$(l,r)^*-$BESSEL NUMBERS AND SOME COMBINATORIAL PROPERTIES

SWITENIA MAE OMPOC*, MARY JOY R. LATAYADA

Department of Mathematics, Caraga State University, Philippines

*Corresponding author: switeniamae.ompoc@carsu.edu.ph

Received Apr. 8, 2026

Abstract. This paper introduces and investigates the $(l,r)^*-$Bessel numbers, a novel combinatorial family that combines the block-size restrictions of Bessel numbers, distinguished elements of $r-$Bessel numbers, and multi-structure enumeration of $(l,r)-$Stirling numbers. These numbers count ordered $l-$tuples of $r-$partitions of an $[n+r]-$element set into $k+r$ blocks of size at most two, where all partitions share the same set of block maxima. We establish some combinatorial properties, including horizontal and vertical recurrence relations, expressed them explicitly as a multiple sum, and present the difference-differential equations satisfied by their row and column generating functions, respectively.

2020 Mathematics Subject Classification. 05A10; 05A18; 05A19; 11B37.

Key words and phrases. $(l,r)^*-$Bessel numbers; $r-$Bessel numbers; block maxima; restricted partitions; generating functions, recurrences.

1. Introduction

The Stirling numbers of the first kind, $\left[{n \atop k}\right]$, and second kind, $\left\{{n \atop k}\right\}$, are among the most fundamental concepts in combinatorics, counting permutations by cycles and partitions by blocks, respectively. A natural restriction emerges when we limit the block sizes. The Bessel numbers of the second kind, $\left\{{n \atop k}\right\}^{\leq 2}$, count partitions of an $[n]-$element set into k blocks each of size at most two [5], [9].

In the 1980's, Broder introduced the $r-$Stirling numbers of the first kind, $\left[{n \atop k}\right]_r$, and second kind, $\left\{{n \atop k}\right\}_r$, which require that the first $r-$ elements must be in distinct cycles or blocks [3]. This distinguished-element parameterization preserves many classical properties while adding a new layer of structure.

The synthesis of block-size restriction and distinguished element led to the $r-$Bessel numbers $\left\{{n \atop k}\right\}_r^{\leq 2}$, also known as restricted Stirling numbers, which count partitions of an $[n+r]-$element set into $k+r$ blocks of size at most two, where $n+1, n+2, \cdots, n+r$ are in distinct blocks [1].

Stirling and $r-$ Stirling numbers are further generalized by the introduction of $(l,r)-$Stirling numbers of the first kind, $\left[{n \atop k}\right]_r^{(l)}$, and second kind, $\left\{{n \atop k}\right\}_r^{(l)}$, which counts the ordered $l-$tuples of permutations

DOI: 10.28924/APJM/13-88

and partitions sharing identical cycle and block leaders, with the first r elements required to be among these leaders [2].

In this paper, we define and explore some combinatorial properties of $(l,r)^*-$Bessel numbers, which combine the block-size restriction of Bessel numbers, the distinguished-element of $r-$Bessel numbers, and the multi-structure enumeration of $(l,r)-$Stirling numbers, with the additional twist of using block maxima instead of minima.

2. Preliminaries

In this section, we recall some of the basic concepts in combinatorics which are necessary for this study.

Definition 1. [1] A partition of the elements $1,2,...,n+r$ is called an $r-$partition if $n+1,n+2,...,n+r$ belong to distinct blocks.

Definition 2. [4] The Addition Principle. Assume that there are

$$\begin{array}{llll} n_1 & \text{ways for the event} & E_1 & \text{to occur,} \\ n_2 & \text{ways for the event} & E_2 & \text{to occur,} \\ & & \vdots & \\ n_k & \text{ways for the event} & E_k & \text{to occur,} \end{array}$$

where $k \geq 1$. If these ways for the different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \cdots$, or E_k to occur is $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$.

Definition 3. [6] The falling factorial $x^{\underline{n}}$ is defined by

$$x^{\underline{n}} = x(x-1)\cdots(x-(n-1))$$

for $n \geq 0$, where $x^{\underline{0}} = 1$.

Theorem 1. [4] The Binomial Theorem. For any integer $n \geq 0$,

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r.\end{aligned}$$

Definition 4. [6] The Stirling number of the second kind, $\left\{{n \atop k}\right\}$, counts the number of partitions of the set $[n]$ into k nonempty blocks.

Definition 5. [6] A recurrence is an equation that expresses the n^{th} term a_n of a sequence in terms of values of earlier terms, a_i for $i < n$.

Definition 6. [7] An ordinary differential equation is an equation that involves the derivatives of one or more dependent variables with respect to a single independent variable.

Definition 7. [8] A difference-differential equation is a two-variable equation consisting of a coupled ordinary differential equation and recurrence relation.

3. The $(l, r)^*-$Bessel Numbers

In this section, we will establish the $(l, r)^*-$ Bessel numbers. We begin by introducing a new concept that serves as the foundation for our new combinatorial family.

