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NEUTROSOPHIC IMPLICATIVE HYPER $\mathfrak{B}$CK-IDEALS IN HYPER $\mathfrak{B}$CK-ALGEBRAS

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Abstract. In this paper, we introduce the notions of neutrosophic implicative hyper $\mathfrak{B}$CK-ideals and neutrosophic weak implicative hyper $\mathfrak{B}$CK-ideals in hyper $\mathfrak{B}$CK-algebras. We investigate their fundamental properties and establish relationships among these new classes and previously known neutrosophic hyper $\mathfrak{B}$CK-ideals. Several characterization theorems are obtained in terms of upper and lower level subsets. Examples are provided to illustrate the concepts and to show that certain converses do not hold in general.

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1. Introduction

Marty [14] introduced the concept of hyperstructure theory, commonly known as multialgebras, at the 8th Congress of Scandinavian Mathematicians in 1934. The notion of hypergroups was first established by Marty, who investigated their properties and their applications to group theory and associated algebraic structures ([3, 5, 6, 10, 12, 22]). Since then, numerous works, including articles and books, have explored this topic, showcasing its broad applicability across various branches of pure and applied science. $\mathfrak{B}$CK-algebras (briefly, $\mathfrak{B}$CK-A) were initially investigated in 1966 when Imai and Iséki [8] established them in the form of a generalization of propositional calculi and set difference operations. Subsequent to their inception, there was a significant amount of scholarly interest in the concept of $\mathfrak{B}$CK-As, especially with regard to studying its ideal characteristics. In their work,

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Borzooei et al. [9] developed the notion of hyper-BCK algebras (briefly, H$\mathfrak{B}$CK-As) by integrating the hyperstructure framework with $\mathfrak{B}$CK-As. In addition, the authors proposed the concepts of weak hyper $\mathfrak{B}$CK-ideals and hyper $\mathfrak{B}$CK-ideals (briefly, H$\mathfrak{B}$CK-Is).

As a generalization of classical sets, Zadeh's fuzzy sets [23], and Atanasov's intuitionistic fuzzy sets [4], the neutrosophic set (briefly, NSS) was introduced by Smarandache ([19,20]) to provide a comprehensive conceptual framework. The theory of NSSs has found applications across several fields of mathematics and has attracted considerable interest from researchers committed to developing, improving, and expanding the theory ([1,2,7,15–18,21]). As presented in [13], Borzooei, Jun, Zahedi, and others defined neutrosophic H$\mathfrak{B}$CK-Is in various forms (strong, weak, and s-weak) and investigated the associated characteristics and relationships between them.

In this article, we introduce and investigate two novel classes of neutrosophic ideals in the framework of hyper $\mathfrak{B}$CK-algebras, namely the *neutrosophic implicative hyper* $\mathfrak{B}$CK-*ideals* (NIH$\mathfrak{B}$CK-Is) and the *neutrosophic weak implicative hyper* $\mathfrak{B}$CK-*ideals* (NWIH$\mathfrak{B}$CK-Is). We establish their defining conditions, examine their interrelations with existing neutrosophic structures, and characterize them through level subsets. Several structural theorems are presented, along with illustrative examples and counterexamples that clarify the distinctions between these ideal types.

2. Preliminaries

In this section, we recall essential definitions and theoretical results related to hyper $\mathfrak{B}$CK-algebras (H$\mathfrak{B}$CK-As) and neutrosophic sets, which are instrumental in formulating the new concepts introduced in this paper. We begin by outlining the structure of H$\mathfrak{B}$CK-As, followed by key types of hyper $\mathfrak{B}$CK-ideals and their properties. Subsequently, we review the notion of neutrosophic sets and their applications in algebraic contexts, laying the groundwork for the definitions of neutrosophic hyper $\mathfrak{B}$CK-ideals and their implicative variants.

Definition 2.1 ([9]). *By an* H$\mathfrak{B}$CK-*A* $(\grave{\mathcal{D}}, \circ, 0)$, *we mean a nonempty set* $\grave{\mathcal{D}}$ *endowed with a hyper operation* $\circ$ *and a constant 0 satisfying the following axioms*

$$(p_0 \circ u_0) \circ (r_0 \circ u_0) \ll p_0 \circ r_0, \tag{2.1}$$

$$(p_0 \circ r_0) \circ u_0 = (p_0 \circ u_0) \circ r_0, \tag{2.2}$$

$$p_0 \circ \grave{\mathcal{D}} \ll \{p_0\}, \tag{2.3}$$

$$p_0 \ll r_0 \text{ and } r_0 \ll p_0 \Rightarrow p_0 = r_0 \text{ for all } p_0, r_0, u_0 \in \grave{\mathcal{D}}. \tag{2.4}$$

We can define a relation $\ll$ *on* $\grave{\mathcal{D}}$ *by letting* $p_0 \ll r_0$ *if and only if* $0 \in p_0 \circ r_0$ *and for every* $A_1, A_2 \subseteq \grave{\mathcal{D}}$, $A_1 \ll A_2$ *is defined by for all* $a \in A_1$ *there exists* $b \in A_2$ *such that* $a \ll b$. *In such case, we call* $\ll$ *the hyper order in* $\grave{\mathcal{D}}$.

Theorem 2.1 ([9]). *Let $\grave{\mathcal{D}}$ be an* H$\mathfrak{B}$CK*-A. Then*

(i) $p_0 \circ 0 \ll \{p_0\}$, $0 \circ p_0 \ll \{0\}$ *and* $0 \circ 0 \ll \{0\}$, *for all* $p_0 \in \grave{\mathcal{D}}$,

(ii) $(\mathcal{B}_1 \circ \mathcal{B}_2) \circ \mathcal{B}_3 = (\mathcal{B}_1 \circ \mathcal{B}_3) \circ \mathcal{B}_2$, $\mathcal{B}_1 \circ \mathcal{B}_2 \ll \mathcal{B}_1$ *and* $0 \circ \mathcal{B}_1 \ll \{0\}$, *for any non-empty subsets* $\mathcal{B}_1$, $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *of* $\grave{\mathcal{D}}$.

Theorem 2.2 ([9]). *In any* H$\mathfrak{B}$CK*-A* $\grave{\mathcal{D}}$, *the followings hold*

(i) $0 \circ 0 = \{0\}, 0 \ll p_0, p_0 \ll p_0$ *and* $\mathcal{B}_1 \ll \mathcal{B}_1$,

(ii) $\mathcal{B}_1 \subseteq \mathcal{B}_2 \Rightarrow \mathcal{B}_1 \ll \mathcal{B}_2$,

(iii) $0 \circ p_0 = \{0\}$ *and* $0 \circ \mathcal{B}_1 = \{0\}$,

(iv) $\mathcal{B}_1 \ll \{0\} \Rightarrow \mathcal{B}_1 = \{0\}$,

(v) $p_0 \in p_0 \circ 0$,

for all $p_0, r_0, u_0 \in \grave{\mathcal{D}}$ *and for any non-empty subsets* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *of* $\grave{\mathcal{D}}$.

Definition 2.2 ([9]). *Let* $\mathcal{J}$ *be a nonempty subset of* H$\mathfrak{B}$CK*-A* $\grave{\mathcal{D}}$. *Then* $\mathcal{J}$ *is said to be*

- *an* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if for all* $p_0, r_0 \in \grave{\mathcal{D}}$, $p_0 \circ r_0 \ll \mathcal{J}$ *and* $r_0 \in \mathcal{J} \Rightarrow p_0 \in \mathcal{J}$,
- *a weak* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if for all* $p_0, r_0 \in \grave{\mathcal{D}}$, $p_0 \circ r_0 \subseteq \mathcal{J}$ *and* $r_0 \in \mathcal{J} \Rightarrow p_0 \in \mathcal{J}$,
- *a strong* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if for all* $p_0, r_0 \in \grave{\mathcal{D}}$, $p_0 \circ r_0 \cap \mathcal{J} \neq \varnothing$ *and* $r_0 \in \mathcal{J} \Rightarrow p_0 \in \mathcal{J}$,
- *a reflexive if* $p_0 \circ p_0 \subseteq \mathcal{J}$ *for all* $p_0 \in \grave{\mathcal{D}}$,
- *an S-reflexive if* $(p_0 \circ r_0) \cap \mathcal{J} \neq \varnothing \Rightarrow p_0 \circ r_0 \ll \mathcal{J}$ *for all* $p_0, r_0 \in \grave{\mathcal{D}}$,
- *closed if* $p_0 \ll r_0$ *and* $r_0 \in \mathcal{J} \Rightarrow p_0 \in \mathcal{J}$ *for all* $p_0 \in \grave{\mathcal{D}}$.

