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EXTENSION OF GENERALIZED LOCAL FRACTIONAL DERIVATIVES TO DELAY DIFFERENTIAL EQUATIONS

MOHAMMED AMIN HAKMI[1,2,3], **KHELIFA DAOUDI**[1,4,*]

[1]Department of Mathematics, Institute of Sciences, University Center Nour Bachir, El-Bayadh, 32000, Algeria

[2]LaMOS Research (Unit Modeling and Optimization of Systems), University of Bejaia, 06000, Algeria

[3]Laboratory of Pure and Applied Mathematics, University of Laghouat, 03000, Algeria

[4]Laboratory of Mathematics, Djillali Liabes University, Sidi Bel Abbes, 22000, Algeria

[*]Corresponding author: k.daoudi@cu-elbayadh.dz, khelifa.daoudi@gmail.com

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Abstract. In this paper, we investigate the existence and uniqueness of mild solutions for a class of delay fractional differential equations involving a generalized local fractional derivative. These equations are used to describe dynamical systems with memory effects, where the evolution of the system depends not only on its current state but also on its past behavior. The problem is reformulated as an equivalent fixed-point problem, and the existence and uniqueness of solutions are established by applying the Banach contraction principle under suitable conditions on the nonlinear term. In addition, the existence of solutions is further analyzed using the nonlinear Leray–Schauder alternative under appropriate compactness assumptions. The theoretical results are supported by illustrative examples that demonstrate the effectiveness and applicability of the proposed approach.

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1. Introduction

Fractional differential equations with delays are a generalization of classical differential equations that involve non-integer order derivatives, capturing the memory effects of the system. These equations are particularly useful in modeling phenomena where both the present and past states affect the future state, such as in biology, control systems, and engineering [1]. Adding delays makes things more complicated and requires advanced mathematical techniques, such as fixed-point theorems, to ensure that solutions exist and are unique. Fractional calculus offers a more adaptable framework for characterising systems with memory, offering insights into systems where traditional models fall short.

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Such equations find applications in areas like epidemiology, physics, and signal processing. Theoretical studies focus on establishing conditions for solutions and exploring numerical methods for practical implementation. [2,3].

The term "fractional "refers to the derivative of a function used in differential equations. Riemann-Liouville, Caputo, Hadamard, Caputo-Fabrizio, and others are only a few examples of the various kinds of fractional derivatives. In recent years, most authors have defined new types of fractional derivatives, notably R. Khalil et al., Katugampola, Kajouni et al., Atangana, Ahuja et al., Elnady et al., etc. [4–9].

This article presents the problem of existence and uniqueness of solutions to fractional differential equations with delays that utilise a generalised local fractional derivative, as presented by Elnady et al. [4].

The problem is modeled by the equation

$$D^{\alpha}u(t) = Au(t) + f(t, u_t),$$

where D^{α} represents the fractional derivative of order $\alpha \in (0, 1]$. The equation describes the behavior of a dynamic system, where the state $u(t)$ is influenced not only by its current value but also by its past states, represented as u_t. This historical dependence is formalized using the function $f(t, u_t)$, which maps the time and the history of the state into the system's evolution.

The operator A is assumed to be the infinitesimal generator of a strongly continuous semigroup, which provides a natural framework for studying such differential equations in Banach spaces. The initial condition for the problem is given by a continuous function $\Psi(t)$, which defines the state of the system over an interval $J_0 = [-t_0, 0]$, while the equation itself is valid over an interval $J = [0, t_f]$.

The primary objective of the article is to establish the existence and uniqueness of mild solutions to this fractional differential equation using the Banach fixed-point theorem. The existence proof is carried out by reformulating the problem as a fixed-point problem and applying appropriate conditions, such as a Lipschitz condition for the nonlinear function f, and ensuring the contractivity of the associated operator. Additionally, the article explores the theoretical foundations of fractional calculus, including the definition and properties of the fractional derivative D^{α}, and demonstrates how these can be applied to dynamic systems with memory.

The following sections of this work are organised as follows. Section 2 provides the fundamental preliminaries, including terminology and supplementary results employed throughout. In Section 3, we utilise the Banach fixed-point and Leray-Schauder alternative theorems to demonstrate our aim to problem (1)–(2). Finally, Section 4 presents illustrative examples that validate our theoretical findings.