Definition 8. Let π be a partition of the set $[n]$ into k blocks $b_1, b_2, ..., b_k$. A set of block maximum of π, denoted by $bm(\pi)$, is the set of the largest elements on their blocks, that is,

$$bm(\pi) = \{\max b_1, \max b_2, ..., \max b_k\}.$$

Definition 9. The $(l, r)^*-$Bessel numbers, $\left\{ {n \atop k} \right\}_r^{\leq 2(l)}$, counts the number of $l-$tuple of $r-$partitions of an $(n + r)-$element set into $k + r$ nonempty blocks with the restriction that each block contains at most two elements, and the partitions have exactly the same set of block maximum, i.e.,

$$bm(\pi_1) = bm(\pi_2) = \cdots = bm(\pi_l).$$

Example 1. Compute $\left\{ {3 \atop 1} \right\}_2^{\leq 2(2)}$. Here $n = 3$, $k = 1, l = r = 2$. Constructing all partitions of the set $[3 + 2] = 5 = \{1, 2, 3, 4, 5\}$ into $1 + 2 = 3$ blocks, such that $3 + 1 = 4$ and $3 + 2 = 5$ are in distinct blocks. There are 12 such partitions:

$$\begin{aligned}
\pi_1 &= \{\{1, 4\}, \{2, 5\}, \{3\}\}, & \pi_7 &= \{\{1, 4\}, \{2, 3\}, \{5\}\},\\
\pi_2 &= \{\{1, 4\}, \{2\}, \{3, 5\}\}, & \pi_8 &= \{\{1, 3\}, \{2, 4\}, \{5\}\},\\
\pi_3 &= \{\{1, 5\}, \{2, 4\}, \{3\}\}, & \pi_9 &= \{\{1, 2\}, \{3, 4\}, \{5\}\},\\
\pi_4 &= \{\{1\}, \{2, 4\}, \{3, 5\}\}, & \pi_{10} &= \{\{1, 5\}, \{2, 3\}, \{4\}\},\\
\pi_5 &= \{\{1, 5\}, \{2\}, \{3, 4\}\}, & \pi_{11} &= \{\{1, 3\}, \{2, 5\}, \{4\}\},\\
\pi_6 &= \{\{1\}, \{2, 5\}, \{3, 4\}\}, & \pi_{12} &= \{\{1, 2\}, \{3, 5\}, \{4\}\},
\end{aligned}$$

and the sets of block maximum are

$$\begin{aligned}
\{1, 4, 5\} &= bm(\pi_4) = bm(\pi_6),\\
\{2, 4, 5\} &= bm(\pi_2) = bm(\pi_5) = bm(\pi_9) = bm(\pi_{12}),\\
\{3, 4, 5\} &= bm(\pi_1) = bm(\pi_3) = bm(\pi_7) = bm(\pi_8) = bm(\pi_{10}) = bm(\pi_{11}).
\end{aligned}$$

Counting the number of ordered 2-tuples of partitions such that they share the same set of block maximum. As there are 2 partitions with the set of block maximum $\{1,4,5\}$, 4 with block maximum set $\{2,4,5\}$, and 6 with block maximum set $\{3,4,5\}$, thus,

$$\left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\}_2^{\leq 2(2)} = 2^2 + 4^2 + 6^2 = 56.$$

Theorem 2. For $1 \leq k \leq n$, $r \geq 0$, and $l \geq 1$, the $(l,r)^*-$Bessel numbers satisfy the horizontal recurrence,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^{\leq 2(l)} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_r^{\leq 2(l)} + (2k+r+1-n)^l \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_r^{\leq 2(l)}. \tag{1}$$

Proof. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_l)$ be an $l-$tuple of $r-$partitions of the set $[n+r] = \{1, \cdots, n+r\}$ into $k+r$ blocks such that the following conditions are satisfied:

(1) each block contains at most two elements,
(2) the distinguished elements $n+1, \ldots, n+r$ lie in distinct blocks,
(3) all partitions share the same set of block maxima.

We consider the smallest element 1. In each partition π_i, the element 1 either form a singleton block or belongs to a two-element block. Since all l partitions share the same block maxima set, the status of 1 as a block maximum or not must be the same across the whole l-tuple. Each $l-$tuple π_i can be classified into two types:

Type A:. The element 1 is alone in its block in every partition. Then 1 is a block maximum. Deleting the block $\{1\}$ from each partition yields an $l-$tuple of $r-$partitions of the set $[n-1+r] = \{2, \ldots, n+r\}$ into $(k-1)+r$ blocks, where conditions (1)–(3) are still satisfied. This reduced tuple is counted by $\left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}_r^{\leq 2(l)}$.

Type B:. The element 1 belongs to a doubleton block with some larger element x_i. Then 1 is not a block maximum and removing 1 from each partition does not change the block maxima set. The resulting $l-$tuple consists of $r-$partitions of $[n-1+r]-$element set into $k+r$ blocks, still satisfying conditions (1)–(3). Consequently, it is counted by $\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^{\leq 2(l)}$.