Definition 2.3 ([5]). *Let* $\grave{\mathcal{D}}$ *be an* H$\mathfrak{B}$CK*-A. Then* $\grave{\mathcal{D}}$ *is said to be a positive implicative* H$\mathfrak{B}$CK*-A if*

$$(p_0 \circ r_0) \circ u_0 = (p_0 \circ u_0) \circ (r_0 \circ u_0)$$

for all $p_0, r_0, u_0 \in \grave{\mathcal{D}}$.

Definition 2.4 ([11]). *Let* $\mathcal{J}$ *be a nonempty subset of* $\grave{\mathcal{D}}$ *and* $0 \in \mathcal{J}$. *Then* $\mathcal{J}$ *is said to be*

(i) *a weak implicative* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if* $(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{J}$ *and* $u_0 \in \mathcal{J}$, *then* $p_0 \in \mathcal{J}$.

(ii) *an implicative* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if* $(p_0 \circ u_0) \circ (r_0 \circ p_0) \ll \mathcal{J}$ *and* $u_0 \in \mathcal{J}$, *then* $p_0 \in \mathcal{J}$.

Theorem 2.3 ([11]). *Let* $\mathcal{J}$ *be a nonempty subset of* $\grave{\mathcal{D}}$. *Then*

(i) *if* $\mathcal{J}$ *is an implicative* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$, *then* $\mathcal{J}$ *is an* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$.

(ii) *if* $\mathcal{J}$ *is an* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$, *then* $\mathcal{J}$ *is an implicative* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *if and only if* $(p_0 \circ r_0) \circ p_0 \ll \mathcal{J} \Rightarrow p_0 \in \mathcal{J}$, *for all* $p_0, r_0 \in \grave{\mathcal{D}}$.

(iii) *if* $\mathcal{J}$ *is an implicative and reflexive* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$, *then* $\mathcal{J}$ *is a positive implicative* H$\mathfrak{B}$CK*-I of type-3.*

Theorem 2.4. *Let* $\mathcal{B}_1$, $\mathcal{B}_2$ *and* $\mathcal{J}$ *be non-empty subsets of* $\grave{\mathcal{D}}$. *Then*

(i) *if* $\mathcal{J}$ *is an* H$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *and* $\mathcal{B}_1 \ll \mathcal{J}$, *then* $\mathcal{B}_1 \subseteq \mathcal{J}$.

(ii) *if* $\mathcal{J}$ *is a reflexive* H$\mathfrak{B}$CK-*I of* $\grave{\mathcal{D}}$*, then* $(p_0 \circ r_0) \cap \mathcal{J} \neq \emptyset \Rightarrow p_0 \circ r_0 \ll \mathcal{J}$ *for all* $p_0, r_0 \in \grave{\mathcal{D}}$*.*

(iii) *if* $\mathcal{J}$ *is an* H$\mathfrak{B}$CK-*I of* $\grave{\mathcal{D}}$ *and* $\mathcal{B}_1 \circ \mathcal{B}_2 \ll \mathcal{J}$ *and* $\mathcal{B}_2 \subseteq \mathcal{J}$*, then* $\mathcal{B}_1 \subseteq \mathcal{J}$*.*

Definition 2.5 ([19])**.** *An NSS in* $\grave{\mathcal{D}}$ *is a structure of the form*

$$\mathcal{M} = \{\langle p_0; \lambda_t(p_0), \lambda_i(p_0), \lambda_f(p_0)\rangle \mid p_0 \in \grave{\mathcal{D}}\}$$

where $\lambda_t : \grave{\mathcal{D}} \to [0,1]$*,* $\lambda_i : \grave{\mathcal{D}} \to [0,1]$*, and* $\lambda_f : \grave{\mathcal{D}} \to [0,1]$ *denote the truth, indeterminacy, and falsity membership functions, respectively.*

For simplicity we use $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *for the NSS* $\mathcal{M} = \{\langle p_0; \lambda_t(p_0), \lambda_i(p_0), \lambda_f(p_0)\rangle \mid p_0 \in \grave{\mathcal{D}}\}$*.*

Definition 2.6 ([13])**.** *An NSS* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *in* $\grave{\mathcal{D}}$ *is said to be a neutrosophic* H$\mathfrak{B}$CK-*I (NH$\mathfrak{B}$CK-I) of* $\grave{\mathcal{D}}$ *if the following conditions are met*

(*NH$\mathfrak{B}$CK-I 1*) $p_0 \ll r_0 \Rightarrow \lambda_t(p_0) \geqslant \lambda_t(r_0), \lambda_i(p_0) \geqslant \lambda_i(r_0)$ *and* $\lambda_f(p_0) \leqslant \lambda_f(r_0)$

(*NH$\mathfrak{B}$CK-I 2*) $\lambda_t(p_0) \geqslant \min\left\{\inf_{a \in p_0 \circ r_0} \lambda_t(a), \lambda_t(r_0)\right\}$

(*NH$\mathfrak{B}$CK-I 3*) $\lambda_i(p_0) \geqslant \min\left\{\inf_{b \in p_0 \circ r_0} \lambda_i(b), \lambda_i(r_0)\right\}$

(*NH$\mathfrak{B}$CK-I 4*) $\lambda_f(p_0) \leqslant \max\left\{\sup_{c \in p_0 \circ r_0} \lambda_f(c), \lambda_f(r_0)\right\}$

for all $p_0, r_0 \in \grave{\mathcal{D}}$*.*

Definition 2.7 ([13])**.** *An NSS* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *in* $\grave{\mathcal{D}}$ *is said to be a neutrosophic strong* H$\mathfrak{B}$CK-*I (NSH$\mathfrak{B}$CK-I) of* $\grave{\mathcal{D}}$ *if it satisfies following*

(*NSH$\mathfrak{B}$CK-I 1*) $\inf_{a \in p_0 \circ p_0} \lambda_t(a) \geqslant \lambda_t(p_0) \geqslant \min\left\{\sup_{b \in p_0 \circ r_0} \lambda_t(b), \lambda_t(r_0)\right\}$

(*NSH$\mathfrak{B}$CK-I 2*) $\inf_{c \in p_0 \circ p_0} \lambda_i(c) \geqslant \lambda_i(p_0) \geqslant \min\left\{\sup_{d \in p_0 \circ r_0} \lambda_i(d), \lambda_i(r_0)\right\}$

(*NSH$\mathfrak{B}$CK-I 3*) $\sup_{e \in p_0 \circ r_0} \lambda_f(e) \leqslant \lambda_f(p_0) \leqslant \max\left\{\inf_{f \in p_0 \circ r_0} \lambda_f(f), \lambda_f(r_0)\right\}$

for all $p_0, r_0 \in \grave{\mathcal{D}}$*.*

Definition 2.8 ([13])**.** *An NSS* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *in* $\grave{\mathcal{D}}$ *is said to be a neutrosophic weak* H$\mathfrak{B}$CK-*I (NWH$\mathfrak{B}$CK-I) of* $\grave{\mathcal{D}}$ *if the following conditions are met*

(*NWH$\mathfrak{B}$CK-I 1*) $\lambda_t(0) \geqslant \lambda_t(p_0) \geqslant \min\left\{\inf_{a \in p_0 \circ r_0} \lambda_t(a), \lambda_t(r_0)\right\}$

(*NWH$\mathfrak{B}$CK-I 2*) $\lambda_i(0) \geqslant \lambda_i(p_0) \geqslant \min\left\{\inf_{b \in p_0 \circ r_0} \lambda_i(b), \lambda_i(r_0)\right\}$

(*NWH$\mathfrak{B}$CK-I 3*) $\lambda_f(0) \leqslant \lambda_f(p_0) \leqslant \max\left\{ \begin{array}{c} \sup \lambda_f(c) \\ c \in p_0 \circ r_0 \end{array}, \lambda_f(r_0) \right\}$

for all $p_0, r_0 \in \grave{\mathcal{D}}$.