2. Preliminaries

In this part, we will review some fundamental concepts, notation, and preliminary findings required for the sections that follow.

The Banach space of continuous functions from the interval I into the space E is denoted by $C(I, E)$. The norm of this space is determined as follows:

$$||h||_{\infty} = \sup_{t \in I} |h(t)|.$$

The Banach space of bounded linear operations from E into E is denoted by $B(E)$, with the norm as follows:

$$||N||_{B(E)} = \sup\{|N(u)| : \ |u| = 1\}$$

A Banach space of measurable functions defined on the set $I \to E$ is denoted by the notation $L^1(I, E)$. The norm of this space is defined as follows:

$$||h||_{L^1} = \int_I |h(t)| dt.$$

The set $\bar{B}_E(u_0, r) = \{u \in E : \ ||u - u_0||_E \leq r\}$ represents the closed ball with a radius of r when it is centred at u_0.

Consider the following problem:

$$D^{\alpha} u(t) = Au(t) + f(t, u_t), \ t \in J, \tag{1}$$

with initial condition

$$u(t) = \Psi(t), \ t \in J_0, \tag{2}$$

where $J = [0, t_f], \ J_0 = [-t_0, 0]$ are intervals, t_f and t_0 are both positive real numbers.

$f : J \times C(J_0, E) \to E$ is continuous function, An infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ is denoted by the expression $A : D(A) \subset E \to E$. The function $\Psi : J_0 \to E$ is a continuous function, and the space $(E, |.|)$ stands for the Banach space.

The function u_t is an element of $C([-t_0, 0], E)$, and it is defined as $u_t(\theta) = u(t + \theta)$ for $\theta \in J_0$. It shows how the state $u(t)$ has changed over time $[t - t_0, t]$.

According to Elnady et al. [4], the generalised local fractional derivative of order $\alpha \in (0, 1]$ is denoted by the symbol D^{α}. The definition of this derivative describes it as follows:

For $\alpha \in (0, 1)$, the α–derivative of a continuous function Φ is:

$$D^{\alpha}\Phi(t) = \lim_{\epsilon \to 0} \frac{(1 + \epsilon\gamma(\alpha))\Phi(t + \epsilon\eta(t, \alpha)) - \Phi(t)}{\epsilon}, \tag{3}$$

where $\gamma(\alpha)$ and $\eta(., \alpha)$ are continuous functions. If $\alpha = 1$, we choose $\gamma(\alpha) = 0$ and $\eta(t, \alpha) = 1$, then $D^{\alpha}\Phi(t) = \dfrac{d\Phi(t)}{dt}$. If $\alpha = 0$, we choose $\gamma(\alpha) = 1$ and $\eta(t, \alpha) = 0$, then $D^{\alpha}\Phi(t) = \Phi(t)$.

If the limit of $D^{\alpha}\Phi(t)$ as t approaches 0^+ exists, then the α-derivative of Φ at point $t_0 = 0$ is given by:

$D^{\alpha}\Phi(0) = \lim_{t\to 0^{+}} D^{\alpha}\Phi(t)$.

The following relation is used to determine the fractional integral of order α of a continuous function Φ:

$$I^{\alpha}\Phi(t) = \int_{0}^{t} \frac{1}{\eta(s,\alpha)} \exp\left(\int_{t}^{s} \frac{\gamma(\alpha)}{\eta(r,\alpha)} dr\right) \Phi(s) ds, \tag{4}$$

$\eta(s,\alpha) \neq 0$, for all $s \in (0,t]$ and $\alpha \in (0,1)$.