Now fix such a reduced tuple α counted by $\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^{\leq 2(l)}$. We count how many Type-B tuples give rise after deleting 1. In each partition α_i we must insert the element 1 into a block. Since each block can have at most two elements, 1 can only be placed into a singleton block. Let us determine the number of singleton blocks in α_i. Let t be the number of doubleton blocks. Since there are $k+r$ blocks and a total of $n-1+r$ elements, then the number of elements is $2t + ((k+r)-t) = k+r+t$. Equating to

$n-1+r$ gives $t=n-1-k$. Hence the number of singleton blocks is

$$(k+r)-t=k+r-(n-1-k)=2k+r+1-n.$$

Denote this number by s. Note that the number of singleton blocks is the same for every α_i in the tuple. This means that for a fixed tuple, there are s choices of a singleton block in α_i, and these choices are independent across the l partitions. Thus, the number of ways to insert 1 into all l partitions simultaneously is $s^l = (2k+r+1-n)^l$. Moreover, each such insertion yields a distinct Type-B tuple, and every Type-B tuple reduces uniquely to some α. Hence the number of Type-B tuples is $(2k+r+1-n)^l \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)}$.

By definition 2,

$$\left\{ {n \atop k} \right\}_r^{\leq 2(l)} = \left\{ {n-1 \atop k-1} \right\}_r^{\leq 2(l)} + (2k+r+1-n)^l \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)}.$$

□

Remark 1. For all $n,k,r\geq 0$, and $l\geq 1$, the boundary conditions

$$\text{a.)}\ \left\{ {n \atop k} \right\}_r^{\leq 2(l)} = 0 \quad \text{when} \quad n<k \quad \text{or} \quad n>2k+r,$$

$$\text{b.)}\ \left\{ {n \atop k} \right\}_r^{\leq 2(l)} = 1 \quad \text{when} \quad n=k.$$

We can also prove a vertical recurrence for $(l,r)^*-$ Bessel numbers.

Theorem 3. For $0\leq k\leq n$, $r\geq 0$, and $l\geq 1$, the $(l,r)^*-$Bessel numbers satisfy the vertical recurrence,

$$\left\{ {n+1 \atop k+1} \right\}_r^{\leq 2(l)} = \sum_{j=k}^{n} \left[(2k+r+1-j)^{\underline{n-j}} \right]^l \left\{ {j \atop k} \right\}_r^{\leq 2(l)}. \tag{2}$$

Proof. Consider an ordered l-tuple $\pi=(\pi_1,\ldots,\pi_l)$ of r-partitions of $[n+1+r]=\{1,\ldots,n+1+r\}$ into $k+1+r$ blocks satisfying the three conditions:

(1) each block contains at most two elements,
(2) the distinguished elements $n+2,\ldots,n+1+r$ lie in distinct blocks,
(3) all partitions share the same set of block maxima.

We count such tuples by conditioning on the smallest block maximum.
Let S be the common set of block maxima. The r distinguished numbers are the largest, so they belong to S. Denote m as the smallest element of S, then m is necessarily non-distinguished.
Note that the non-distinguished numbers are $1,\ldots,n+1$, where $m\in\{1,\cdots,n+1\}$.

Let j be the number of non-distinguished elements larger than m, then $j = n + 1 - m$.

Note that there are $k + 1$ non-distinguished blocks, then besides the block containing m, there are other k non-distinguished maxima, all larger than m, so $j \geq k$ and the smallest possible value of $m = 1$, giving $j = n$. This implies that the possible values of j range from k to n, that is $k \leq j \leq n$.

Fix the value of j. The elements larger than m are the j non-distinguished numbers $m+1, \ldots, n+1$ and the r distinguished numbers $n+2, \ldots, n+1+r$. In each partition π_i, these elements form exactly $k + r$ blocks. Removing m restricts each π_i to an r-partition of the set of size $j + r$ into $k + r$ blocks that satisfies conditions (1)-(3). Thus, the number of ways to choose this restriction is $\left\{ {j \atop k} \right\}_r^{\leq 2(l)}$, by definition 9. Since the element m itself is a block maximum, it cannot be paired with any larger element, so it appears as a singleton block in every π_i.

Now consider the elements smaller than m. That is, $1, \cdots, m-1$, so there are $m - 1 = n - j$ of them. Note that each partition contains a fixed number of singleton blocks. For a single partition, let t be the number of two-element blocks. The total number of elements for a $j+r$ set is $2t + ((k+r) - t) = k + r + t$ which implies $t = j - k$. Consequently, the number of singleton blocks is $(k + r) - (j - k) = 2k + r - j$. Adding the singleton block of m gives a total of

$$s = 2k + r + 1 - j$$

singleton blocks in each partition.