Theorem 2.5 ([13]). *An NSS* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *be a NWH$\mathfrak{B}$CK-I of* $\grave{\mathcal{D}}$ *if and only if the non-empty sets* $\mathcal{U}(\lambda_t, \mathfrak{r})$, $\mathcal{U}(\lambda_i, \mathfrak{s})$, *and* $\mathcal{L}(\lambda_f, \mathfrak{t})$ *are weak* H$\mathfrak{B}$CK-*Is of* $\grave{\mathcal{D}}$ *for all* $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$.

Definition 2.9 ([13]). *An NSS* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *in* $\grave{\mathcal{D}}$ *is said to be a reflexive neutrosophic* H$\mathfrak{B}$CK-*I* (*RNH$\mathfrak{B}$CK-I*) *of* $\grave{\mathcal{D}}$ *if the following conditions are met*

(*RNH$\mathfrak{B}$CK-I 1*) $\begin{array}{c} \inf \lambda_t(a) \\ a \in p_0 \circ p_0 \end{array} \geqslant \lambda_t(r_0)$, $\begin{array}{c} \inf \lambda_i(b) \\ b \in p_0 \circ p_0 \end{array} \geqslant \lambda_i(r_0)$ *and* $\begin{array}{c} \sup \lambda_f(c) \\ c \in p_0 \circ p_0 \end{array} \leqslant \lambda_f(r_0)$

(*RNH$\mathfrak{B}$CK-I 2*) $\lambda_t(p_0) \geqslant \min\left\{ \begin{array}{c} \sup \lambda_t(a) \\ a \in p_0 \circ r_0 \end{array}, \lambda_t(r_0) \right\}$

(*RNH$\mathfrak{B}$CK-I 3*) $\lambda_i(p_0) \geqslant \min\left\{ \begin{array}{c} \sup \lambda_i(b) \\ b \in p_0 \circ r_0 \end{array}, \lambda_i(r_0) \right\}$

(*RNH$\mathfrak{B}$CK-I 4*) $\lambda_f(p_0) \leqslant \max\left\{ \begin{array}{c} \inf \lambda_f(c) \\ c \in p_0 \circ r_0 \end{array}, \lambda_f(r_0) \right\}$

for all $p_0, r_0 \in \grave{\mathcal{D}}$.

Theorem 2.6. *Let* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *is an NSS in* $\grave{\mathcal{D}}$, *then the following statements hold: If for all* $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$, *the non-empty sets* $\mathfrak{U}(\lambda_t; \mathfrak{r})$, $\mathfrak{U}(\lambda_i; \mathfrak{s})$ *and* $\mathfrak{L}(\lambda_f; \mathfrak{t})$ *are reflexive positive implicative* H$\mathfrak{B}$CK-*Is of type* 2(3) *and* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *satisfies sup-sup-inf property, then* $\mathcal{M}$ *is a neutrosophic positive implicative* H$\mathfrak{B}$CK-*I of type* 2(3).

3. Neutrosophic Implicative Hyper $\mathfrak{B}$CK-Ideals

Building upon the foundational definitions of neutrosophic hyper $\mathfrak{B}$CK-ideals, this section introduces two novel classes: the *neutrosophic implicative hyper* $\mathfrak{B}$CK-*ideals* (NIH$\mathfrak{B}$CK-Is) and the *neutrosophic weak implicative hyper* $\mathfrak{B}$CK-*ideals* (NWIH$\mathfrak{B}$CK-Is). These structures are motivated by the desire to extend implicative properties from classical and fuzzy contexts into the neutrosophic framework, within the setting of hyper $\mathfrak{B}$CK-algebras.

We formally define these ideal types using neutrosophic membership functions and investigate their structural behavior. Several fundamental theorems are presented to characterize their relationships with existing neutrosophic ideals, particularly with respect to inclusion, level subsets, and implicative conditions. Additionally, illustrative examples and counterexamples are constructed to demonstrate the distinctions between the two classes and to show that certain converses do not hold in general.

Definition 3.1. *Let* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *be an NSS of* $\grave{\mathcal{D}}$ *and* $\lambda_t(0) \geqslant \lambda_t(p_0)$, $\lambda_i(0) \geqslant \lambda_i(p_0)$, *and* $\lambda_f(0) \leqslant \lambda_f(p_0)$ *for all* $p_0 \in \grave{\mathcal{D}}$. *Then,* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *is called a*

(a) *neutrosophic weak implicative hyper* $\mathfrak{B}$CK*-ideal* (*NWIH*$\mathfrak{B}$CK*-I*) *of* $\grave{\mathcal{D}}$ *if*

(*NWIH*$\mathfrak{B}$CK*-I 1*) $\lambda_t(p_0) \geqslant \min\left\{\inf_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0)\right\}$

(*NWIH*$\mathfrak{B}$CK*-I 2*) $\lambda_i(p_0) \geqslant \min\left\{\inf_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0)\right\}$

(*NWIH*$\mathfrak{B}$CK*-I 3*) $\lambda_f(p_0) \leqslant \max\left\{\sup_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0)\right\}$

(b) *neutrosophic implicative hyper* $\mathfrak{B}$CK*-ideal* (*NIH*$\mathfrak{B}$CK*-I*) *of* $\grave{\mathcal{D}}$ *if*

(*NIH*$\mathfrak{B}$CK*-I 1*) $\lambda_t(p_0) \geqslant \min\left\{\sup_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0)\right\}$

(*NIH*$\mathfrak{B}$CK*-I 2*) $\lambda_i(p_0) \geqslant \min\left\{\sup_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0)\right\}$

(*NIH*$\mathfrak{B}$CK*-I 3*) $\lambda_f(p_0) \leqslant \max\left\{\inf_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0)\right\}$

for all $p_0, r_0, u_0 \in \grave{\mathcal{D}}$.

Theorem 3.1. *In an* H$\mathfrak{B}$CK*-A* $\grave{\mathcal{D}}$*, every NIH*$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *is a NWIH*$\mathfrak{B}$CK*-I.*

Proof. Suppose $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a NIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and for $p_0, r_0, u_0 \in \grave{\mathrm{D}}$, we have

$$\begin{aligned}
\inf_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a) &\leqslant \sup_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a) \\
\inf_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b) &\leqslant \sup_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b) \\
\inf_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c) &\leqslant \sup_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c) .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_t(p_0) &\geqslant \min\left\{\sup_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0)\right\} \geqslant \min\left\{\inf_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0)\right\} \\
\lambda_i(p_0) &\geqslant \min\left\{\sup_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0)\right\} \geqslant \min\left\{\inf_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0)\right\} \\
\lambda_f(p_0) &\leqslant \max\left\{\inf_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0)\right\} \leqslant \max\left\{\sup_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0)\right\}.
\end{aligned}$$

Thus, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a NWH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. □

Example 3.1. *Consider an* H$\mathfrak{B}$CK*-A* $\grave{\mathcal{D}} = \{0, a, b\}$ *as defined in the Table 1.*

Define an NSS $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *on* $\grave{\mathcal{D}}$ *as shown in Table 2.*

Then $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *is a NWIH*$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$ *but it is not a NIH*$\mathfrak{B}$CK*-I of* $\grave{\mathcal{D}}$*, because*

$$\lambda_t(a) = 0.13 < 0.95 = \lambda_t(0) = \min\left\{\sup_{t \in (a \circ 0) \circ (a \circ a)} \lambda_t(t), \lambda_t(0)\right\}$$

