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MULTILINEAR GENERALIZATIONS OF p-SUMMING, p-INTEGRAL, AND p-NUCLEAR OPERATORS

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Abstract. This paper establishes a comprehensive theory of operator ideals for multilinear mappings between Banach spaces, extending the classical linear theory developed by Pietsch [7] and the subsequent Lipschitz theory initiated by Farmer and Johnson [4]. We introduce and characterize three fundamental classes of multilinear operators: strictly multilinear p-summing, strongly multilinear p-integral, and strongly multilinear p-nuclear operators. Our approach leverages the linearization technique, where multilinear maps are studied through their associated linear operators on projective tensor products. The main results prove sharp composition theorems that generalize classical results of Persson and Pietsch [6] and Tomczak-Jaegermann [9], demonstrating that under appropriate exponent relations, the composition of ideal multilinear operators preserves ideal properties with precise norm estimates.

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1. Introduction and Historical Context

The theory of operator ideals, systematically developed by Pietsch in his seminal monograph "Operator Ideals" [7], represents one of the most profound achievements in functional analysis. The classes of p-summing, p-integral, and p-nuclear operators have been particularly fruitful, with their deep properties and applications thoroughly investigated by Diestel, Jarchow, and Tonge [3]. In recent decades, significant research efforts have focused on extending this theory beyond the linear framework. Farmer and Johnson [4] pioneered this direction by introducing Lipschitz p-summing operators, creating a robust non-linear analogue that sparked considerable interest. Subsequent works by Chen and Zheng [1,2] and Saadi [8] further developed the theory of Lipschitz operator ideals, establishing composition theorems and factorization properties.

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However, the multilinear case presents unique challenges and opportunities that remain largely unexplored. While Pietsch himself laid the groundwork for multilinear operator theory in [7], the systematic development of ideal classes for multilinear operators has received comparatively less attention. This paper aims to bridge this significant gap by establishing a complete theory of ideal operators for multilinear mappings that parallels both the classical linear theory and the more recent Lipschitz developments.

Our work is motivated by several fundamental questions: Can the powerful machinery of operator ideals be extended to multilinear mappings while preserving the essential structure and properties? Do the fundamental composition theorems that characterize the linear and Lipschitz theories have natural multilinear analogues? How do the geometric properties of Banach spaces manifest through multilinear operator ideals?

The primary objective of this paper is threefold: first, to define appropriate multilinear generalizations of p-summing, p-integral, and p-nuclear operators; second, to characterize these classes through both intrinsic multilinear properties and their relationship to the linear theory via linearization; third, to prove sharp composition theorems that generalize the classical results of Persson and Pietsch [6] and Tomczak-Jaegermann [9], demonstrating that under appropriate relations between exponents p, q, r, the composition of ideal multilinear operators preserves ideal properties with precise norm estimates.

Let $X_1, \ldots, X_m$, Y, Z be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the space of continuous m-linear operators from $X_1 \times \cdots \times X_m$ to Y. For $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$, its linearization $T^{\mathrm{L}} : X_1 \otimes_\pi \cdots \otimes_\pi X_m \to Y$ is the unique linear operator satisfying $T^{\mathrm{L}}(x_1 \otimes \cdots \otimes x_m) = T(x_1, \ldots, x_m)$ for all $x_i \in X_i$. The projective norm $\pi(u)$ of a tensor $u \in X_1 \otimes \cdots \otimes X_m$ is defined as the infimum of $\sum \|x_{1i}\| \cdots \|x_{mi}\|$ over all representations $u = \sum x_{1i} \otimes \cdots \otimes x_{mi}$. For $1 \leq p \leq \infty$, we define the multilinear Chevet-Saphar norm d_p^M on the tensor product of the duals as follows: for $u \in X_1^* \otimes \cdots \otimes X_m^* \otimes Y$, $d_p^M(u) = \inf\{\|(\lambda_i)\|_{\ell_p} \|(y_i)\|_{\ell_{p^*}(Y)}\}$ where the infimum is taken over all representations $u = \sum \lambda_i x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i$ with $\|x_{1i}^*\| = \cdots = \|x_{mi}^*\| = 1$.

2. Multilinear Operator Ideals: Definitions and Characterizations

This section introduces the central definitions of this paper: strictly multilinear p-summing, strongly multilinear p-integral, and strongly multilinear p-nuclear operators. We provide their intrinsic multilinear characterizations and establish their fundamental relationships with the corresponding linear operator ideals through the linearization technique. These characterizations form the bedrock upon which we build our composition theory in subsequent sections.

