cases} 2, & \text{if } n = 1, \\ k^2 + 4, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases} \tag{22}$$

Proof. We consider three cases according to the size of $H_n^{(k)}$.
Case $n = 1$: The matrix $H_1^{(k)}$ is the 1×1 matrix $[L_0^{(k)}] = [2]$, hence $\det(H_1^{(k)}) = 2$.
Case $n = 2$: By Theorem 3.5, $\det(H_2^{(k)}) = k^2 + 4$ for all $k \geq 1$.
Case $n \geq 3$: By Theorem 3.6, the rows of $H_n^{(k)}$ are linearly dependent for all $n \geq 3$, hence $\det(H_n^{(k)}) = 0$. □

Table 2 shows the determinants of k-Lucas Hankel matrices $H_n^{(k)}$ for varying values of $k \geq 1$ and $n \geq 1$, confirming Theorem 3.7.

TABLE 2. Determinants of k-Lucas Hankel matrices $H_n^{(k)}$.

k	$\det(H_1^{(k)})$	$\det(H_2^{(k)})$	$\det(H_3^{(k)})$	$\det(H_4^{(k)})$	$\det(H_5^{(k)})$	$\cdots$	$\det(H_n^{(k)})$
1	2	5	0	0	0	$\cdots$	0
2	2	8	0	0	0	$\cdots$	0
3	2	13	0	0	0	$\cdots$	0
4	2	20	0	0	0	$\cdots$	0
5	2	29	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Formula	2	$k^2 + 4$	0	0	0	$\cdots$	$0\ (n \geq 3)$

The three theorems established in this section together yield a complete and explicit characterization of the determinants of k-Lucas Hankel matrices. Specifically, for $n = 1$, $\det(H_n^{(k)}) = 2$, $k^2 + 4$ for $n = 2$, and vanishes for all $n \geq 3$, reflecting the linear dependence of rows induced by the second-order recurrence of the k-Lucas sequence. This singular behavior for large n motivates the study of the Hankel transform, which we develop in the following section.

3.2. k-Lucas Hankel Transform.

Theorem 3.8. Let $\{L_n^{(k)}\}_{n\geq 0}$ be the k-Lucas sequence, and let $\{h_n^{(k)}\}_{n\geq 1}$ be its k-Lucas Hankel transform where $h_n^{(k)} = \det(H_n^{(k)})$. Then

$$h_n^{(k)} = \begin{cases} 2, & \text{if } n = 1, \\ k^2 + 4, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases} \tag{23}$$

Equivalently, the Hankel transform of the k-Lucas sequence is the sequence

$$\{h_n^{(k)}\}_{n\geq 1} = \{2,\ k^2+4,\ 0,\ 0,\ 0,\ \ldots\},$$

which is zero for all $n \geq 3$.

Proof. We verify each case of (23) directly.
Case $n = 1$*:* The Hankel matrix $H_1^{(k)} = [L_0^{(k)}] = [2]$ is a 1×1 matrix, so

$$h_1^{(k)} = \det\big(H_1^{(k)}\big) = 2.$$

Case $n = 2$*:* The 2×2 k-Lucas Hankel matrix is

$$H_2^{(k)} = \begin{bmatrix} L_0^{(k)} & L_1^{(k)} \\ L_1^{(k)} & L_2^{(k)} \end{bmatrix} = \begin{bmatrix} 2 & k \\ k & k^2+2 \end{bmatrix}.$$

By Theorem 3.5,

$$h_2^{(k)} = \det\big(H_2^{(k)}\big) = k^2 + 4.$$

Case $n \geq 3$*:* By Theorem 3.6, all k-Lucas Hankel matrices of size $n \times n$ with $n \geq 3$ are singular, hence

$$h_n^{(k)} = \det\big(H_n^{(k)}\big) = 0 \quad \text{for all } n \geq 3.$$ □

Example 3.1.7. Let $k = 2$. The first few terms of the 2-Lucas sequence are

$$\{L_n^{(2)}\}_{n\geq 0} = \{2, 2, 6, 14, 34, 82, \ldots\}.$$

By Theorem 3.8, the Hankel transform of this sequence is

$$\{h_n^{(k)}\}_{n\geq 1} = \{2,\ 2^2+4,\ 0,\ 0,\ 0,\ \ldots\} = \{2,\ 8,\ 0,\ 0,\ 0,\ \ldots\}.$$

We verify each term directly, that is,

(*i*) $h_1^{(k)} = \det[2] = 2$,

(*ii*) $h_2^{(k)} = \det \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} = 12 - 4 = 8 = 2^2 + 4$,

(*iii*) $h_n^{(k)} = \det(H_n^{(2)}) = 0$ for all $n \geq 3$, since $H_n^{(2)}$ is singular by Theorem 3.6.

Thus, the complete Hankel transform of the 2-Lucas sequence is $\{2, 8, 0, 0, 0, \ldots\}$, confirming Theorem 3.8.

Table 3 presents the Hankel transform $\{h_n^{(k)}\}_{n\geq 1}$ of the k-Lucas sequence, illustrating Theorem 3.8.

Table 3. Hankel transform $\{h_n^{(k)}\}_{n\geq 1}$ of the k-Lucas sequence for $k \geq 1$.

k	$h_1^{(k)}$	$h_2^{(k)}$	$h_3^{(k)}$	$h_4^{(k)}$	$h_5^{(k)}$	$\cdots$	$h_n^{(k)}$
1	2	5	0	0	0	$\cdots$	0
2	2	8	0	0	0	$\cdots$	0
3	2	13	0	0	0	$\cdots$	0
4	2	20	0	0	0	$\cdots$	0
5	2	29	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
General Formula	2	k^2+4	0	0	0	$\cdots$	$0\ (n \geq 3)$

Now, from Table 3, we formulate the following proposition.

