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COMPARISON PRINCIPLE APPROACH FOR TWO MEASURE STABILITY OF FRACTIONAL DYNAMIC SYSTEMS ON TIME SCALE WITH APPLICATION TO LIBRARY INFORMATION SYSTEMS

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Abstract. This work develops a stability analysis framework for Caputo fractional dynamic equations on time scales using the comparison principle approach. By leveraging the comparison theorem, we derive sufficient conditions for two-measure (m, m_0) stability, ensuring that system solutions remain bounded or converge to a desired equilibrium under specific measures. The framework employs vector Lyapunov functions and the generalized Caputo fractional delta Dini derivative to establish rigorous stability criteria, unifying the analysis of fractional differential equations, difference equations, and hybrid systems with mixed continuous-discrete dynamics. To demonstrate applicability, we apply is results to library information science problems, where a quadratic Lyapunov function and the comparison principle confirm stability with respect to two measures; the sum of squares and the Euclidean norm. Numerical simulations for a two-category library support the theoretical predictions.

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1. Introduction

The study of dynamic equations on time scales has emerged as a powerful framework for unifying continuous and discrete dynamical systems, offering a versatile approach to modeling systems with mixed continuous-discrete behaviors ([12,18]). Within this framework, fractional dynamic equations have gained significant attention due to their ability to capture memory effects and non-local dynamics, which are prevalent in real-world phenomena such as biological systems, control theory, and network dynamics. Among fractional derivatives, the Caputo fractional derivative stands out for its compatibility

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with initial conditions, making it particularly suitable for stability analysis. However, the stability analysis of Caputo fractional dynamic equations on time scales presents unique challenges, as traditional methods for differential and difference equations often fail to seamlessly translate to this hybrid setting.

In this work, we address these challenges by developing a stability analysis framework for Caputo fractional dynamic equations on time scales using the comparison principle approach. Building on the established a comparison theorem in [13], we extend the stability results in [7,8,10,19,20] to the concept of two-measure stability. The two-measure stability framework provides a unified and flexible approach to stability analysis, allowing us to simultaneously consider different measures of system behavior and unify various stability concepts under a single theoretical umbrella.

The foundation of our analysis lies in the comparison theorem, which relates the behavior of the original fractional dynamic system on time scales to that of a simpler comparison system. Leveraging this theorem, we derive sufficient conditions for two-measure (m, m_0) stability, ensuring that the system's solutions remain bounded or converge to a desired equilibrium under specific measures. Our approach uses vector Lyapunov functions and the generalized Caputo fractional delta Dini derivative (introduced in [11,13]) to establish rigorous stability criteria, unifying the stability analysis of fractional differential and difference equations while extending to hybrid systems with mixed continuous-discrete dynamics. To demonstrate applicability, we turn to Library Information Science (LIS), which deals with the acquisition, organization, preservation, and dissemination of information resources [15–17]. Dynamic models are increasingly used in LIS to understand how resource levels evolve under different policies, user demands, and external perturbations; fractional calculus captures the memory and hereditary effects typical of human-driven systems, while time scales unify discrete and continuous dynamics. The contributions of this work are twofold: first, we introduce and prove a two-measure stability theorem that generalizes existing stability results and offers a more nuanced understanding of system behavior; second, we support these theoretical findings with an illustrative application to an LIS model, demonstrating the applicability of our framework to systems with complex temporal dynamics.

Consider the Caputo fractional dynamic system for $0 < \alpha < 1$:

$$\begin{aligned} {}^{C}\Delta^{\alpha} \upsilon &= \Xi(t, \upsilon), \quad t \in \mathbb{T}, \\ \upsilon(t_0) &= \upsilon_0, \quad t_0 \geq 0, \end{aligned} \tag{1}$$

where $\Xi \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$ with $\Xi(t, 0) \equiv 0$, and ${}^{C}\Delta^{\alpha}\upsilon$ denotes the Caputo fractional delta derivative (FrΔD) of $\upsilon \in \mathbb{R}^N$ of order α. Let $\upsilon(t) = \upsilon(t, t_0, \upsilon_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^N]$ (where the fractional derivative of order α exists and is rd-continuous) be the unique solution of (1) – existence and uniqueness results can be found in [14]. To establish the (m_0, m)-stability of (1), we consider the comparison system of the form:

$$ {}^{C}\Delta^{\alpha}\chi = \Phi(t,\chi), \quad \chi(t_0) = \chi_0 \geq 0, \tag{2} $$

where $\Phi : \mathbb{T} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with $\Phi(t,0) \equiv 0$. Assuming unique solutions $\chi(t) = \chi(t;t_0,\chi_0) \in C^{\alpha}_{rd}[\mathbb{T},\mathbb{R}^n_+]$ exist (see [14]).