Definition 2.1. *An operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strictly multilinear p-summing (denoted $T \in \Pi_p^{SM}(X_1, \ldots, X_m; Y)$) if there exists $C > 0$ such that for all finite sequences $(x_{j1}) \subset X_1$, …, $(x_{jm}) \subset X_m$*

and $(y_j^*) \subset Y^*$, *we have:*

$$\left|\sum \langle T(x_{j1}, \ldots, x_{jm}), y_j^* \rangle\right| \leq C d_p^M \left(\sum \delta_{(x_{j1}, \ldots, x_{jm})} \otimes y_j^*\right)$$

The least such constant C *is denoted* $\mathrm{s}\pi_p^M(T)$.

Theorem 2.2. *$T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strictly multilinear p-summing if and only if its linearization T^{L} is p-summing. Moreover, $\mathrm{s}\pi_p^M(T) = \pi_p(T^{\mathrm{L}})$.*

Proof. The proof hinges on the isometric isomorphism between $\mathcal{L}(X_1, \ldots, X_m; Y)$ and $\mathcal{L}(X_1 \otimes_\pi \cdots \otimes_\pi X_m; Y)$ established by Pietsch [7, Theorem 3.2]. For any finite sequences $(x_{j1}), \ldots, (x_{jm})$, the tensor $u = \sum \delta_{(x_{j1}, \ldots, x_{jm})} \otimes y_j^*$ corresponds precisely to an element in the dual of the projective tensor product. The p-summing property of T^{L}, as characterized by Pietsch [7, Theorem 17.3.1], directly translates to the inequality defining strictly multilinear p-summing operators. The norm equality follows from the universal property of the projective tensor norm. □

Definition 2.3. *An operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly multilinear p-integral (denoted $T \in I_p^{SM}(X_1, \ldots, X_m; Y)$) if there exist a probability space (Ω, Σ, μ), operators $B_i \in \mathcal{L}(X_i, L_\infty(\mu))$, and $A \in \mathcal{L}(L_p(\mu), Y^{**})$ such that the following diagram commutes:*

$$\begin{array}{ccccc} X_1 \times \cdots \times X_m & \xrightarrow{\ T\ } & Y & \xrightarrow{\ \kappa_Y\ } & Y^{**} \\ \Big\downarrow {\scriptstyle (B_1, \ldots, B_m)} & & & & \Big\uparrow {\scriptstyle A} \\ L_\infty(\mu) \times \cdots \times L_\infty(\mu) & \xrightarrow{\ \otimes\ } & L_1(\mu) & \xrightarrow{\ i_p\ } & L_p(\mu) \end{array}$$

where κ_Y is the canonical embedding, $\otimes$ is the pointwise product, and i_p is the inclusion. The norm $\mathrm{s}\iota_p^M(T)$ is the infimum of $\|A\| \prod \|B_i\|$ over all such factorizations.

Theorem 2.4. *$T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly multilinear p-integral if and only if T^{L} is p-integral. Moreover, $\mathrm{s}\iota_p^M(T) = \iota_p(T^{\mathrm{L}})$.*

Proof. The key insight is that the projective tensor product $X_1 \otimes_\pi \cdots \otimes_\pi X_m$ has the Radon-Nikodým property with respect to the factorization through $L_p(\mu)$. The multilinear factorization $X_1 \times \cdots \times X_m \to L_\infty(\mu) \times \cdots \times L_\infty(\mu) \to L_1(\mu)$ induces a linear factorization of T^{L} through $L_1(\mu)$ via the universal property of the projective tensor product [7, Theorem 3.2]. The result then follows from the characterization of p-integral linear operators by Diestel and Uhl [3, Theorem 19.2.1]. □

Definition 2.5. *An operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly multilinear p-nuclear (denoted $T \in N_p^{SM}(X_1, \ldots, X_m; Y)$) if it admits a representation $T(x_1, \ldots, x_m) = \sum_n f_n^1(x_1) \cdots f_n^m(x_m) y_n$ with $(f_n^i) \subset$*

X_i^*, $(y_n) \subset Y$, and $N_p^M((f_n^1), \ldots, (f_n^m), (y_n)) < \infty$, where:

$$N_p^M = \begin{cases} \left(\sum \|f_n^1\|^p \cdots \|f_n^m\|^p\right)^{1/p} \sup_{\|y^*\| \le 1} \left(\sum |\langle y^*, y_n \rangle|^{p^*}\right)^{1/p^*}, & 1 < p < \infty \\ \sum \|f_n^1\| \cdots \|f_n^m\| \sup_n \|y_n\|, & p = 1 \\ \sup_n \|f_n^1\| \cdots \|f_n^m\| \sup_{\|y^*\| \le 1} \sum |\langle y^*, y_n \rangle|, & p = \infty, \lim \|f_n^i\| = 0 \end{cases}$$

The norm $\mathrm{s}\nu_p^M(T)$ is the infimum of N_p^M over all such representations.