We will then insert the $n - j$ smallest elements into the singleton blocks, one element per block, thereby, turning each such block into a two-element block. For a single partition, there are s singleton blocks. Since there are $n-j$ elements to insert to the first singleton blocks, then $n-j-1$ to the second singleton blocks. Following this step, the number of ways to insert $n - j$ distinct elements to s distinct singleton blocks is the falling factorial $s^{\underline{n-j}}$. Since the l partitions are independent, the total number of ways to insert the smaller elements into all l partitions simultaneously is $(s^{\underline{n-j}})^l = \left((2k + r + 1 - j)^{\underline{n-j}}\right)^l$. Thus, for a fixed j we obtain,

$$\left((2k + r + 1 - j)^{\underline{n-j}}\right)^l \left\{ {j \atop k} \right\}_r^{\leq 2(l)}.$$

Summing over all possible j from k to n to account for every possible smallest block maximum m, hence,

$$\left\{ {n+1 \atop k+1} \right\}_r^{\leq 2(l)} = \sum_{j=k}^{n} \left((2k + r + 1 - j)^{\underline{n-j}}\right)^l \left\{ {j \atop k} \right\}_r^{\leq 2(l)}.$$

$\square$

Using Theorems 2 and 3, Tables 1 and 2 show the values of $(l, r)-$Bessel numbers for $n, k \leq 9$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	9	1	0	0	0	0	0	0	0	0
2	36	25	1	0	0	0	0	0	0	0
3	36	261	50	1	0	0	0	0	0	0
4	0	1080	1061	86	1	0	0	0	0	0
5	0	1080	10629	3211	135	1	0	0	0	0
6	0	0	43596	62005	8071	199	1	0	0	0
7	0	0	43596	601641	263780	17822	280	1	0	0
8	0	0	0	2450160	4822121	905372	35742	380	1	0
9	0	0	0	2450160	45849249	27456421	2656730	66522	501	1

TABLE 1. $(2, 3)^*$-Bessel Numbers

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	8	1	0	0	0	0	0	0	0	0
2	8	35	1	0	0	0	0	0	0	0
3	0	288	99	1	0	0	0	0	0	0
4	0	288	2961	224	1	0	0	0	0	0
5	0	0	23976	17297	440	1	0	0	0	0
6	0	0	23976	490995	72297	783	1	0	0	0
7	0	0	0	3951936	5118003	241425	1295	1	0	0
8	0	0	0	3951936	142138017	35296128	685610	2024	1	0
9	0	0	0	0	1141056072	2401090209	183387888	1721898	3024	1

TABLE 2. $(3, 2)^*$-Bessel Numbers

Example 2. Consider $\left\{ {7 \atop 4} \right\}_3^{\leq 2(2)}$, that is $n = 7, k = 4, r = 3$, and $l = 2$. Using Theorem 2, and the values in Table 1,

$$\begin{aligned} \left\{ {7 \atop 4} \right\}_3^{\leq 2(2)} &= \left\{ {6 \atop 3} \right\}_3^{\leq 2(2)} + (5)^2 \left\{ {6 \atop 4} \right\}_3^{\leq 2(2)} \\ &= 62005 + 25(8071) \\ &= 263780. \end{aligned}$$

Using the same values, and Theorem 3 gives

$$\begin{aligned}
\left\{ {n+1 \atop k+1} \right\}_r^{\leq 2(l)} &= \left\{ {7 \atop 4} \right\}_3^{\leq 2(2)} \\
&= \sum_{j=3}^{6} \left[(10-j)^{\underline{6-j}} \right]^2 \left\{ {j \atop 3} \right\}_3^{\leq 2(2)} \\
&= \left[(7)^{\underline{3}} \right]^2 \left\{ {3 \atop 3} \right\}_3^{\leq 2(2)} + \left[(6)^{\underline{2}} \right]^2 \left\{ {4 \atop 3} \right\}_3^{\leq 2(2)} + \left[(5)^{\underline{1}} \right]^2 \left\{ {5 \atop 3} \right\}_3^{\leq 2(2)} + \left[(4)^{\underline{0}} \right]^2 \left\{ {6 \atop 3} \right\}_3^{\leq 2(2)} \\
&= (210)^2(1) + (30)^2(86) + (5)^2(3211) + (1)^2(62005) \\
&= 263780.
\end{aligned}$$

Proposition 1. For $n, r \geq 0$ and $l \geq 1$,

a. $$\left\{ {n \atop n-1} \right\}_r^{\leq 2(l)} = \sum_{j=r}^{n+r-1} j^l.$$

b. $$\left\{ {n \atop 0} \right\}_r^{\leq 2(l)} = (r^{\underline{n}})^l = \left(\frac{r!}{(r-n)!} \right)^l.$$

Proof.

a. Let S be the set of block maxima. Since there are $n-1+r$ blocks, S has $n-1+r$ elements. The r distinguished elements are always maxima, so they are contained in S. This means that exactly one non-distinguished element is *not* a block maximum Let x be this element. In each partition π_i, the element x must belong to a two-element block whose other element is a block maximum. All other non-distinguished elements are maxima and therefore appear either as singletons or as the larger element of a two-element block.

Consider a fixed $x \in \{1, \ldots, n\}$. For a single partition, the possible partners for x are:

- any of the r distinguished elements;
- any non-distinguished maximum that is larger than x.