Proposition 3.9. For all $n \geq 1$,

$$\sum_{k=1}^{n} h_n^{(k)} = \begin{cases} 2n, & n = 1, \\ \dfrac{n(n+1)(2n+1)}{6} + 4n, & n = 2, \\ 0, & n \geq 3. \end{cases} \tag{24}$$

Proof. By Theorem 3.8, $h_1^{(k)} = 2$, $h_2^{(k)} = k^2 + 4$, and $h_n^{(k)} = 0$ for all $n \geq 3$ and all $k \geq 1$.

Case $n = 1$: Since $h_1^{(k)} = 2$ for all $k \geq 1$,

$$\sum_{k=1}^{n} h_1^{(k)} = \sum_{k=1}^{n} 2 = 2n.$$

Case $n = 2$: Since $h_2^{(k)} = k^2 + 4$,

$$\sum_{k=1}^{n} h_2^{(k)} = \sum_{k=1}^{n} (k^2 + 4)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 4.$$

Note that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$\sum_{k=1}^{n} 4 = 4n.$$

Thus,

$$\sum_{k=1}^{n} h_2^{(k)} = \frac{n(n+1)(2n+1)}{6} + 4n.$$

Case $n \geq 3$*:* Since $h_n^{(k)} = 0$ for all $k \geq 1$ and all $n \geq 3$,

$$\sum_{k=1}^{n} h_n^{(k)} = \sum_{k=1}^{n} 0 = 0.$$

□

Example 3.1.8. We verify Proposition 3.9 for $n = 4$.

Case $n = 1$*:*

$$\sum_{k=1}^{4} h_1^{(k)} = 2 + 2 + 2 + 2 = 8 = 2(4).$$

Case $n = 2$*:*

$$\sum_{k=1}^{4} h_2^{(k)} = 5 + 8 + 13 + 20 = 46,$$

and,

$$\frac{4(5)(9)}{6} + 4(4) = 30 + 16 = 46.$$

Case $n \geq 3$*:*

$$\sum_{k=1}^{4} h_3^{(k)} = 0 + 0 + 0 + 0 = 0,$$

and similarly for all $n \geq 3$, confirming Proposition 3.9.

Recall that in Remark 3.3, we showed that k can be recovered from three consecutive entries of the k-Lucas Hankel matrix. We now establish an analogous result at the level of the Hankel transform.

Proposition 3.10. Let $\{h_n^{(k)}\}_{n\geq 1} = \{2,\ k^2+4,\ 0,\ 0,\ \ldots\}$ be the Hankel transform of the k-Lucas sequence. Then the parameter k is uniquely determined by the second term $h_2^{(k)}$ through the relation

$$k = \sqrt{h_2^{(k)} - 4}.$$

Proof. By Theorem 3.8, $h_2^{(k)} = k^2 + 4$ for all $k \geq 1$. Solving for k gives

$$k^2 = h_2^{(k)} - 4 \implies k = \pm\sqrt{h_2^{(k)} - 4}.$$

Since $k \geq 1$ by definition, we take the positive square root:

$$k = \sqrt{h_2^{(k)} - 4}.$$

Note that $h_2^{(k)} = k^2 + 4 \geq 5 > 4$ for all $k \geq 1$, so the expression under the square root is always positive. This uniquely recovers k from the second term of the k-Lucas Hankel transform. □

Example 3.1.9. For $k = 3$, the Hankel transform gives $h_2^{(3)} = 3^2 + 4 = 13$. Applying Proposition 3.10,

$$\begin{aligned} k &= \sqrt{h_2^{(3)} - 4} \\ &= \sqrt{13 - 4} \\ &= \sqrt{9} = 3, \end{aligned}$$

correctly recovering $k = 3$.

4. Conclusion

In this paper, we conducted a systematic study of k-Lucas Hankel matrices $H_n^{(k)} = [L_{i+j}^{(k)}]_{0 \leq i,j \leq n-1}$, constructed from the k-Lucas sequence $\{L_n^{(k)}\}_{n \geq 0}$. We established that every row and column of $H_n^{(k)}$ satisfies the same second-order recurrence relation as the underlying k-Lucas sequence, and derived a third-order recurrence identity $L_n^{(k)} = (k+1)L_{n-1}^{(k)} - (k-1)L_{n-2}^{(k)} - L_{n-3}^{(k)}$ for $n \geq 3$. A complete characterization of the determinants was obtained: $\det(H_n^{(k)}) = 2$ for $n = 1$, $\det(H_n^{(k)}) = k^2 + 4$ for $n = 2$, and $\det(H_n^{(k)}) = 0$ for all $n \geq 3$, the last case arising from the linear dependence of rows induced by the recurrence relation. Furthermore, the complete Hankel transform of the k-Lucas sequence was explicitly computed as $\{h_n^{(k)}\}_{n \geq 1} = \{2, k^2 + 4, 0, 0, \ldots\}$, summation formulas for its terms were derived, and the invertibility of the transform with respect to the parameter k was established through the closed-form relation $k = \sqrt{h_2^{(k)} - 4}$.

These results collectively provide a theoretical framework for the study of k-Lucas Hankel matrices and contribute to the growing literature on structured matrices defined by linear recurrence sequences. For future work, researchers could investigate the spectral properties and norms of k-Lucas Hankel matrices, to extend the present study to generalized k-Lucas Hankel matrices of higher order, and to explore potential connections between the algebraic structure of these matrices and their applications.

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