TABLE 1. Hyper $\mathfrak{B}$CK-algebra

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

TABLE 2. Neutrosophic set

$\grave{\mathcal{D}}$	$\lambda_t(p_0)$	$\lambda_i(p_0)$	$\lambda_f(p_0)$
0	0.95	0.88	0.01
a	0.13	0.21	0.97
b	0.75	0.73	0.47

$$\lambda_i(a) = 0.21 < 0.88 = \lambda_i(0) = \min\left\{ \sup_{t\in(a\circ 0)\circ(a\circ a)} \lambda_i(t) \ , \lambda_i(0)\right\}$$

$$\lambda_f(a) = 0.97 > 0.01 = \lambda_f(0) = \max\left\{ \inf_{t\in(a\circ 0)\circ(a\circ a)} \lambda_f(t) \ , \lambda_f(0)\right\}.$$

Theorem 3.2. *In an* H$\mathfrak{B}$CK-*A* $\grave{\mathcal{D}}$,

(i) *every NIH$\mathfrak{B}$CK-I of $\grave{\mathcal{D}}$ is an NSH$\mathfrak{B}$CK-I.*

(ii) *every NWIH$\mathfrak{B}$CK-I of $\grave{\mathcal{D}}$ is an NWH$\mathfrak{B}$CK-I.*

Proof. Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NIH$\mathfrak{B}$CK-I of $\grave{\mathcal{D}}$. By putting $r_0 = 0$ and $u_0 = r_0$ in Definition 3.1(b), we obtain

$$\left(\begin{array}{l} \lambda_t(p_0) \geqslant \min\left\{ \sup\limits_{a\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_t(a) \ , \lambda_t(r_0)\right\} = \min\left\{ \sup\limits_{a\in p_0\circ r_0} \lambda_t(a), \lambda_t(r_0)\right\} \\ \lambda_i(p_0) \geqslant \min\left\{ \sup\limits_{b\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_i(b) \ , \lambda_i(r_0)\right\} = \min\left\{ \sup\limits_{b\in p_0\circ r_0} \lambda_i(b), \lambda_i(r_0)\right\} \\ \lambda_f(p_0) \leqslant \max\left\{ \inf\limits_{c\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_f(c) \ , \lambda_f(r_0)\right\} = \max\left\{ \inf\limits_{c\in p_0\circ r_0} \lambda_f(c), \lambda_f(r_0)\right\} \end{array}\right) \tag{3.1}$$

First, we show that for $p_0, r_0 \in \grave{\mathcal{D}}$ if $p_0 \ll r_0 \Rightarrow \lambda_t(p_0) \geqslant \lambda_t(r_0)$, $\lambda_i(p_0) \geqslant \lambda_i(r_0)$, and $\lambda_f(p_0) \leqslant \lambda_f(r_0)$. For this let $p_0, r_0 \in \grave{\mathcal{D}}$ be such that $p_0 \ll r_0$, then $0 \in p_0 \circ r_0$ and so from (3.1), we get

$$\left(\begin{array}{l} \lambda_t(p_0) \geqslant \min\left\{ \sup\limits_{a\in p_0\circ r_0} \lambda_t(a), \lambda_t(r_0)\right\} = \min\{\lambda_t(0), \lambda_t(r_0)\} = \lambda_t(r_0) \\ \lambda_i(p_0) \geqslant \min\left\{ \sup\limits_{b\in p_0\circ r_0} \lambda_i(b), \lambda_i(r_0)\right\} = \min\{\lambda_i(0), \lambda_i(r_0)\} = \lambda_i(r_0) \\ \lambda_f(p_0) \leqslant \max\left\{ \inf\limits_{c\in p_0\circ r_0} \lambda_f(c), \lambda_f(r_0)\right\} = \max\{\lambda_f(0), \lambda_f(r_0)\ \} = \lambda_f(r_0) \end{array}\right) \tag{3.2}$$

Let $p_0 \in \grave{\mathrm{D}}$ and $a \in p_0 \circ p_0$. Since $p_0 \circ p_0 \ll p_0$, we have $a \ll p_0$ for all $a \in p_0 \circ p_0$. So, by (3.2), we have $\lambda_t(a) \geqslant \lambda_t(p_0)$, $\lambda_i(a) \geqslant \lambda_i(p_0)$, and $\lambda_f(a) \leqslant \lambda_f(p_0)$ for all $a \in p_0 \circ p_0$. Hence,

$$\inf_{a\in p_0\circ p_0} \lambda_t(a) \geqslant \lambda_t(p_0),\ \inf_{a\in p_0\circ p_0} \lambda_i(a) \geqslant \lambda_i(p_0),\ \text{and} \sup_{a\in p_0\circ p_0} \lambda_f(a) \leqslant \lambda_f(p_0) \tag{3.3}$$

Combining (3.1) and (3.3), we get

$$\inf_{a\in p_0\circ p_0} \lambda_t(a) \geqslant \lambda_t(p_0) \geqslant \min\left\{\sup_{b\in p_0\circ r_0} \lambda_t(b), \lambda_t(r_0)\right\}$$

$$\inf_{c\in p_0\circ p_0} \lambda_i(c) \geqslant \lambda_i(p_0) \geqslant \min\left\{\sup_{d\in p_0\circ r_0} \lambda_i(d), \lambda_i(r_0)\right\}$$

$$\sup_{e\in p_0\circ p_0} \lambda_f(e) \leqslant \lambda_f(p_0) \leqslant \max\left\{\inf_{f\in p_0\circ r_0} \lambda_f(f), \lambda_f(r_0)\right\}$$

for all $p_0, r_0 \in \grave{\mathrm{D}}$. Thus, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NSH$\mathfrak{B}$CK-I.

(ii) Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be a NWIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. By putting $r_0 = 0$ and $u_0 = r_0$ in Definition 3.1(a), we get

$$\lambda_t(p_0) \geqslant \min\left\{\inf_{a\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_t(a), \lambda_t(r_0)\right\} = \min\left\{\inf_{a\in p_0\circ r_0} \lambda_t(a), \lambda_t(r_0)\right\}$$

$$\lambda_i(p_0) \geqslant \min\left\{\inf_{b\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_i(b), \lambda_i(r_0)\right\} = \min\left\{\inf_{b\in p_0\circ r_0} \lambda_i(b), \lambda_i(r_0)\right\}$$

$$\lambda_f(p_0) \leqslant \max\left\{\sup_{c\in(p_0\circ r_0)\circ(0\circ p_0)} \lambda_f(c), \lambda_f(r_0)\right\} = \max\left\{\sup_{c\in p_0\circ r_0} \lambda_f(c), \lambda_f(r_0)\right\}.$$

Therefore,

$$\lambda_t(0) \geqslant \lambda_t(p_0) \geqslant \min\left\{\inf_{a\in p_0\circ r_0} \lambda_t(a), \lambda_t(r_0)\right\}$$

$$\lambda_i(0) \geqslant \lambda_i(p_0) \geqslant \min\left\{\inf_{b\in p_0\circ r_0} \lambda_i(b), \lambda_i(r_0)\right\}$$

$$\lambda_f(0) \leqslant \lambda_f(p_0) \leqslant \max\left\{\sup_{c\in p_0\circ r_0} \lambda_f(c), \lambda_f(r_0)\right\}$$

for all $p_0, r_0 \in \grave{\mathrm{D}}$. Thus, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a NWIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. □

Example 3.2. *Consider an* H$\mathfrak{B}$CK-A $\grave{\mathcal{D}} = \{0, a, b, c\}$ *as defined in the Table 3.*

Define an NSS $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *on* $\grave{\mathcal{D}}$ *as shown in the Table 4. Then* $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ *is an NSH$\mathfrak{B}$CK-I, but it is not an NIH$\mathfrak{B}$CK-I of* $\grave{\mathcal{D}}$ *because*

$$\lambda_t(b) = 0.32 < 0.95 = \lambda_t(0) = \min\left\{\sup_{t\in(b\circ 0)\circ(c\circ b)} \lambda_t(t), \lambda_t(0)\right\}$$

$$\lambda_i(b) = 0.53 < 0.87 = \lambda_i(0) = \min\left\{\sup_{t\in(b\circ 0)\circ(c\circ b)} \lambda_i(t), \lambda_i(0)\right\}$$

Table 3. Hyper $\mathfrak{B}$CK-algebra

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Table 4. Neutrosophic set

$\grave{\mathcal{D}}$	$\lambda_t(p_0)$	$\lambda_i(p_0)$	$\lambda_f(p_0)$
0	0.95	0.87	0.19
a	0.95	0.87	0.19
b	0.32	0.53	0.72
c	0.32	0.53	0.72

$$\lambda_f(b) = 0.72 > 0.19 = \lambda_f(0) = \max\left\{\inf_{t\in(b\circ 0)\circ(c\circ b)} \lambda_f(t)\ , \lambda_f(0)\right\}.$$

From Example 3.2, we conclude that the converse of Theorem 3.2 is not true in general.