Since the non-distinguished maxima are all elements of $\{1, \ldots, n\}$ except x, the ones larger than x are exactly $x+1, \ldots, n$, so there are $n-x$ of them. Thus, the number of possible partners in one partition is

$$n - x + r.$$

Denote this number by $a_x = n - x + r$. For a given x, the block maxima set is completely determined, so the condition that all l partitions share the same set of block maxima is automatically satisfied once x is fixed.

In an l-tuple of partitions with this common x, the partner of x in each π_i can be chosen independently from the a_x elements. Thus, the number of ordered l-tuples with the given x is $(a_x)^l$.

Now x can be any of the n non-distinguished numbers. Consequently,

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}_r^{\leq 2(l)} = \sum_{x=1}^{n} (n-x+r)^l.$$

Let $j = n - x + r$. Reversing the order of summation gives

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}_r^{\leq 2(l)} = \sum_{j=r}^{n+r-1} j^{\,l}.$$

$\square$

b. By definition, $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_r^{\leq 2(l)}$ counts the ordered l-tuples $(\pi_1, \ldots, \pi_l)$ of r-partitions of $[n+r] = \{1, \ldots, n+r\}$ into $0 + r = r$ blocks satisfying:

(1) each block contains at most two elements,

(2) the distinguished elements $n+1, \ldots, n+r$ lie in distinct blocks,

(3) all partitions share the same set of block maxima.

Since there are exactly r blocks and the r distinguished elements must be in distinct blocks, each block must contain exactly one distinguished element. Thus, the set of block maxima consists precisely of the r distinguished elements, which implies that no non-distinguished element can be a block maximum. Then every non-distinguished element must be placed in a block that already contains a distinguished element.

Note that n cannot be greater than r, as there cannot be more than elements to the number of blocks.

Assume $n \leq r$. In a single partition, the assignment of the n non-distinguished elements to the r blocks is an injection. We must choose a distinct block for each non-distinguished element. The number of such injections is the falling factorial

$$r^{\underline{n}} = r(r-1)\cdots(r-n+1) = \frac{r!}{(r-n)!}.$$

The set of block maxima is always the set of distinguished elements. Thus, for a fixed injection, all l partitions automatically satisfy condition (3), and the choices for different partitions are independent. Hence the number of ordered l-tuples is the l^{th} power of the number of single partitions:

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_r^{\leq 2(l)} = \left(r^{\underline{n}}\right)^l = \left(\frac{r!}{(r-n)!}\right)^l.$$

$\square$

Theorem 4. The $(l, r)^*-$Bessel numbers can be expressed explicitly as

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^{\leq 2(l)} = \sum_{1 \leq j_1 < j_2 < \cdots < j_{n-k} \leq n} \prod_{i=1}^{n-k} (j_i - 2i + r + 1)^l. \tag{3}$$

Proof. Let $m = n - k$ and define

$$F(n,k,r,l) = \sum_{1 \le j_1 < \cdots < j_m \le n} \prod_{i=1}^{m} (j_i - 2i + r + 1)^l.$$

We will show that F satisfies the same horizontal recurrence and boundary conditions as $\left\{ {n \atop k} \right\}_r^{\le 2(l)}$. Since $\left\{ {n \atop k} \right\}_r^{\le 2(l)}$ is uniquely determined by these conditions, equality follows.

Boundary conditions.

- If $k = n$ then $m = 0$. Thus,

$$F(n,n,r,l) = \sum_{1 \le j_1 < j_2 < \cdots < j_m \le n} \prod_{i=1}^{0} (j_i - 2i + r + 1)^l = 1$$

- If $k = 0$ then $m = n$. The only increasing sequence is $j_i = i$ for $i = 1, \ldots, n$. Thus,

$$F(n,0,r,l) = \sum_{1,2,\cdots,n} \prod_{i=1}^{n} (i - 2i + r + 1)^l = \prod_{i=1}^{n} (r + 1 - i)^l = (r^{\underline{n}})^l.$$

- If $n < k$, then

$$F(n,k,r,l) = \sum_{1 \le j_1 < j_2 < \cdots < j_m \le n} \prod_{i=1}^{-m} (j_i - 2i + r + 1)^l = 0.$$

This implies that F agrees with $\left\{ {n \atop k} \right\}_r^{\le 2(l)}$ in these cases.

Horizontal recurrence. Separate the sum according to whether $j_m = n$ or $j_m \le n - 1$.

- Case 1: When $j_m = n$. Then $j_1, \ldots, j_{m-1}$ is an increasing sequence of length $m - 1$ from $\{1, \ldots, n-1\}$. The last factor is

$$(n - 2m + r + 1)^l = (n - 2(n-k) + r + 1)^l = (2k + r + 1 - n)^l.$$

  The product over the first $m - 1$ indices gives $F(n-1,k,r,l)$. Hence, this part contributes $(2k + r + 1 - n)^l F(n-1,k,r,l)$.