By bridging the gap between fractional calculus, time scale theory, and stability analysis, this work advances the understanding of dynamic systems with complex temporal behaviors. The proposed framework not only enriches the theoretical foundations of stability analysis but also provides practical tools for analyzing and designing systems in engineering, biology, and other disciplines where fractional dynamics and hybrid behaviors are prevalent.

In the next section, we review the necessary preliminaries on time scale calculus, Caputo fractional derivatives, and two-measure stability concepts, and we restate the comparison theorem established in [11]. In Section 3, we develop $(m, m_0)-$stability criteria for system (1). Following this, in Section 4, we present an illustrative example to validate the theoretical results. Finally in Section 5, we apply the obtained results in solving library and information problem.

2. Preliminaries, Definitions, and Notations

Definition 2.1 ([5]). *For a time scale $\mathbb{T}$, the forward and backward jump operators are respectively given by*

$$ \sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, $$

for all $t \in \mathbb{T}$.

A point $t \in \mathbb{T}$ is classified according to the behavior of these operators as follows:

(i) *t is* right-scattered *whenever $\sigma(t) > t$;*
(ii) *t is* left-scattered *whenever $\rho(t) < t$;*
(iii) *t is* right-dense *if $t < \max\mathbb{T}$ and $\sigma(t) = t$;*
(iv) *t is* left-dense *if $t > \min\mathbb{T}$ and $\rho(t) = t$.*

Definition 2.2 ([5]). *The graininess function $\mu : \mathbb{T} \to [0,\infty)$ for $t \in \mathbb{T}$ is defined as*

$$ \mu(t) = \sigma(t) - t. $$

Definition 2.3 ([5]). *Let $C_{rd}(\mathbb{T})$ denote the set of all functions $\psi : \mathbb{T} \to \mathbb{R}$ that are continuous at right-dense points and possess finite left-sided limits at left-dense points of $\mathbb{T}$.*

Definition 2.4 ([5]). *A function $\phi : [0,r] \to [0,\infty)$ is of class $\mathcal{K}$ if it is continuous, and strictly increasing on $[0,r]$ with $\phi(0) = 0$.*

Definition 2.5. *We define the Caputo FrΔDiD of the Lyapunov function,* $\mathcal{L}(t,\upsilon) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N_+]$ *(which is locally Lipschitz with respect to its second argument and satisfies* $\mathcal{L}(t,0)\equiv 0$*) along the trajectories of solutions of the system (1) as:*

$$
\begin{aligned}
{}^{C}\Delta^{\alpha}_{+}\mathcal{L}(t,\upsilon) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{&\mathcal{L}(\sigma(t),\upsilon(\sigma(t))-\mathcal{L}(t_0,\upsilon_0) \\
&-\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^{r+1}\binom{\alpha}{r}\left[\mathcal{L}(\sigma(t)-r\mu,\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon(t))-\mathcal{L}(t_0,\upsilon_0)\right]\Big\},
\end{aligned}
\tag{3}
$$

where $t\in\mathbb{T}$, $\upsilon,\upsilon_0\in\mathbb{R}^N$, $\mu=\sigma(t)-t$, *and* $\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon)\in\mathbb{R}^N$.
and can be represented as

$$
\begin{aligned}
{}^{C}\Delta^{\alpha}_{+}\mathcal{L}(t,\upsilon) \;=\; &\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{\mathcal{L}(\sigma(t),\upsilon(\sigma(t)) \\
&+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^{r}\binom{\alpha}{r}\left[\mathcal{L}(\sigma(t)-r\mu,\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon(t))\right]\Big\} \qquad (4)\\
&-\frac{\mathcal{L}(t_0,\upsilon_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \qquad (5)
\end{aligned}
$$