Definition 3.2. *Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NSS of $\grave{\mathcal{D}}$. We define the following sets*

$$\mathcal{U}(\lambda_t; \mathfrak{r}) = \left\{p_0 \in \grave{\mathcal{D}} \mid \lambda_t(p_0) \geqslant \mathfrak{r}\right\}$$
$$\mathcal{U}(\lambda_i; \mathfrak{s}) = \left\{p_0 \in \grave{\mathcal{D}} \mid \lambda_i(p_0) \geqslant \mathfrak{s}\right\}$$
$$\mathcal{L}(\lambda_f; \mathfrak{t}) = \left\{p_0 \in \grave{\mathcal{D}} \mid \lambda_f(p_0) \leqslant \mathfrak{t}\right\}$$

where $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$.

Definition 3.3. *An NSS $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ of $\grave{\mathcal{D}}$ is said to satisfy sup-sup-inf property if for any subset F of $\grave{\mathcal{D}}$, there exist l_1, l_2, and l_3 such that*

$$\lambda_t(l_1) = \sup_{p_0\in F} \lambda_t(p_0),\ \lambda_i(l_2) = \sup_{p_0\in F} \lambda_i(p_0),\ \textit{and}\ \lambda_f(l_3) = \inf_{p_0\in F} \lambda_f(p_0). \tag{3.4}$$

Definition 3.4. *An NSS $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ of $\grave{\mathcal{D}}$ is said to satisfy inf-inf-sup property if for any subset F of $\grave{\mathcal{D}}$, there exist l_1, l_2, and l_3 such that*

$$\lambda_t(l_1) = \inf_{p_0\in F} \lambda_t(p_0),\ \lambda_i(l_2) = \inf_{p_0\in F} \lambda_i(p_0),\ \textit{and}\ \lambda_f(l_3) = \sup_{p_0\in F} \lambda_f(p_0). \tag{3.5}$$

Theorem 3.3. *Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NSS of $\grave{\mathcal{D}}$. Then the following statements hold*

(i) *$\mathcal{M}$ is a NWIH$\mathfrak{B}$CK-I of $\grave{\mathcal{D}}$ if and only if for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$, the non-empty sets $\mathcal{U}(\lambda_t, \mathfrak{r})$, $\mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are weak implicative H$\mathfrak{B}$CK-Is of $\grave{\mathcal{D}}$.*

(*ii*) *If* $\mathcal{M}$ *is a NIH*$\mathfrak{B}$*CK-I of* $\grave{\mathrm{D}}$*, then for all* $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$*, the non-empty sets* $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$*, and* $\mathcal{L}(\lambda_f, \mathfrak{t})$ *are implicative* H$\mathfrak{B}$CK*-Is of* $\grave{\mathrm{D}}$*.*

(*iii*) *If for all* $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$*, the non-empty sets* $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$*, and* $\mathcal{L}(\lambda_f, \mathfrak{t})$ *are S-reflexive implicative* H$\mathfrak{B}$CK*-Is of* $\grave{\mathrm{D}}$*, and* $\mathcal{M}$ *satisfies the sup-sup-inf property, then* $\mathcal{M}$ *is a NIH*$\mathfrak{B}$*CK-I of* $\grave{\mathrm{D}}$*.*

Proof. (i) Assume $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$.

Let $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$ and $p_0, r_0, u_0 \in \grave{\mathrm{D}}$ be such that $(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{U}(\lambda_t, \mathfrak{r})$ and $u_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$. Then $a \in \mathcal{U}(\lambda_t, \mathfrak{r})$ for all $a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$. Thus, $\lambda_t(a) \geqslant \mathfrak{r}$, for all $a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ and $\lambda_t(u_0) \geqslant \mathfrak{r}$, so $\inf\limits_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a) \geqslant \mathfrak{r}$ and $\lambda_t(u_0) \geqslant \mathfrak{r}$. Thus, by the hypothesis,

$$\begin{aligned} \lambda_t(p_0) &\geqslant \min\left\{ \inf_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0) \right\} \\ &\geqslant \min\{\mathfrak{r}, \mathfrak{r}\} = \mathfrak{r}, \\ \Rightarrow p_0 &\in \mathcal{U}(\lambda_t, \mathfrak{r}). \end{aligned}$$

Let $p_0, r_0, u_0 \in \grave{\mathrm{D}}$ be such that $(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{U}(\lambda_i, \mathfrak{s})$ and $u_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$. Then $b \in \mathcal{U}(\lambda_i, \mathfrak{s})$ for all $b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$. Thus $\lambda_i(b) \geqslant \mathfrak{s}$, for all $b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ and $\lambda_i(u_0) \geqslant \mathfrak{s}$, so $\inf\limits_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b) \geqslant \mathfrak{s}$ and $\lambda_i(u_0) \geqslant \mathfrak{s}$. Thus, by the hypothesis,

$$\begin{aligned} \lambda_i(p_0) &\geqslant \min\left\{ \inf_{b \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0) \right\} \\ &\geqslant \min\{\mathfrak{s}, \mathfrak{s}\} = \mathfrak{s}, \\ \Rightarrow p_0 &\in \mathcal{U}(\lambda_i, \mathfrak{s}). \end{aligned}$$

Again, let $p_0, r_0, u_0 \in \grave{\mathrm{D}}$ be such that $(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{L}(\lambda_f, \mathfrak{t})$ and $u_0 \in \mathcal{L}(\lambda_f, \mathfrak{t})$. Then $c \in \mathcal{L}(\lambda_f, \mathfrak{t})$ for all $c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$. Thus, $\lambda_f(c) \leqslant \mathfrak{t}$, for all $c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ and $\lambda_f(u_0) \leqslant \mathfrak{t}$, so $\sup\limits_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c) \leqslant \mathfrak{t}$ and $\lambda_f(u_0) \leqslant \mathfrak{t}$. Thus, by the hypothesis,

$$\begin{aligned} \lambda_f(p_0) &\leqslant \max\left\{ \sup_{c \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0) \right\} \\ &\leqslant \max\{\mathfrak{t}, \mathfrak{t}\} = \mathfrak{t}, \\ \Rightarrow p_0 &\in \mathcal{L}(\lambda_f, \mathfrak{t}). \end{aligned}$$

Therefore, $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are weak implicative H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$, for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$.

Conversely, let for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$, the non-empty sets $\mathcal{U}(\lambda_t, \mathfrak{r})$, $\mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ be weak implicative H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$. Let $p_0, r_0, u_0 \in \grave{\mathrm{D}}$, and put

$$\mathfrak{r} = \min\left\{ \inf_{a \in (p_0 \circ u_0) \circ (r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0) \right\}.$$

Then

$$\inf_{a\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_t(a) \geqslant \mathfrak{r} \text{ and } \lambda_t(u_0) \geqslant \mathfrak{r}.$$

So that

$$\lambda_t(a) \geqslant \mathfrak{r} \text{ for all } a \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } \lambda_t(u_0) \geqslant \mathfrak{r}.$$

Hence,

$$a \in \mathcal{U}(\lambda_t, \mathfrak{r}) \text{ for all } a \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } u_0 \in \mathcal{U}(\lambda_t, \mathfrak{r}).$$

That is,

$$(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{U}(\lambda_t, \mathfrak{r}) \text{ and } u_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$$

and so, by the hypothesis, $p_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$. Thus,

$$\lambda_t(p_0) \geqslant \mathfrak{r} = \min\left\{\inf_{a\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_t(a), \lambda_t(u_0)\right\}.$$

Put $\mathfrak{s} = \min\left\{\inf_{b\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_i(b), \lambda_i(u_0)\right\}$. Then

$$\inf_{b\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_i(b) \geqslant \mathfrak{s} \text{ and } \lambda_i(u_0) \geqslant \mathfrak{s}.$$