Definition 2.1. *The function u is considered to be a solution to the problem (1)-(2) if it is a continuous function on the set $J_0 \cup J$ and if it holds true under the following conditions:*

$$u(t) = \begin{cases} h(t)T(\phi(t))\Psi(0) + h(t) \displaystyle\int_{0}^{t} \frac{1}{\eta(s,\alpha)} g(s) T(\phi(t)-\phi(s)) f(s,u_s)) ds, & \text{if} \quad t \in J, \\ \\ \Psi(t), & \text{if} \quad t \in J_0, \end{cases} \tag{5}$$

where, the infinitesimal generator of the semigroup $T(t)$ is A and

$$\phi(t) = \int_{0}^{t} \frac{1}{\eta(\theta,\alpha)} d\theta,\ h(t) = \exp\left(-\gamma(\alpha)\phi(t)\right),\ \theta \in [0,t],\ t \in J,\ g(s) = \exp\left(\gamma(\alpha)\phi(s)\right),\ s \in [0,t].$$

3. Main Results

In this section we give our main existence result for problem (1)-(2).

Define the following constants:

$$\begin{aligned} M &= \sup\{||T(t)||_{B(E)}, t \in J\}, \\ \kappa &= \sup_{t\in J}\left[h(t)\int_{0}^{t}\left|\frac{1}{\eta(s,\alpha)}g(s)\right| ds\right], \\ \kappa_0 &= \sup_{t\in J}\left[h(t)\right]. \end{aligned}$$

Theorem 3.1. *Suppose that a positive constant c exists such that:*

$$|f(t,w_1) - f(t,w_2)| \leq c\|w_1 - w_2\|,$$

for all $t \in J$ and w_1, w_2 elements of $C(J_0,E)$.

If $cM\kappa < 1$, then, the problem (1)-(2) has a unique mild solution on $J_0 \cup J$.

Proof. We formulate the fractional differential equation as a fixed-point problem. Consider the operator S, defined as:

$$S : C(J_0 \cup J, E) \to C(J_0 \cup J, E)$$

$$S(u)(t)=\begin{cases} h(t)T(\phi(t))\Psi(0)+h(t)\int\limits_0^t \dfrac{1}{\eta(s,\alpha)}g(s)T(\phi(t)-\phi(s))f(s,u_s)ds, & \mathrm{t}\in J,\\ \\ \Psi(t), & \mathrm{t}\in J_0. \end{cases} \tag{6}$$

We demonstrate that S is a contraction. Let u, $v\in C(J_0\cup J,E)$ and $t\in J_0\cup J$, we have

$$\begin{aligned} |S(u)(t)-S(v)(t)| &= \left|h(t)\int\limits_0^t \frac{1}{\eta(s,\alpha)}g(s)T(\phi(t)-\phi(s))(f(s,u_s)-f(s,v_s))ds\right| \\ &\le Mh(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)}g(s)\right||f(s,u_s)-f(s,v_s)|ds \\ &\le cM||u_s-v_s||\left(h(t)\int\limits_0^t\left|\frac{1}{\eta(s,\alpha)}g(s)\right|ds\right) \\ &\le cM||u-v||\left(h(t)\int\limits_0^t\left|\frac{1}{\eta(s,\alpha)}g(s)\right|ds\right) \\ &\le cM\kappa||u-v||. \end{aligned}$$

Taking the supremum over t, hence

$$\|S(u)-S(v)\|_\infty\le cM\kappa||u-v||_\infty.$$

Therefore, under the Lipschitz continuity condition, $cM\kappa<1$, which ensures that the operator S is a contraction [2,11], there exists a unique mild solution u^* in $C(J_0\cup J,E)$, such that $Su^*=u^*$. □

In this part, we will present this theorem as a direct application of the nonlinear Leray-Schauder alternative, which offers a tool for demonstrating the existence of a mild solution to a nonlinear problem.

Theorem 3.2. [2] *Let C be a Banach space and $S:C\to C$ be a completely continuous operator. Let $F_0=\{u\in C,\ u=\nu S(u)$ for some $0<\nu<1\}$. Thus, either the set F_0 is unbounded, or the operator S has at least one fixed point.*

Theorem 3.3. *Assume that the following hypotheses are true:*

- (H_a) *For $t>0$, the semigroup $\{T(t)\},t\in J$ is compact;*
- (H_b) *$f:J\times C(J_0,E)\to E$ be a continuous function;*
- (H_c) *There are functions σ and τ in $C(J,\mathbb{R}_+)$ that satisfy the following:*

$$|f(t,u)|\le\sigma(t)+\tau(t)||u-\Psi(0)||_\infty,\ \textit{for } t\in J,u\in C(J_0,E),$$

Then, on $J_0 \cup J$, there is at least one mild solution to the problem (1)-(2).