Given that $\lim\limits_{N\to\infty}\sum_{r=0}^{N}(-1)^r\binom{\alpha}{r}=0$ where $\alpha\in(0,1)$, and $\lim\limits_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}]=\infty$ then it is easy to see that

$$
\lim_{\mu\to 0^+}\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^r\binom{\alpha}{r} = -1 \tag{6}
$$

Also

$$
{}^{C\mathbb{T}}D^{\alpha}_{+}\psi^{\Delta}(t) = {}^{RL\mathbb{T}}\,D^{\alpha}_{+}\psi^{\Delta}(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^r\binom{\alpha}{r}[\psi(\sigma(t)-r\mu)],\quad t\ge t_0, \tag{7}
$$

so that,

$$
{}^{C\mathbb{T}}D^{\alpha}_{+}\psi^{\Delta}(t) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^r\binom{\alpha}{r} = {}^{RL\mathbb{T}}\,D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)},\quad t\ge t_0. \tag{8}
$$

For the purpose of this work, the inequality between vectors is taken as component-wise inequality.

Definition 2.6. *The fractional dynamic system (1) is said to be*

(M_1) $(m_0,m)-$*stable if, for each* $\epsilon>0$, $t_0\in\mathbb{T}$, $\exists$ *a positive function* $\delta=\delta(t_0,\epsilon)$ *that is rd-continuous in* t_0 *for each* ϵ *such that* $m_0(t_0,\upsilon)<\delta \implies m(t,\upsilon(t))<\epsilon$, $t\ge t_0$, *where* $\upsilon(t)=\upsilon(t;t_0,\upsilon_0)$ *is any solution of system (1).*

Definition 2.7. *Let* $m_0,m\in\Lambda$. *Then we say that*

(i) m_0 *is finer than* m *if there exists a* $\gamma>0$ *and a function* $\tau\in\mathcal{CK}$ *such that* $m_0(t,\upsilon)<\gamma$ *implies* $m(t,\upsilon)\le\tau(t,m_0(t,\upsilon))$;

(ii) m_0 *is uniformly finer than* m *if there exists a* $\gamma > 0$ *and a function* $\tau \in \mathcal{CK}$ *such that* $m_0(t, \upsilon) < \gamma$ *implies* $m(t, \upsilon) \leq \tau(m_0(t, \upsilon))$.

Definition 2.8. *The Lyapunov function* $\mathcal{L} \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *is said to be*

(i) $m-$*positive definite if there exists a* $\gamma > 0$ *and a function* $\phi \in \|$ *such that* $\phi(m(t, \upsilon)) \leq \mathcal{L}(t, \upsilon)$ *whenever* $m(t, \upsilon) < \gamma$;

(ii) $m-$*decrescent if there exists a* $\gamma > 0$ *and a function* $\chi \in \mathcal{K}$ *such that* $\mathcal{L}(t, \upsilon) \leq \chi(m(t, \upsilon))$ *whenever* $m(t, \upsilon) < \gamma$;

(iii) $m-$*weakly decrescent if there exists a* $\gamma > 0$ *and a function* $\chi \in \mathcal{CK}$ *such that* $\mathcal{L}(t, \upsilon) \leq \chi(m(t, \upsilon))$ *whenever* $m(t, \upsilon) < \gamma$.

Lemma 2.1 (**Comparison Theorem**). [13] *Assume that*

(i) $\Phi \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ *and* $\Phi(t, \chi)\mu$ *is non-decreasing in* χ.

(ii) $\mathcal{L} \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *is locally Lipschitz in the second variable such that*

$$^{C}\Delta^{\alpha}_{+}\mathcal{L}(t, \upsilon) \leq \Phi(t, \mathcal{L}(t, \upsilon)), \quad (t, \upsilon) \in \mathbb{T} \times \mathbb{R}^N. \tag{9}$$

(iii) $z(t) = z(t; t_0, \chi_0)$ *is the maximal solution of* (2) *existing on* $\mathbb{T}$.

Then

$$\mathcal{L}(t, \upsilon(t)) \leq z(t), \quad t \geq t_0 \tag{10}$$

provided that

$$\mathcal{L}(t_0, \upsilon_0) \leq \chi_0, \tag{11}$$

where $\upsilon(t) = \upsilon(t; t_0, \upsilon_0)$ *is any solution of* (1), $t \in \mathbb{T}, \ \ t \geq t_0$.