- Case 2: When $j_m \le n - 1$. Then all j_i lie in $\{1, \ldots, n-1\}$ and the length is $m = (n-1) - (k-1)$. This is exactly the sum defining $F(n-1,k-1,r,l)$.

Adding the two cases yields

$$F(n,k,r,l) = F(n-1,k-1,r,l) + (2k + r + 1 - n)^l \, F(n-1,k,r,l). \tag{4}$$

Now, we prove by induction on n that $F(n,k,r,l) = 0$ whenever $2k + r < n$.

If $2k + r < n$, the smallest possible value of $n = 2k + r + 1$. Using Equation 4,

$$F(2k+r+1,k,r,l) = F(2k+r,k-1,r,l) + 0^l \cdot F(2k+r,k,r,l) = F(2k+r,k-1,r,l).$$

Now consider $(n', k') = (2k + r,\, k - 1)$. We have $2(k-1) + r = 2k + r - 2 < 2k + r = n'$, so the condition $2k' + r < n'$ holds.

Assume that $n > 2k + r + 1$ and that for all smaller n', Equation 4 is true.

$$F(n,k,r,l) = F(n-1,k-1,r,l) + (2k + r + 1 - n)^l\, F(n-1,k,r,l).$$

For the first term,

$$2(k-1) + r = 2k + r - 2.$$

Since $n > 2k + r + 1$, we have $n - 1 \geq 2k + r + 1 > 2k + r - 2$, thus $n - 1 > 2(k-1) + r$. By the induction hypothesis, $F(n-1,k-1,r,l) = 0$.

For the second term, since $n > 2k + r + 1$, then $n - 1 > 2k + r$, so again by the induction hypothesis $F(n-1,k,r,l) = 0$. The coefficient is non-zero but multiplies zero. Hence $F(n,k,r,l) = 0$. Thus $F(n,k,r,l) = 0$ for all n, k with $2k + r < n$.

Since F satisfies the same recurrence and boundary conditions as $\left\{ {n \atop k} \right\}_r^{\leq 2(l)}$, and these conditions uniquely determine the numbers, we conclude

$$\left\{ {n \atop k} \right\}_r^{\leq 2(l)} = F(n,k,r,l) = \sum_{1 \leq j_1 < \cdots < j_{n-k} \leq n} \prod_{i=1}^{n-k} (j_i - 2i + r + 1)^l.$$

□

Example 3. Using the same values in Example 2, and Theorem 4,

$$\begin{aligned}
\left\{ {7 \atop 4} \right\}_3^{\leq 2(2)} &= \sum_{1 \leq j_1 < j_2 < \cdots < j_{n-k} \leq 7} \prod_{i=1}^{3} (j_i - 2i + 4)^2 \\
&= \sum_{1 \leq j_1 < j_2 < j_3 \leq 7} \left[(j_1 + 2)^2 (j_2)^2 (j_3 - 2)^2 \right] \\
&= (1+2)^2(2)^2(3-2)^2 + (1+2)^2(3)^2(4-2) + \cdots + (5+2)^2(6)^2(7-2)^2 \\
&= 263780.
\end{aligned}$$

Theorem 5. For $n, r \geq 0$, and $l \geq 1$, the row generating function

$$R_n^{(l,r)}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\leq 2(l)} x^k$$

of $(l,r)^*-$Bessel numbers, which is a polynomial in x of degree n satisfies the difference-differential equation,

$$R_n^{(l,r)}(x) = x R_{n-1}^{(l,r)}(x) + \sum_{j=0}^{l} \binom{l}{j} 2^j (r + 1 - n)^{l-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} x^i \frac{d^i}{dx^i} \left(R_{n-1}^{(l,r)}(x) \right),$$

where $R_0^{(l,r)}(x) = 1$.

Proof. Suppose $R_n^{(l,r)}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\leq 2(l)} x^k$ be the row generating function of $(l,r)^*-$ Bessel numbers. By Theorem 2,

$$\begin{aligned} R_n^{(l,r)}(x) &= \sum_{k=0}^{n} \left[\left\{ {n-1 \atop k-1} \right\}_r^{\leq 2(l)} + (2k+r+1-n)^l \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} \right] x^k \\ &= \sum_{k=0}^{n} \left\{ {n-1 \atop k-1} \right\}_r^{\leq 2(l)} x^k + \sum_{k=0}^{n} (2k+r+1-n)^l \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^k. \end{aligned}$$

Let $j = k-1$, then

$$\begin{aligned} \sum_{k=0}^{n} \left\{ {n-1 \atop k-1} \right\}_r^{\leq 2(l)} x^k &= \sum_{j=-1}^{n} \left\{ {n-1 \atop j} \right\}_r^{\leq 2(l)} x^{j+1} \\ &= \sum_{j=0}^{n} \left\{ {n-1 \atop j} \right\}_r^{\leq 2(l)} x^j x \\ &= x \sum_{j=0}^{n} \left\{ {n-1 \atop j} \right\}_r^{\leq 2(l)} x^j \\ &= x R_{n-1}^{(l,r)}(x). \end{aligned}$$