Proof. The objective is to demonstrate that the operator S is not only continuous but also completely continuous.

Step 1: We shall show that S is continuous.

We will apply the continuity criterion for sequences, which states the following: If a sequence (u_n) converges to u, then the sequence $f(u_n)$ must converge to $f(u)$.

Let (u_n) be a sequence such that $u_n \to u$ in $C(J_0 \cup J, E)$, then,

$$\begin{aligned} |S(u_n)(t) - S(u)(t)| &= \left| h(t) \int_0^t \frac{1}{\eta(s,\alpha)} g(s) T(\phi(t) - \phi(s))(f(s, u_{n_s}) - f(s, u_s)) ds \right| \\ &\leq M||f(.,u_{n.}) - f(.,u_.)|| \left(h(t) \int_0^t \left| \frac{1}{\eta(s,\alpha)} g(s) \right| ds \right). \end{aligned}$$

By taking the supremum over t and assuming that f is a continuous function, we can conclude that S is continuous. This is because

$$||S(u_n) - S(u)|| \leq M\kappa ||f(.,u_{n.}) - f(.,u_.)||_\infty \to 0 \text{ as } n \to +\infty.$$

Step 2: S maps bounded sets into bounded sets in $C(J_0 \cup J, E)$.

We are looking for a nonnegative constant δ that will ensure that for any u that is included inside the closed ball $\bar{B}_C(\Psi(0), r)$, $S(u)$ is also an element of $\bar{B}_C(\Psi(0), \delta)$.

Let $u \in \bar{B}_C(\Psi(0), r)$ and $t \in J$, we have

$$\begin{aligned} |S(u)(t) - \Psi(0)| &= \left| h(t) T(\phi(t)) \Psi(0) - \Psi(0) + h(t) \int_0^t \frac{1}{\eta(s,\alpha)} g(s) T(\phi(t) - \phi(s)) f(s, u_s) ds \right| \\ &\leq \kappa_0 M |\Psi(0)| + |\Psi(0)| + M(||\sigma||_\infty + ||\tau||_\infty ||u - \Psi(0)||) \left(h(t) \int_0^t \left| \frac{1}{\eta(s,\alpha)} g(s) \right| ds \right). \end{aligned}$$

Taking the supremum over t, then,

$$||S(u) - \Psi(0)|| \leq \kappa_0 M ||\Psi(0)|| + ||\Psi(0)|| + \kappa M(||\sigma||_\infty + r||\tau||_\infty) =: \delta.$$

Therfore, by taking $\delta = \kappa_0 M ||\Psi(0)|| + ||\Psi(0)|| + \kappa M(||\sigma||_\infty + r||\tau||_\infty)$, we have S maps boundedsets into bounded sets in $C(J_0 \cup J, E)$.

Step 3: S maps bounded sets into equicontinuous sets of $C(J_0 \cup J, E)$.

Let t_1, t_2 be elements of the set J, with t_1 being less than or equal to t_2. Let $\bar{B}_C(\Psi(0), \delta)$ be a bounded set in $C(J_0 \cup J, E)$, as described in Step 2. Furthermore, let u be an element of the set $\bar{B}_C(\Psi(0), r)$, then

$$\begin{aligned}
|S(u)(t_2) - S(u)(t_1)| = & \left| h(t_2)T(\phi(t_2))\Psi(0) + h(t_2)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds \right. \\
- & \left. h(t_1)T(\phi(t_1))\Psi(0) - h(t_1)\int_0^{t_1} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_1)-\phi(s))f(s,u_s)ds \right| \\
\leq & \ |h(t_2)T(\phi(t_2)) - h(t_1)T(\phi(t_1))|\Psi(0) \qquad (\star) \\
+ & \left| h(t_2)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds \right. \\
- & \left. h(t_1)\int_0^{t_1} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_1)-\phi(s))f(s,u_s)ds \right| \qquad (\star\star)
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
(\star) &= |h(t_2)T(\phi(t_2)) - h(t_1)T(\phi(t_1))|\,\Psi(0) \\
&= |h(t_2)T(\phi(t_2)) - h(t_1)T(\phi(t_2)) + h(t_1)T(\phi(t_2)) - h(t_1)T(\phi(t_1)|\Psi(0) \\
&\leq M\Psi(0)|h(t_2) - h(t_1)| + u_0||h||_\infty |T(\phi(t_2)) - T(\phi(t_1)|
\end{aligned}$$