So that

$$\lambda_i(b) \geqslant \mathfrak{s} \text{ for all } b \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } \lambda_i(u_0) \geqslant \mathfrak{s}.$$

Hence,

$$b \in \mathcal{U}(\lambda_i, \mathfrak{s}) \text{ for all } b \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } u_0 \in \mathcal{U}(\lambda_i, \mathfrak{s}).$$

That is,

$$(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{U}(\lambda_i, \mathfrak{s}) \text{ and } u_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$$

and so, by the hypothesis, $p_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$. Thus,

$$\lambda_i(p_0) \geqslant \mathfrak{s} = \min\left\{\inf_{b\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_i(b), \lambda_i(u_0)\right\}.$$

Put $\mathfrak{t} = \max\left\{\sup_{c\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_f(c), \lambda_f(u_0)\right\}$. Then

$$\sup_{c\in(p_0\circ u_0)\circ(r_0\circ p_0)}\lambda_f(c) \leqslant \mathfrak{t} \text{ and } \lambda_f(u_0) \leqslant \mathfrak{t}.$$

So that

$$\lambda_f(c) \leqslant \mathfrak{t} \text{ for all } c \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } \lambda_f(u_0) \leqslant \mathfrak{t}.$$

Hence,

$$c \in \mathcal{L}(\lambda_f, \mathfrak{t}) \text{ for all } c \in (p_0 \circ u_0) \circ (r_0 \circ p_0) \text{ and } u_0 \in \mathcal{L}(\lambda_f, \mathfrak{t}).$$

That is,

$$(p_0 \circ u_0) \circ (r_0 \circ p_0) \subseteq \mathcal{L}(\lambda_f, \mathfrak{t}) \text{ and } u_0 \in \mathcal{L}(\lambda_f, \mathfrak{t})$$

and so, by the hypothesis, $p_0 \in \mathcal{L}(\lambda_f, \mathfrak{t})$. Thus,

$$\lambda_f(p_0) \leqslant \mathfrak{t} = \max\left\{ \sup_{c\in(p_0\circ u_0)\circ(r_0\circ p_0)} \lambda_f(c), \lambda_f(u_0) \right\}.$$

Therefore, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$.

(ii) Suppose $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a NIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. By Theorem 3.2(i), $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NSH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and, therefore, it is an NH$\mathfrak{B}$CK-ideal of $\grave{\mathrm{D}}$. By Theorem 2.5, for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$, the non-empty sets $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$ and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$. Now by Theorem 2.3(ii), it is sufficient to show that if $\begin{pmatrix} p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_t, \mathfrak{r}), \\ p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_i, \mathfrak{s}), \text{ and} \\ p_0 \circ (r_0 \circ p_0) \ll \mathcal{L}(\lambda_f, \mathfrak{t}) \end{pmatrix}$, then $p_0 \in \mathcal{U}(\lambda_t, \mathfrak{r}) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) \cap \mathcal{L}(\lambda_f, \mathfrak{t})$. For this, let $p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_t, \mathfrak{r})$ for $p_0, r_0 \in \grave{\mathrm{D}}$. Then, for all $a \in p_0 \circ (r_0 \circ p_0)$ there exists $b \in \mathcal{U}(\lambda_t, \mathfrak{r})$ such that $a \ll b$. Then

$$\begin{aligned} \lambda_t(a) \geqslant \lambda_t(b) \geqslant \mathfrak{r} &\Rightarrow \lambda_t(a) \geqslant \mathfrak{r} \text{ for all } a \in p_0 \circ (r_0 \circ p_0) \\ &\Rightarrow \sup_{a\in p_0\circ(r_0\circ p_0)} \lambda_t(a) \geqslant \mathfrak{r}. \end{aligned}$$

Hence, by the hypothesis,

$$\begin{aligned} \lambda_t(p_0) &\geqslant \min\left\{ \sup_{a\in(p_0\circ 0)\circ(r_0\circ p_0)} \lambda_t(a), \lambda_t(0) \right\} \\ &= \sup_{a\in p_0\circ(r_0\circ p_0)} \lambda_t(a) \geqslant \mathfrak{r}. \end{aligned}$$

Thus, $p_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$. Let $p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_i, \mathfrak{s})$ for $p_0, r_0 \in \grave{\mathrm{D}}$. Then, for all $c \in p_0 \circ (r_0 \circ p_0)$ there exists $d \in \mathcal{U}(\lambda_i, \mathfrak{s})$ such that $c \ll d$. Then

$$\begin{aligned} \lambda_i(c) \geqslant \lambda_i(d) \geqslant \mathfrak{s} &\Rightarrow \lambda_i(c) \geqslant \mathfrak{s} \text{ for all } c \in p_0 \circ (r_0 \circ p_0) \\ &\Rightarrow \sup_{c\in p_0\circ(r_0\circ p_0)} \lambda_i(c) \geqslant \mathfrak{s}. \end{aligned}$$

Hence, by the hypothesis,

$$\lambda_i(p_0) \geqslant \min\left\{ \sup_{c\in(p_0\circ 0)\circ(r_0\circ p_0)} \lambda_i(c) \;, \lambda_i(0) \right\}$$

$$= \sup_{c\in p_0\circ(r_0\circ p_0)} \lambda_i(c) \;\geqslant \mathfrak{s}.$$

Thus, $p_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$. Let $p_0 \circ (r_0 \circ p_0) \ll \mathcal{L}(\lambda_f, \mathfrak{t})$ for $p_0, r_0 \in \grave{\mathrm{D}}$. Then, for all $e \in p_0 \circ (r_0 \circ p_0)$ there exists $f \in \mathcal{L}(\lambda_f, \mathfrak{t})$ such that $e \ll f$. Then

$$\lambda_f(e) \leqslant \lambda_f(f) \leqslant \mathfrak{t} \Rightarrow \lambda_f(e) \leqslant \mathfrak{t} \text{ for all } e \in p_0 \circ (r_0 \circ p_0)$$

$$\Rightarrow \inf_{e\in p_0\circ(r_0\circ p_0)} \lambda_f(e) \;\leqslant \mathfrak{t}.$$

Hence, by the hypothesis,

$$\lambda_f(p_0) \leqslant \max\left\{ \inf_{e\in(p_0\circ 0)\circ(r_0\circ p_0)} \lambda_f(e) \;, \lambda_f(0) \right\}$$

$$= \inf_{e\in p_0\circ(r_0\circ p_0)} \lambda_f(e) \;\leqslant \mathfrak{t}.$$

$$\Rightarrow p_0 \in \mathcal{L}(\lambda_f, \mathfrak{t}).$$

Thus, for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$, the non-empty sets $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are implicative H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$.

(iii) Assume that for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0, 1]$, the non-empty sets $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are S-reflexive implicative H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$. Let $p_0, r_0, u_0 \in \grave{\mathrm{D}}$.

$$\text{Put } \mathfrak{r} = \min\left\{ \sup_{a\in(p_0\circ u_0)\circ(r_0\circ p_0)} \lambda_t(a) \;, \lambda_t(u_0) \right\}$$

$$\Rightarrow \sup_{a\in(p_0\circ u_0)\circ(r_0\circ p_0)} \lambda_t(a) \;\geqslant \mathfrak{r} \text{ and } \lambda_t(u_0) \geqslant \mathfrak{r}.$$

Since λ_t satisfies the sup property, we have there exists $a_0 \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ such that $\lambda_t(a_0) = \sup_{a\in(p_0\circ u_0)\circ(r_0\circ p_0)} \lambda_t(a) \;\geqslant \mathfrak{r}$, and so $a_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$. Hence, by using (2.2), we have

$$((p_0 \circ (r_0 \circ p_0)) \circ u_0) \cap \mathcal{U}(\lambda_t, \mathfrak{r}) = ((p_0 \circ u_0) \circ (r_0 \circ p_0)) \cap \mathcal{U}(\lambda_t, \mathfrak{r}) \neq \varnothing.$$