3. Main Results

Theorem 3.1. *Assume the following conditions are satisfied:*

(1) $\mathcal{L}(t, \upsilon(t)) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *be such that*
   (i) $\mathcal{L}$ *is locally Lipschitzian in* υ *with* $\mathcal{L}(t, 0) \equiv 0$,
   (ii) $\phi(m(t, \upsilon)) \leq \mathcal{L}_0(t, \upsilon)$ *where* $\mathcal{L}_0(t, \upsilon) = \sum_{j=1}^{N} \mathcal{L}_j(t, \upsilon(t))$ *and* $\phi \in \mathcal{K}$,

(2) *for* $m_0, m \in \Lambda$,
   (i) m_0 *is uniformly finer than* m,
   (ii) $\mathcal{L}(t, \upsilon)$ *is* m_0*- decresecent*,

(3) $\Phi \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ *is quasimonotone nondecreasing with respect to* χ *at all* $t \in \mathbb{T}$, $\Phi(t, 0) \equiv 0$, *and*

$$^{C}\Delta^{\alpha}_{+}\mathcal{L}(t, \upsilon(t)) \leq \Phi(t, \mathcal{L}(t, \upsilon(t)));$$

(4) *the zero solution of the comparison equation* (2) *is stable.*

Then the system (1) *is* $(m_0, m)-$*stable.*

Proof. Let $\epsilon > 0$ be an arbitrary small number, the stability of the zero solution, $\chi = 0$ of (2) implies that for $\phi(\epsilon) > 0$, and $t_0 \in \mathbb{T}$, we can find a $\lambda = \lambda(t_0, \epsilon) > 0$ such that

$$\sum_{j=1}^{n} \chi_j(t; t_0, \chi_0) < \phi(\epsilon), \quad \text{at all } t \geq t_0, \tag{12}$$

whenever $\sum_{j=1}^{n} \chi_{0_j} < \lambda$, where $\chi(t) = \chi(t, t_0, \chi_0)$ is any solution of the comparison system (2). By the m_0-decrescent property of $\mathcal{L}(t, \upsilon(t))$ and since m_0 is uniformly finer that m, then we can find a positive number γ and functions $\chi \in \mathcal{K}$ and $\beta \in \mathcal{CK}$, such that

$$\mathcal{L}_0(t_0, \upsilon_0) \leq \chi(m_0(t_0, \upsilon_0)) \text{ if } m_0(t_0, \upsilon_0) < \gamma, \text{and } m(t_0, \upsilon_0) \leq \beta(m_0(t_0, \upsilon_0)). \tag{13}$$

Combining (13) and assumption 1(ii) of the theorem, we have that for $(t_0, \upsilon_0) \in (\mathbb{T}, \mathbb{R}^N)$,

$$\phi(m(t_0, \upsilon_0)) \leq \mathcal{L}_0(t_0, \upsilon_0) \leq \chi(m_0(t_0, \upsilon_0)), \tag{14}$$

whenever $m_0(t_0, \upsilon_0) < \gamma$.

Now, we claim that for any solution $\upsilon(t) = \upsilon(t; t_0, \upsilon_0)$, and numbers $\delta = \delta(t_0, \epsilon) \in (0, \gamma]$, $\chi(\delta) < \lambda$, such that

$$m(t, \upsilon(t)) < \epsilon, \text{ whenever } m_0(t_0, \upsilon_0) < \delta. \tag{15}$$

If this claim were false, then there would exists a time $t_1 > t_0$ such that

$$m(t_1, \upsilon(t_1)) \geq \epsilon \text{ and } m(t, \upsilon(t)) < \epsilon, \ t \in [t_0, t_1). \tag{16}$$

However, from Lemma 2.1, we have that

$$\mathcal{L}(t, \upsilon(t)) \leq z(t), \tag{17}$$

for $t \in [t_0, t_1)$, where $z(t) = z(t; t_0, \chi_0)$ is the maximal solution of (2).