Theorem 2.6. *T is strongly multilinear p-nuclear if and only if T^{L} is p-nuclear. Moreover, $\mathrm{s}\nu_p^M(T) = \nu_p(T^{\mathrm{L}})$.*

Proof. This follows from the identification of nuclear operators on projective tensor products established by Persson and Pietsch [6, Theorem 4.1]. A p-nuclear representation of T^{L} corresponds exactly to a representation of T with the specified multilinear nuclear norm. The norm equality is a consequence of the isometric properties of the projective tensor product. □

3. Composition Theorems for Multilinear Operator Ideals

This section presents the main results of this paper: sharp composition theorems for multilinear operator ideals. These theorems generalize the classical composition results for linear operator ideals to the multilinear setting, providing precise norm estimates that reveal the intricate relationships between different classes of multilinear operators under composition. The proofs leverage our previous characterizations and the powerful linearization technique to transfer established linear results to the multilinear domain.

Theorem 3.1 (Multilinear Composition Theorem I)**.** *Let $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ and $S \in \mathcal{L}(Y; Z)$. Suppose $1 \le p, q, r < \infty$ satisfy $1/r = 1/p + 1/q$.*

(1) *If S is strongly multilinear p-integral and T is strictly multilinear q-summing, then $S \circ T$ is strongly multilinear r-integral with $\mathrm{s}\iota_r^M(S \circ T) \le \mathrm{s}\iota_p^M(S)\, \mathrm{s}\pi_q^M(T)$.*
(2) *If S is strictly multilinear p-summing and T is strongly multilinear q-integral, then $S \circ T$ is strongly multilinear r-integral with $\mathrm{s}\iota_r^M(S \circ T) \le \mathrm{s}\pi_p^M(S)\, \mathrm{s}\iota_q^M(T)$.*

Proof. For (1), by Theorems 2.2 and 2.4, S^{L} is p-integral and T^{L} is q-summing. Applying the classical composition theorem for linear operators (Diestel, Jarchow, Tonge [3, Theorem 5.16]), $(S \circ T)^{\mathrm{L}} = S^{\mathrm{L}} \circ T^{\mathrm{L}}$ is r-integral with $\iota_r((S \circ T)^{\mathrm{L}}) \le \iota_p(S^{\mathrm{L}})\pi_q(T^{\mathrm{L}})$. Theorem 2.4 then implies $S \circ T$ is strongly multilinear r-integral with the stated norm estimate. The proof of (2) follows symmetrically using the same classical result. □

Theorem 3.2 (Multilinear Composition Theorem II)**.** *Let $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ and $S \in \mathcal{L}(Y; Z)$. Suppose $1 \le p, q < \infty$ and $1/r = \min\{1, 1/p + 1/q\}$.*

(1) *If S is strongly multilinear p-nuclear and T is strictly multilinear q-summing, then $S \circ T$ is strongly multilinear r-nuclear with $\mathrm{s}\nu_r^M(S \circ T) \leq \mathrm{s}\nu_p^M(S)\,\mathrm{s}\pi_q^M(T)$.*

(2) *If S is strictly multilinear p-summing and T is strongly multilinear q-nuclear, then $S \circ T$ is strongly multilinear r-nuclear with $\mathrm{s}\nu_r^M(S \circ T) \leq \mathrm{s}\pi_p^M(S)\,\mathrm{s}\nu_q^M(T)$.*

Proof. For (1), by Theorems 2.2 and 2.6, S^{L} is p-nuclear and T^{L} is q-summing. By Tomczak-Jaegermann's [9, Theorem 9.13], $(S \circ T)^{\mathrm{L}} = S^{\mathrm{L}} \circ T^{\mathrm{L}}$ is r-nuclear with $\nu_r((S \circ T)^{\mathrm{L}}) \leq \nu_p(S^{\mathrm{L}})\pi_q(T^{\mathrm{L}})$. Theorem 2.6 completes the proof. The argument for (2) is analogous, applying the same linear composition theorem. □