Thus, there exists $a \in p_0 \circ (r_0 \circ p_0)$ such that $(a \circ u_0) \cap \mathcal{U}(\lambda_t, \mathfrak{r}) \neq \varnothing$. By Theorem 2.3(i), we have $\mathcal{U}(\lambda_t, \mathfrak{r})$ is a hyper $\mathfrak{B}$CK-ideal of $\grave{\mathrm{D}}$, and by hypothesis it is S-reflexive. Therefore, it is a reflexive H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and, consequently, a strong H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. Since $\lambda_t(u_0) \geqslant \mathfrak{r}$, we have $u_0 \in \mathcal{U}(\lambda_t, \mathfrak{r})$.

$$(a \circ u_0) \cap \mathcal{U}(\lambda_t, \mathfrak{r}) \neq \varnothing\ ,\ u_0 \in \mathcal{U}(\lambda_t, \mathfrak{r}) \Rightarrow a \in \mathcal{U}(\lambda_t, \mathfrak{r})$$

$$\Rightarrow (p_0 \circ (r_0 \circ p_0)) \cap \mathcal{U}(\lambda_t, \mathfrak{r}) \neq \varnothing.$$

$$\Rightarrow p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_t, \mathfrak{r}) \quad \text{(By Theorem 2.4(ii))}$$

$$\Rightarrow p_0 \in \mathcal{U}(\lambda_t, \mathfrak{r}) \;\; \text{(By Definition 2.4(ii))}$$

$$\Rightarrow \lambda_t(p_0) \geqslant \mathfrak{r} = \min\left\{ \sup_{a \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_t(a), \lambda_t(u_0) \right\}.$$

Put $\mathfrak{s} = \min\left\{ \sup_{b \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0) \right\}$. Then $\sup_{b \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_i(b) \geqslant \mathfrak{s}$ and $\lambda_i(u_0) \geqslant \mathfrak{s}$. Since λ_i satisfies the sup property, we have there exists $b_0 \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ such that $\lambda_i(b_0) = \sup_{b \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_i(b) \geqslant \mathfrak{s}$, and so $b_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$. Hence, by using (2.2), we have

$$((p_0 \circ (r_0 \circ p_0)) \circ u_0) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) = ((p_0 \circ u_0) \circ (r_0 \circ p_0)) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) \neq \varnothing.$$

Thus, there exists $b \in p_0 \circ (r_0 \circ p_0)$ such that $(b \circ u_0) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) \neq \varnothing$. By Theorem 2.3(i), we have $\mathcal{U}(\lambda_i, \mathfrak{s})$ is a H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$, and according to hypothesis it is S-reflexive. Therefore, it is a reflexive H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and, consequently, a strong H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. Since $\lambda_i(u_0) \geqslant \mathfrak{s}$, we have $u_0 \in \mathcal{U}(\lambda_i, \mathfrak{s})$.

$$(b \circ u_0) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) \neq \varnothing \,,\; u_0 \in \mathcal{U}(\lambda_i, \mathfrak{s}) \Rightarrow b \in \mathcal{U}(\lambda_i, \mathfrak{s})$$

$$\Rightarrow (p_0 \circ (r_0 \circ p_0)) \cap \mathcal{U}(\lambda_i, \mathfrak{s}) \neq \varnothing.$$

$$\Rightarrow p_0 \circ (r_0 \circ p_0) \ll \mathcal{U}(\lambda_i, \mathfrak{s}) \;\; \text{(By Theorem 2.4(ii))}$$

$$\Rightarrow p_0 \in \mathcal{U}(\lambda_i, \mathfrak{s}) \;\; \text{(By Definition 2.4(ii))}$$

$$\Rightarrow \lambda_i(p_0) \geqslant \mathfrak{s} = \min\left\{ \sup_{b \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_i(b), \lambda_i(u_0) \right\}.$$

Put $\mathfrak{t} = \max\left\{ \inf_{c \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0) \right\}$. Then $\inf_{c \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_f(c) \leqslant \mathfrak{t}$ and $\lambda_f(u_0) \leqslant \mathfrak{t}$. Since λ_f satisfies the inf property, there exists $c_0 \in (p_0 \circ u_0) \circ (r_0 \circ p_0)$ such that $\lambda_f(c_0) = \inf_{c \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_f(c) \leqslant \mathfrak{t}$, and so $c_0 \in \mathcal{L}(\lambda_f, \mathfrak{t})$. Hence, by using (2.2), we have

$$((p_0 \circ (r_0 \circ p_0)) \circ u_0) \cap \mathcal{L}(\lambda_f, \mathfrak{t}) = ((p_0 \circ u_0) \circ (r_0 \circ p_0)) \cap \mathcal{L}(\lambda_f, \mathfrak{t}) \neq \varnothing.$$

Thus, there exists $c \in p_0 \circ (r_0 \circ p_0)$ such that $(c \circ u_0) \cap \mathcal{L}(\lambda_f, \mathfrak{t}) \neq \varnothing$. By Theorem 2.3(i), we have $\mathcal{L}(\lambda_f, \mathfrak{t})$ is a H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and according to hypothesis it is S-reflexive. Therefore, it is a reflexive H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$, consequently, a strong H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. Since $\lambda_f(u_0) \leqslant \mathfrak{t}$, we have $u_0 \in \mathcal{L}(\lambda_f, \mathfrak{t})$.

$$(c \circ u_0) \cap \mathcal{L}(\lambda_f, \mathfrak{t}) \neq \varnothing \,,\; u_0 \in \mathcal{L}(\lambda_f, \mathfrak{t}) \Rightarrow c \in \mathcal{L}(\lambda_f, \mathfrak{t})$$

$$\Rightarrow (p_0 \circ (r_0 \circ p_0)) \cap \mathcal{L}(\lambda_f, \mathfrak{t}) \neq \varnothing.$$

$$\Rightarrow p_0 \circ (r_0 \circ p_0) \ll \mathcal{L}(\lambda_f, \mathfrak{t}) \;\; \text{(By Theorem 2.4(ii))}$$

$$\Rightarrow p_0 \in \mathcal{L}(\lambda_f, \mathfrak{t}) \;\; \text{(By Definition 2.4(ii))}$$

$$\Rightarrow \lambda_f(p_0) \leqslant \mathfrak{t} = \max\left\{ \inf_{c \in (p_0 \circ u_0)\circ(r_0 \circ p_0)} \lambda_f(c), \lambda_f(u_0) \right\}.$$

Hence, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. $\square$

Theorem 3.4. *Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NSS of $\grave{\mathcal{D}}$.*

- (i) *If $\mathcal{M}$ satisfies the sup-sup-inf property, and for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$, $\mathcal{U}(\lambda_t, \mathfrak{r})$, $\mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are reflexive, and $\mathcal{M}$ is a neutrosophic implicative* H$\mathfrak{B}$CK-*I of $\grave{\mathcal{D}}$, then $\mathcal{M}$ is a neutrosophic positive implicative* H$\mathfrak{B}$CK-*I of type-3.*
- (ii) *Let $\grave{\mathcal{D}}$ be a positive implicative* H$\mathfrak{B}$CK-*A. If $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a NWIH$\mathfrak{B}$CK-I of $\grave{\mathcal{D}}$, then $\mathcal{M}$ is a neutrosophic positive implicative* H$\mathfrak{B}$CK-*I of type-1.*

Proof. (i) Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be a NIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$. Then, by Theorem 3.3(ii), for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in [0,1]$, $\mathcal{U}(\lambda_t, \mathfrak{r}), \mathcal{U}(\lambda_i, \mathfrak{s})$, and $\mathcal{L}(\lambda_f, \mathfrak{t})$ are implicative H$\mathfrak{B}$CK-Is of $\grave{\mathrm{D}}$.

Since these non-empty sets are reflexive, then by Theorem 2.3(iii), they are positive implicative H$\mathfrak{B}$CK-Is of type-3. So, by Theorem 2.6, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a neutrosophic positive implicative H$\mathfrak{B}$CK-I of type-3.