By Binomial Theorem,

$$\begin{aligned} (2k+r+1-n)^l &= \sum_{j=0}^{l} \binom{l}{j} (2k)^j (r+1-n)^{l-j} \\ &= \sum_{j=0}^{l} \binom{l}{j} 2^j k^j (r+1-n)^{l-j}. \end{aligned}$$

$$\begin{aligned} \text{Let} \quad D &= \sum_{k=0}^{n} (2k+r+1-n)^l \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^k \\ &= \sum_{k=0}^{n} \sum_{j=0}^{l} \binom{l}{j} 2^j k^j (r+1-n)^{l-j} \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^k \\ &= \sum_{j=0}^{l} \binom{l}{j} 2^j (r+1-n)^{l-j} \sum_{k=0}^{n} k^j \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^k \\ &= \sum_{j=0}^{l} \binom{l}{j} 2^j (r+1-n)^{l-j} \sum_{k=0}^{n} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} k^{\underline{i}} \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^k \\ &= \sum_{j=0}^{l} \binom{l}{j} 2^j (r+1-n)^{l-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} \sum_{k=0}^{n} \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} k^{\underline{i}} x^k \\ &= \sum_{j=0}^{l} \binom{l}{j} 2^j (r+1-n)^{l-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} \sum_{k=0}^{n} \left\{ {n-1 \atop k} \right\}_r^{\leq 2(l)} x^i \frac{d^i}{dx^i} x^k \end{aligned}$$

$$=\sum_{j=0}^{l}\binom{l}{j}2^j(r+1-n)^{l-j}\sum_{i=0}^{j}\left\{{j \atop i}\right\}x^i\frac{d^i}{dx^i}\sum_{k=0}^{n}\left\{{n-1 \atop k}\right\}_r^{\leq 2(l)}x^k$$

$$=\sum_{j=0}^{l}\binom{l}{j}2^j(r+1-n)^{l-j}\sum_{i=0}^{j}\left\{{j \atop i}\right\}x^i\frac{d^i}{dx^i}\left(R_{n-1}^{(l,r)}(x)\right).$$

Thus,

$$\begin{aligned}R_n^{(l,r)}(x) =& xR_{n-1}^{(l,r)}(x)+D\\ =& xR_{n-1}^{(l,r)}(x)+\sum_{j=0}^{l}\binom{l}{j}2^j(r+1-n)^{l-j}\sum_{i=0}^{j}\left\{{j \atop i}\right\}x^i\frac{d^i}{dx^i}\left(R_{n-1}^{(l,r)}(x)\right).\end{aligned}$$

□

Theorem 6. For $k,r\geq 0$ and $l\geq 1$, the column generating function

$$C_k^{(l,r)}(z)=\sum_{n=0}^{2k+r}\left\{{n \atop k}\right\}_r^{\leq 2(l)}z^n$$

of $(l,r)^*-$Bessel numbers satisfies the difference-differential equation

$$C_k^{(l,r)}(z)=zC_{k-1}^{(l,r)}(z)+\sum_{j=0}^{l}\binom{l}{j}(-1)^j(2k+r+1)^{l-j}\sum_{i=0}^{j}\left\{{j \atop i}\right\}z^i\frac{d^i}{dz^i}\left(zC_k^{(l,r)}(z)\right),$$

where $C_0^{(l,r)}(z)=1$.

Proof. Suppose $C_k^{(l,r)}(z)=\sum_{n=0}^{2k+r}\left\{{n \atop k}\right\}_r^{\leq 2(l)}z^n$ be the column generating function of $(l,r)^*-$ Bessel numbers. By Theorem 2,

$$\begin{aligned}C_k^{(l,r)}(z)&=\sum_{n=0}^{2k+r}\left[\left\{{n-1 \atop k-1}\right\}_r^{\leq 2(l)}+(2k+r+1-n)^l\left\{{n-1 \atop k}\right\}_r^{\leq 2(l)}\right]z^n\\ &=\sum_{n=0}^{2k+r}\left\{{n-1 \atop k-1}\right\}_r^{\leq 2(l)}z^n+\sum_{n=0}^{2k+r}(2k+r+1-n)^l\left\{{n-1 \atop k}\right\}_r^{\leq 2(l)}z^n.\end{aligned}$$

Let $j=n-1$, then

$$\begin{aligned}\sum_{n=0}^{2k+r}\left\{{n-1 \atop k-1}\right\}_r^{\leq 2(l)}z^n&=\sum_{j=-1}^{2k+r}\left\{{j \atop k-1}\right\}_r^{\leq 2(l)}z^{j+1}\\ &=\sum_{j=0}^{2k+r}\left\{{j \atop k-1}\right\}_r^{\leq 2(l)}z^jz\\ &=z\sum_{j=0}^{2k+r}\left\{{j \atop k-1}\right\}_r^{\leq 2(l)}z^j\\ &=zC_{k-1}^{(l,r)}(z).\end{aligned}$$