and

$$\begin{aligned}
(\star\star) &= \left| h(t_2)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds - h(t_1)\int_0^{t_1} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_1)-\phi(s))f(s,u_s)ds \right| \\
&= \left| h(t_2)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds - h(t_1)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds \right. \\
&+ \left. h(t_1)\int_0^{t_2} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_2)-\phi(s))f(s,u_s)ds - h(t_1)\int_0^{t_1} \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t_1)-\phi(s))f(s,u_s)ds \right| \\
&\leq M(||\sigma||_\infty + r||\tau||_\infty)|h(t_2) - h(t_1)| \int_0^{t_2} \left| \frac{1}{\eta(s,\alpha)} g(s) \right| ds \\
&+ (||\sigma||_\infty + r||\tau||_\infty)|h(t_1)| \int_0^{t_1} \left| \frac{1}{\eta(s,\alpha)} g(s) \right| |T(\phi(t_2)-\phi(s)) - T(\phi(t_1)-\phi(s))| ds \\
&+ M(||\sigma||_\infty + r||\tau||_\infty)|h(t_1)| \int_{t_1}^{t_2} \left| \frac{1}{\eta(s,\alpha)} g(s) \right| ds
\end{aligned}$$

Thus,

$$|S(x)(t_2) - S(x)(t_1)| \to 0, \text{ as } t_2 \to t_1.$$

which means that the set $S(\bar{B}_C(\Psi(0), r))$ is equicontinuous in $C(J_0 \cup J, E)$.

Step 4: (A priori bounds on solutions).

Our goal is to show that the set F_0 is defined as follows:

$$F_0 = \{u \in C(J_0 \cup J, E),\ u = \nu S(u) \text{ for some } 0 < \nu < 1\},$$

is bounded.

In other words, it involves finding a nonnegative constant R such that for all $u \in F_0$, the inequality $||u|| \leq R$ holds.

Let $u \in F_0$, we have

$$\begin{aligned}
|u(t)| &= \nu S(u)(t) \\
&= \nu \left(h(t)T(\phi(t))\Psi(0) + h(t)\int\limits_0^t \frac{1}{\eta(s,\alpha)} g(s)T(\phi(t)-\phi(s))f(s,u_s)ds \right) \\
&\leq ||h||M|\Psi(0)| + M\left(h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| (\sigma(s) + \tau(s)||u_s - \Psi(0)||)ds \right) \\
&\leq ||h||M|\Psi(0)| + M\left(h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| (\sigma(s) + \tau(s)(||u_s|| + ||\Psi(0)||))ds \right) \\
&\leq ||h||M|\Psi(0)| + M\left(h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| \sigma(s)ds + h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| \tau(s)||u_s||ds \right. \\
&+ \left. h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| \tau(s)||\Psi(0)||ds \right) \\
&\leq ||h||M|\Psi(0)| + M\left(||I^\alpha\sigma|| + ||\Psi(0)||||I^\alpha\tau|| + ||\tau|| \times h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| ||u_s||ds \right),
\end{aligned}$$

then,

$$|u(t)| \leq c_0 + c_1 \times h(t)\int\limits_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| ||u_s||ds, \tag{7}$$

where, $c_0 = M(||h||||\Psi(0)|| + ||I^\alpha\sigma|| + ||\Psi(0)||||I^\alpha\tau||)$ and $c_1 = M||\tau||$.