(ii) Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be a NWIH$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$ and for $p_0, r_0, u_0 \in \grave{\mathrm{D}}$. Put $\mathfrak{r} = \min\left\{\inf\limits_{a \in (p_0 \circ r_0) \circ u_0} \lambda_t(a), \inf\limits_{b \in r_0 \circ u_0} \lambda_t(b)\right\}$. Thus, for all $a \in (p_0 \circ r_0) \circ u_0$ and $b \in r_0 \circ u_0$, $\lambda_t(a) \geqslant \mathfrak{r}$ and $\lambda_t(b) \geqslant \mathfrak{r}$. Since $\grave{\mathrm{D}}$ is a positive implicative H$\mathfrak{B}$CK-A, we have

$$(p_0 \circ u_0) \circ (r_0 \circ u_0) = (p_0 \circ r_0) \circ u_0 \subseteq \mathcal{U}(\lambda_t, \mathfrak{r})$$

$$r_0 \circ u_0 \subseteq \mathcal{U}(\lambda_t, \mathfrak{r}).$$

By Theorems 3.2(ii) and 2.5, we have $\mathcal{U}(\lambda_t, \mathfrak{r})$ is a weak H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$, then $p_0 \circ u_0 \subseteq \mathcal{U}(\lambda_t, \mathfrak{r})$. Therefore, for all $u \in p_0 \circ u_0$, $\lambda_t(u) \geqslant \mathfrak{r} = \min\left\{\inf\limits_{a \in (p_0 \circ r_0) \circ u_0} \lambda_t(a), \inf\limits_{b \in r_0 \circ u_0} \lambda_t(b)\right\}$. Put $\mathfrak{s} = \min\left\{\inf\limits_{c \in (p_0 \circ r_0) \circ u_0} \lambda_i(c), \inf\limits_{d \in r_0 \circ u_0} \lambda_i(d)\right\}$. Thus, for all $c \in (p_0 \circ r_0) \circ u_0$ and $d \in r_0 \circ u_0$, $\lambda_i(c) \geqslant \mathfrak{s}$ and $\lambda_i(d) \geqslant \mathfrak{s}$. Since $\grave{\mathrm{D}}$ is a positive implicative H$\mathfrak{B}$CK-A, we have

$$(p_0 \circ u_0) \circ (r_0 \circ u_0) = (p_0 \circ r_0) \circ u_0 \subseteq \mathcal{U}(\lambda_i, \mathfrak{s})$$

$$r_0 \circ u_0 \subseteq \mathcal{U}(\lambda_i, \mathfrak{s}).$$

By Theorems 3.2(ii) and 2.5, we have $\mathcal{U}(\lambda_i, \mathfrak{s})$ is a weak H$\mathfrak{B}$CK-I of $\grave{\mathrm{D}}$, then $p_0 \circ u_0 \subseteq \mathcal{U}(\lambda_i, \mathfrak{s})$. Therefore, for all $v \in p_0 \circ u_0$, $\lambda_i(v) \geqslant \mathfrak{s} = \min\left\{\inf\limits_{c \in (p_0 \circ r_0) \circ u_0} \lambda_i(c), \inf\limits_{d \in r_0 \circ u_0} \lambda_i(d)\right\}$. Put $\mathfrak{t} = \max\left\{\sup\limits_{e \in (p_0 \circ r_0) \circ u_0} \lambda_f(e), \sup\limits_{f \in r_0 \circ u_0} \lambda_f(f)\right\}$. Thus, for all $e \in (p_0 \circ r_0) \circ u_0$ and $f \in r_0 \circ u_0$, $\lambda_f(e) \leqslant \mathfrak{t}$ and $\lambda_f(f) \leqslant \mathfrak{t}$. Since $\grave{\mathrm{D}}$ is a positive implicative H$\mathfrak{B}$CK-A, we have

$$(p_0 \circ u_0) \circ (r_0 \circ u_0) = (p_0 \circ r_0) \circ u_0 \subseteq \mathcal{L}(\lambda_f, \mathfrak{t})$$

$$r_0 \circ u_0 \subseteq \mathcal{L}(\lambda_f, \mathfrak{t}).$$

By Theorems 3.2(ii) and 2.5, $\mathcal{L}(\lambda_f, \mathrm{t})$ is a weak H$\mathfrak{B}$CK-I of $\grave{\mathfrak{D}}$, then $p_0 \circ u_0 \subseteq \mathcal{L}(\lambda_f, \mathrm{t})$. Therefore, for all $w \in p_0 \circ u_0$, $\lambda_f(w) \leqslant \mathrm{t} = \max\left\{\sup_{e\in(p_0\circ r_0)\circ u_0} \lambda_f(e), \sup_{f\in r_0\circ u_0} \lambda_f(f)\right\}$. Hence, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is a neutrosophic positive implicative H$\mathfrak{B}$CK-I of type-1. □

Corollary 3.1. *Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NSS of $\grave{\mathfrak{D}}$. Then $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I (NIH$\mathfrak{B}$CK-I) if and only if the fuzzy sets λ_t, λ_i, and ${\lambda_f}^c$ are fuzzy weak IH$\mathfrak{B}$CK-Is (fuzzy IH$\mathfrak{B}$CK-Is) of $\grave{\mathfrak{D}}$.*

Proof. Let $\lambda_f^c = 1 - \lambda_f$. The conditions on λ_t and λ_i are exactly the defining conditions of fuzzy weak IH$\mathfrak{B}$CK-Is. Also,

$$1 - \sup \lambda_f = \inf(1 - \lambda_f) \quad \text{and} \quad 1 - \inf \lambda_f = \sup(1 - \lambda_f).$$

Hence the condition on λ_f is equivalent to the corresponding fuzzy weak IH$\mathfrak{B}$CK-ideal condition on λ_f^c. Therefore $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I if and only if λ_t, λ_i and λ_f^c are fuzzy weak IH$\mathfrak{B}$CK-Is.

The proof for NIH$\mathfrak{B}$CK-Is is obtained in the same way by replacing the weak conditions with the corresponding non-weak conditions. □

Corollary 3.2. *Let $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ be an NSS of $\grave{\mathfrak{D}}$. Then $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I (NIH$\mathfrak{B}$CK-I) if and only if the NSSs $\boxdot\mathcal{M} = (\lambda_t, \lambda_i, \lambda_t^c)$ and $\odot\mathcal{M} = (\lambda_f^c, \lambda_i, \lambda_f)$ are NWIH$\mathfrak{B}$CK-Is (NIH$\mathfrak{B}$CK-Is) of $\grave{\mathfrak{D}}$.*

Proof. By Corollary 3.1, $\mathcal{M} = (\lambda_t, \lambda_i, \lambda_f)$ is an NWIH$\mathfrak{B}$CK-I if and only if λ_t, λ_i and λ_f^c are fuzzy weak IH$\mathfrak{B}$CK-Is.

Now

$$\boxdot\mathcal{M} = (\lambda_t, \lambda_i, \lambda_t^c) \quad \text{and} \quad \odot\mathcal{M} = (\lambda_f^c, \lambda_i, \lambda_f).$$

Applying Corollary 3.1 to these two neutrosophic sets shows that $\boxdot\mathcal{M}$ and $\odot\mathcal{M}$ are NWIH$\mathfrak{B}$CK-Is if and only if λ_t, λ_i and λ_f^c are fuzzy weak IH$\mathfrak{B}$CK-Is. Hence the desired equivalence follows.

The NIH$\mathfrak{B}$CK-I case is proved similarly. □

4. Conclusion

In this paper, we introduced the notions of neutrosophic implicative hyper $\mathfrak{B}$CK-ideals and neutrosophic weak implicative hyper $\mathfrak{B}$CK-ideals in hyper $\mathfrak{B}$CK-algebras. Several fundamental properties of these new structures were investigated, and their relationships with previously known neutrosophic hyper $\mathfrak{B}$CK-ideals were established. In particular, we obtained characterization theorems in terms of upper and lower level subsets. Moreover, examples were provided to illustrate the new concepts and to show that some converse statements do not hold in general.

These results extend the study of neutrosophic structures in hyperalgebraic systems and provide a foundation for further research on implicative-type ideals under uncertainty. Future work may investigate other classes of neutrosophic hyper ideals and their applications to uncertain reasoning, algebraic decision-making, and knowledge representation.

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