By Binomial Theorem,

$$\begin{aligned}(2k+r+1-n)^l &= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-n)^j\\ &= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j n^j.\end{aligned}$$

$$\begin{aligned}\text{Let}\quad S &= \sum_{n=0}^{2k+r}(2k+r+1-n)^l \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^n\\
&= \sum_{n=0}^{2k+r}\sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j n^j \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^n\\
&= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j \sum_{n=0}^{2k+r} n^j \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^n\\
&= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j \sum_{n=0}^{2k+r}\sum_{i=0}^{j} \left\{ j \atop i \right\} n^{\underline{i}} \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^n\\
&= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j \sum_{i=0}^{j} \left\{ j \atop i \right\} \sum_{n=0}^{2k+r} \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} n^{\underline{i}} z^n\\
&= \sum_{j=0}^{l}\binom{l}{j}(2k+r+1)^{l-j}(-1)^j \sum_{i=0}^{j} \left\{ j \atop i \right\} \sum_{n=0}^{2k+r} \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^i \frac{d^i}{dz^i} z^n\\
&= \sum_{j=0}^{l}\binom{l}{j}(-1)^j(2k+r+1)^{l-j} \sum_{i=0}^{j} \left\{ j \atop i \right\} z^i \frac{d^i}{dz^i} \sum_{n=0}^{2k+r} \left\{ n-1 \atop k \right\}_r^{\leq 2(l)} z^n\\
S &= \sum_{j=0}^{l}\binom{l}{j}(-1)^j(2k+r+1)^{l-j} \sum_{i=0}^{j} \left\{ j \atop i \right\} z^i \frac{d^i}{dz^i}\left(zC_k^{(l,r)}(z)\right).\end{aligned}$$

Hence,

$$\begin{aligned}C_k^{(l,r)}(z) =& zC_{k-1}^{(l,r)}(z) + S\\
=& zC_{k-1}^{(l,r)}(z) + \sum_{j=0}^{l}\binom{l}{j}(-1)^j(2k+r+1)^{l-j}\sum_{i=0}^{j}\left\{ j \atop i \right\} z^i \frac{d^i}{dz^i}\left(zC_k^{(l,r)}(z)\right).\end{aligned}$$

□

4. Conclusion

We have introduced $(l,r)^*$-Bessel numbers as a combinatorial family encompassing block-size restrictions, distinguished element parameterization, and multi-structure enumeration. Our main contributions include:

(1) A combinatorial definition of $(l,r)^*$-Bessel numbers as counting ordered l-tuples of r-partitions with shared block maxima;
(2) Vertical and horizontal recurrence relations that generalize classical recurrences;
(3) Explicit formulas including closed forms for special cases;
(4) Row and column generating functions with associated difference-differential equations;

These numbers reduce to known sequences for special parameter values:

- When $l=1$, we recover $r-$Bessel numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^{\leq 2}$;
- When $l=1$, and $r=0$, we obtain Bessel numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{\leq 2}$;
- When $k=n-1$, we obtain sums of powers $\sum_{j=r}^{n+r-1} j^l$;
- When $k=0$, we obtain falling factorials $\left(\frac{r!}{(r-n)!}\right)^l$.

Future research directions include extending to blocks of size at most m for $m>2$.

Authors' Contributions. Both authors have read and approved the final version of the manuscript and contributed equally to this work.

Acknowledgement. The first author acknowledges the Department of Science and Technology – Science Education Institute (DOST-SEI) for their financial assistance.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Arnoczki, G. Nyul, On Restricted r-Stirling Numbers, Also Known as r-Bessel Numbers, Australas. J. Combin. 89 (2024), 49–60.

[2] H. Belbachir, Y. Djemmada, The (l,r)-Stirling Numbers: A Combinatorial Approach, Filomat 37 (2023), 2587–2598. https://doi.org/10.2298/fil2308587b.

[3] A.Z. Broder, The r-Stirling Numbers, Discret. Math. 49 (1984), 241–259. https://doi.org/10.1016/0012-365X(84)90161-4.

[4] C.C. Chen, K.M. Koh, Principles and Techniques in Combinatorics, World Scientific, 1992. https://doi.org/10.1142/9789814355162.

[5] G.S. Cheon, J.H. Jung, L.W. Shapiro, Generalized Bessel Numbers and Some Combinatorial Settings, Discret. Math. 313 (2013), 2127–2138. https://doi.org/10.1016/j.disc.2013.05.001.

[6] D. Guichard, An Introduction to Combinatorics and Graph Theory, Whitman College, (2017).

[7] S.L. Ross, Differential Equations, Third Edition, John Wiley & Sons, (2004).

[8] E.W. Weisstein, Difference-Differential Equation, MathWorld-A Wolfram Resource, https://mathworld.wolfram.com/Difference-DifferentialEquation.html.

[9] S.L. Yang, Z.K. Qiao, The Bessel Numbers and Bessel Matrices, J. Math. Res. Appl. 31 (2011), 627–636. https://doi.org/10.3770/j.issn:1000-341X.2011.04.006.