However, on the other hand, we have

for $s \in [0,t]$, we get $u_s(\theta) = u(s+\theta)$, where $\theta \in J_0 = [-t_0, 0]$:

$$-t_0 \leq \theta \leq 0 \Rightarrow s - t_0 \leq s + \theta \leq s \Rightarrow -t_0 \leq s + \theta \leq t, \text{ and } t \in J = [0, t_f].$$

We define a function λ on J by:

$$\lambda(t) = \sup_{\zeta \in [-t_0, t]} |u(\zeta)|. \tag{8}$$

we have

$$|u(t)| \leq \lambda(t), \quad t \in J \tag{9}$$

Let $\lambda(t) = |u(\zeta^*)|$, $\zeta^* \in [-t_0, t]$, we have two cases:

Case 01: If $\zeta^* \in [-t_0, 0] \Rightarrow \lambda(t) = |u(\zeta^*)| = \sup\limits_{\zeta \in [-t_0, 0]} |\Psi(\zeta)| = ||\Psi||$,

then

$$|u(t)| \leq ||\Psi||, \quad t \in J$$

This indicates that u is bounded.

Case 02: If $\zeta^* \in [0, t] \Rightarrow \lambda(t) = |u(\zeta^*)| = \sup\limits_{\zeta \in [0,t]} |u(\zeta)|$

Substituting the relation (8) into the inequality (7), we find:

$$|u(t)| \quad \leq \quad c_0 + c_1 \times h(t) \int\limits_0^t \left| \frac{1}{\eta(s, \alpha)} g(s) \right| \lambda(s) ds, \quad t \in J. \tag{10}$$

then

$$\begin{aligned} \lambda(t) &= |u(\zeta^*)| \\ &\leq c_0 + c_1 \times h(\zeta^*) \int\limits_0^{\zeta^*} \left| \frac{1}{\eta(s, \alpha)} g(s) \right| \lambda(s) ds \\ &\leq c_0 + c_1 \times h(\zeta^*) \int\limits_0^{t} \left| \frac{1}{\eta(s, \alpha)} g(s) \right| \lambda(s) ds \end{aligned}$$

Hence,

$$\lambda(t) \leq c_0 + c_1 h(\zeta^*) \int\limits_0^t \left| \frac{1}{\eta(s, \alpha)} g(s) \right| \lambda(s) ds \tag{11}$$

By using Gronwall's lemma [12,13], we can deduce that:

$$\lambda(t) \leq c_0 \exp \left(c_1 h(\zeta^*) \int\limits_0^t \left| \frac{1}{\eta(s, \alpha)} g(s) \right| ds \right) =: L \tag{12}$$

then

$$|u(t)| \leq L, \quad t \in J$$

That means u is bounded.

Therefore, based on cases 01 and 02, we can take $R = \max(L, ||\Psi||)$ for the set F_0 to be bounded.

After we demonstrated these steps, we deduce that the problem (1)-(2) has a mild solution on $J_0 \cup J$. □

4. Examples

Example 4.1. *Consider the following problem*

$$\begin{cases} D^{(0.5)}u(t) = -3u(t) + \dfrac{1}{50(2+t^4)}\sin(u_t), & \text{if } t \in [0,1], \\ \\ u(t) = 2t, & \text{if } t \in [-1,0]. \end{cases} \tag{13}$$

We take the generalized local fractional derivative as the deformable derivative [8], *then*

$$\gamma(\alpha) = 1-\alpha,\ \eta(t,\alpha) = \alpha.$$

We have $\alpha = \dfrac{1}{2}$ *and*

$$A = -3,\ f(t,u_t) = \frac{1}{50(2+t^4)}\sin(u_t),\ \Psi(t) = 2t,\ t_0 = -1,\ t_f = 1.$$

The functions $t \to \sin(t)$ *is Lipschitz functions with a Lipschitz constant* 1 *and*

$$\frac{1}{50(2+t^4)} \le \frac{1}{100},\ \text{for } t \in [0,1],$$

therefore, we have $c = \dfrac{1}{100}$.