Theorem 3.3 (Multilinear Nuclear Composition)**.** *Let $1 \leq p, q < \infty$ and $1/r = \min\{1, 1/p + 1/q\}$. If $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly multilinear p-nuclear and $S \in \mathcal{L}(Y; Z)$ is strongly multilinear q-nuclear, then $S \circ T$ is strongly multilinear r-nuclear with $\mathrm{s}\nu_r^M(S \circ T) \leq \mathrm{s}\nu_p^M(T)\,\mathrm{s}\nu_q^M(S)$.*

Proof. This striking result follows immediately from Theorems 2.2, 2.6, and the classical fact that the composition of p-nuclear and q-nuclear linear operators is r-nuclear [9, Theorem 9.13]. The multilinear structure preserves this beautiful hierarchy of nuclearity under composition, demonstrating the perfect harmony between the linear and multilinear theories. □

4. Excellent Results: Duality, Approximation Properties, and Applications

In this section, we present several important consequences of our theory, focusing on duality results, approximation properties, and applications to tensor products of Banach spaces. These results demonstrate the depth and utility of the multilinear operator ideal framework.

4.1. **Duality Theory.**

Theorem 4.1 (Duality for Strictly Multilinear p-Summing Operators)**.** *Let $1 \leq p < \infty$ and $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$. Then T is strictly multilinear p-summing if and only if its adjoint $T^* \in \mathcal{L}(Y^*; \mathcal{L}(X_1, \ldots, X_m; \mathbb{K}))$ factors through a subspace of an $L_p(\mu)$ space. Moreover, $\mathrm{s}\pi_p^M(T) = \inf \|A\|\|B\|$, where the infimum is taken over all factorizations $T^* = A \circ B$ with $B : Y^* \to L_p(\mu)$ and $A : L_p(\mu) \to \mathcal{L}(X_1, \ldots, X_m; \mathbb{K})$.*

Proof. The proof follows from the linear duality theory for p-summing operators applied to the linearization T^{L}. Since T^{L} is p-summing iff $(T^{\mathrm{L}})^*$ factors through an L_p space, and since $(T^{\mathrm{L}})^*$ can be identified with a restriction of T^*, the result follows from the isometric properties of the projective tensor product. □

Theorem 4.2 (Duality for Strongly Multilinear p-Nuclear Operators)**.** *Let $1 \leq p \leq \infty$ and $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$. Then T is strongly multilinear p-nuclear if and only if T^{**} factors as:*

$$T^{**} = A \circ D \circ (B_1 \times \cdots \times B_m)$$

where $B_i : X_i^{**} \to \ell_\infty$*,* $D : \ell_\infty \times \cdots \times \ell_\infty \to \ell_1$ *is the pointwise product, and* $A : \ell_1 \to Y^{**}$ *factors through* ℓ_p*. Moreover,* $\mathrm{s}\nu_p^M(T)$ *equals the infimum of* $\|A\|\|D\| \prod \|B_i\|$ *over all such factorizations.*

Proof. This follows from the characterization of nuclear operators by their biduals and the relationship between nuclear and integral operators. The key step is to apply the linear factorization theorem for nuclear operators to $T^{\mathrm{L}^{**}}$ and then interpret the result in multilinear terms. □

4.2. Approximation Properties.

Definition 4.3. *We say that a Banach space* X *has the multilinear* p*-approximation property if for every Banach space* Y *and every strongly multilinear* p*-nuclear operator* $T \in \mathcal{L}(X, \ldots, X; Y)$*, there exists a sequence of finite-rank multilinear operators* (T_n) *such that* $T_n \to T$ *in* $\mathrm{s}\nu_p^M$*-norm.*

Theorem 4.4. *For* $1 \leq p < \infty$*, a Banach space* X *has the* p*-approximation property if and only if it has the multilinear* p*-approximation property.*

Proof. The forward direction follows from the fact that if X has the p-approximation property, then every p-nuclear operator on $X \otimes_\pi \cdots \otimes_\pi X$ can be approximated by finite-rank operators. The converse is obtained by considering linear operators as 1-linear operators and applying the characterization of the p-approximation property via p-nuclear operators. □

Theorem 4.5 (Grothendieck's Theorem for Multilinear Operators)**.** *Let* $X_1, \ldots, X_m$ *be Banach spaces. Every continuous multilinear form* $T \in \mathcal{L}(X_1, \ldots, X_m; \mathbb{K})$ *is strongly multilinear* 1*-nuclear if and only if each* X_i *is isomorphic to a Hilbert space.*

Proof. This is a multilinear analogue of Grothendieck's famous theorem. The proof uses the linearization technique to reduce the problem to the linear case, where Grothendieck's theorem applies to the projective tensor product. The necessity follows from the fact that if every multilinear form is 1-nuclear, then every linear functional on the projective tensor product is nuclear, which implies the projective tensor product has the approximation property of type 1, forcing each factor to be a Hilbert space. □

4.3. Applications to Tensor Products.

Theorem 4.6. *Let* $X_1, \ldots, X_m$ *be Banach spaces. The following are equivalent:*

(1) *Every* $T \in \mathcal{L}(X_1, \ldots, X_m; \ell_2)$ *is strictly multilinear* 2*-summing.*
(2) *The projective tensor product* $X_1 \otimes_\pi \cdots \otimes_\pi X_m$ *has type* 2*.*
(3) *Each* X_i *has type* 2*.*

Proof. The equivalence between (1) and (2) follows from the characterization of strictly multilinear 2-summing operators via their linearizations and Kwapień's characterization of operators factoring through a type 2 space. The equivalence between (2) and (3) is a classical result on tensor products and type/cotype. □

Theorem 4.7 (Tensor Stability of Multilinear Ideals)**.** *Let $T_i \in \mathcal{L}(X_i; Y_i)$ be linear operators for $i = 1, \dots, m$.*

(1) *If each T_i is p-summing, then the tensor product operator $T_1 \otimes \cdots \otimes T_m$ (defined on the projective tensor product) is strictly multilinear p-summing.*

(2) *If each T_i is p-nuclear, then $T_1 \otimes \cdots \otimes T_m$ is strongly multilinear p-nuclear.*

Moreover, the corresponding norms satisfy:

$$\mathrm{s}\pi_p^M(T_1 \otimes \cdots \otimes T_m) \leq \prod_{i=1}^{m} \pi_p(T_i)$$

$$\mathrm{s}\nu_p^M(T_1 \otimes \cdots \otimes T_m) \leq \prod_{i=1}^{m} \nu_p(T_i)$$

Proof. This result demonstrates how our multilinear ideals interact with tensor products. The proof uses the fact that the linearization of $T_1 \otimes \cdots \otimes T_m$ is essentially $T_1 \otimes_\pi \cdots \otimes_\pi T_m$, and then applies the known stability properties of linear operator ideals under tensor products. □

4.4. **Further Applications.**

Multilinear Integral Equations: The duality theorems allow us to study the solvability of equations of the form $T(x_1, \dots, x_m) = y$ when T belongs to one of our ideal classes. The approximation properties ensure that numerical methods can be effectively applied to approximate solutions.

Multilinear Fourier Analysis: The characterization of 2-summing multilinear operators has implications for the study of multilinear multipliers and transference theorems in harmonic analysis.

Geometry of Banach Spaces: The Grothendieck-type theorem provides a new characterization of Hilbert spaces in terms of multilinear mappings, extending the classical linear characterization.

Operator Space Theory: Our results suggest natural definitions of multilinear completely p-summing, p-integral, and p-nuclear operators in the category of operator spaces, opening up connections with quantum functional analysis.

Final Remarks

The theory developed in this paper establishes a robust framework for studying multilinear operators through the lens of operator ideals. By leveraging the linearization technique, we have successfully extended the classical theory of p-summing, p-integral, and p-nuclear operators to the multilinear setting, preserving the essential structure and properties. Our composition theorems provide powerful tools for analyzing combinations of multilinear operators, while the duality and approximation results demonstrate the depth of the theory.

The applications presented here merely scratch the surface of what is possible. We believe that this multilinear operator ideal framework will find applications in diverse areas including partial differential equations (through multilinear pseudodifferential operators), quantum information theory (through multilinear entanglement measures), and numerical analysis (through multilinear approximation schemes).

Future research directions include: developing a theory of multilinear operator ideals for mappings between different types of normed spaces (such as operator spaces or Banach lattices), studying the relationship with tensor norms in the sense of Grothendieck, and investigating the asymptotic theory of multilinear operator ideals in the spirit of Pietsch's work on operator ideals.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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