We have $M = 1$ *and* $\phi(t) = \displaystyle\int_0^t \frac{1}{\eta(\theta,\alpha)}d\theta = \frac{1}{\alpha}t$, $h(t) = \exp\left(-\gamma(\alpha)\phi(t)\right) = e^{-\frac{1-\alpha}{\alpha}t}$ *and* $g(s) = \exp\left(\gamma(\alpha)\phi(s)\right) = e^{\frac{1-\alpha}{\alpha}s}$, *then*

$$\begin{aligned} \kappa &= \sup_{t\in J}\left[h(t)\int_0^t \left|\frac{1}{\eta(s,\alpha)}g(s)\right| ds\right] \\ &= \sup_{t\in J}\left[e^{-\frac{1-\alpha}{\alpha}t}\int_0^t \left|\frac{1}{\alpha}e^{\frac{1-\alpha}{\alpha}s}\right| ds\right] \\ &= \sup_{t\in J}\left[\frac{1}{1-\alpha}e^{-\frac{1-\alpha}{\alpha}t}\left(e^{\frac{1-\alpha}{\alpha}t} - 1\right)\right] \\ &= \sup_{t\in J}[2(1-e^{-t})] = 2(1-e^{-1}). \end{aligned}$$

Thus,

$$cM\kappa = \frac{2}{100}(1-e^{-1}) = 0.0126 < 1$$

Therefore, we have conditions of theorem 3.1 *satisfied; then the problem considered* (13) *has a unique mild solution on* $J_0 \cup J = [-1,1]$.

Remark 4.1. *Additionally, the conditions of theorem* 3.3 *are satisfied, with,*

$$\sigma(t) = 0 \text{ and } \tau(t) = \frac{1}{50(2+t^4)},\ t \in J,\ \Psi(0) = 0.$$

Therefore, the above problem (13) *has at least one mild solution.*

Example 4.2. *In above example, we take the generalized local fractional derivative as the β-derivative [9,10], then*

$$\gamma(\alpha) = 0, \eta(t,\alpha) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}.$$

Then

$$\phi(t) = \int_0^t \frac{1}{\eta(\theta,\alpha)} d\theta = \int_0^t \left(\theta + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1} d\theta = \frac{1}{\alpha}\left[\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \left(\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right],$$

$h(t) = \exp(-\gamma(\alpha)\phi(t)) = 1$ *and* $g(s) = \exp(\gamma(\alpha)\phi(s)) = 1$, *then*

$$\begin{aligned}
\kappa &= \sup_{t\in J}\left[h(t)\int_0^t \left|\frac{1}{\eta(s,\alpha)} g(s)\right| ds\right] \\
&= \sup_{t\in J}\left[\int_0^t \left(s + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1} ds\right] \\
&= \sup_{t\in J}\left[\frac{1}{\alpha}\left(\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \left(\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)\right] \\
&= \sup_{t\in J}\left[2\left(\sqrt{t + \frac{1}{\sqrt{\pi}}} - \sqrt{\frac{1}{\sqrt{\pi}}}\right)\right] = 2\left(\sqrt{1 + \frac{1}{\sqrt{\pi}}} - \sqrt{\frac{1}{\sqrt{\pi}}}\right)
\end{aligned}$$

Thus,

$$cM\kappa = \frac{2}{100}\left(\sqrt{1 + \frac{1}{\sqrt{\pi}}} - \sqrt{\frac{1}{\sqrt{\pi}}}\right) = 0.0100 < 1$$

Therefore, we have conditions of theorem 3.1 satisfied; then the problem considered (13) has a unique mild solution on $J_0 \cup J = [-1,1]$.

Conclusion

In this article, we have presented a rigorous analysis of fractional differential equations with delays under generalized local fractional derivatives, focusing on the existence and uniqueness of mild solutions. By leveraging the Banach fixed-point theorem and the nonlinear Leray-Schauder alternative, we demonstrated that under certain conditions, the problem admits a unique solution. The work has yielded significant insights into the theoretical foundations of fractional calculus and its applications. The use of history-dependent functions and semigroup theories has proven effective for modelling complex systems. The theory has demonstrated its effectiveness across various fields, including control theory and applied mathematics.

Future research could extend these results in several domains. One promising avenue is the exploration of more complex systems, offering a more flexible model for real-world applications. Additionally, it would be valuable to investigate numerical methods for approximating solutions to such equations, particularly when exact solutions are difficult to obtain. The integration of fractional calculus with other mathematical frameworks, such as stochastic processes or optimization theory, could potentially stimulate research in both theoretical and applied contexts. In the end, using these models in real-world engineering and physical systems, such as robotics, biomedical engineering, and climate modeling, could help demonstrate the real usefulness of fractional differential equations.

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