Combining assumption 1(ii), (17), (16), and (12), at time t_1, we get

$$\phi(\epsilon) \leq \mathcal{L}_0(t_1, \upsilon(t_1)) \leq z_0(t_1) < \phi(\epsilon), \text{ where, } z_0(t_1) = \sum_{j=1}^{n} z_i(t_1)$$

which is a contradiction so the claim (15) is true, and therefore, (1) is (m_0, m)-stable. □

4. Illustration

Consider the system

$$\begin{aligned} {}^{C}\Delta^{\alpha} x(t) &= \quad -3x_1 - x_1^2 \cos^2 x_2 - x_2^2 \sin^2 x_2 \\ {}^{C}\Delta^{\alpha} x_2(t) &= -x_2^2 \cos^2(x_1) - 3x_2 - x_1^2 \sin^2(x_2) \end{aligned} \tag{18}$$

for $t \geq t_0$ and $\alpha \in (0, 1)$, with initial conditions

$$x_1(t_0) = x_{10} \quad \text{and} \quad x_2(t_0) = x_{20},$$

where $x = (x_1, x_2)$ and $f = (f_1, f_2)$.

Consider a vector $V = (\mathcal{V}_1, \mathcal{V}_2)^T$, where
$\mathcal{V}_1(t, x_1, x_2) = |x_1|$ and $\mathcal{V}_2(t, x_1, x_2) = |x_2|$, with $x = (x_1, x_2) \in \mathbb{R}^2$, so that the associated norm $\|x\| = \sqrt{x_1^2 + x_2^2}$.
Now

$$V_0(t,x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = |x_1| + |x_2|$$

and so $b(\|x\|) \leq V_0(t,x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, where $a, b \in \mathcal{K}$. From (4), we compute the Caputo fractional Dini derivative for $\mathcal{V}_1(t, x_1, x_2) = |x_1|$ as follows

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathcal{V}_1(t,x) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{\mathcal{V}_1(\sigma(t), x(\sigma(t))\\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} [\mathcal{V}_1(\sigma(t) - r\mu, x(\sigma(t)) - \mu^\alpha f_1(t, x(t))] \Bigg\}\\
&\quad - \frac{\mathcal{V}_1(t_0, x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{|x(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} [|x(\sigma(t)) - \mu^\alpha f_1(t,x)|]\right\}\\
&\quad - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
&\leq \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{|x(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} [|x(\sigma(t))| + |\mu^\alpha f_1(t,x)|]\right\}\\
&\quad - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
&\leq \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{|x(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} |x(\sigma(t))|\\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r)|\mu^\alpha f_1(t,x)|\Bigg\} - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
&\leq |x(\sigma(t))| \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r}\\
&\quad + |f_1(t,x)| \limsup_{\mu\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

Applying (6) and (8) we have

$${}^{C}\Delta_{+}^{\alpha}\mathcal{V}_1(t,x) = \frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_1(t;x)| - \frac{|x_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$$

$$^{C}\Delta_{+}^{\alpha}\mathcal{V}_1 \leq \frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_1(t;x)|.$$

As $t \to \infty$, $\frac{|x(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \to 0$, then

$$\begin{aligned}
^{C}\Delta_{+}^{\alpha}\mathcal{V}_1 &\leq -|f_1(t;x)| \\
&= -\left[-3x_1 - x_1^2\cos^2 x_2 - x_2^2 \sin^2 x_2\right] \\
&\leq -[|-3x|] - \left[|x_1^2\cos^2 x_2 - x_2^2\sin^2 x_2|\right] \\
&\leq -3|x| \\
^{C}\Delta_{+}^{\alpha}\mathcal{V}_1 &\leq -3\mathcal{V}_1 + 0\mathcal{V}_2. \qquad (19)
\end{aligned}$$

Similarly, compute the Caputo fractional Dini derivative for $\mathcal{V}_2(t,x,y) = |y|$ as follows:

$$\begin{aligned}
^{C}\Delta_{+}^{\alpha}\mathcal{V}_2(t,y) &= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\mathcal{V}_2(\sigma(t),y(\sigma(t))) \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r}\left[\mathcal{V}_2(\sigma(t)-r\mu, y(\sigma(t)) - \mu^\alpha f_2(t,y(t))\right]\Bigg\} \\
&\quad -\frac{\mathcal{V}_2(t_0,y_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{|y(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r}[|y(\sigma(t)) - \mu^\alpha f_2(t,y)|]\right\} \\
&\quad -\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{|y(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r}[|y(\sigma(t))| + |\mu^\alpha f_2(t,y)|]\right\} \\
&\quad -\frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{|y(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r}|y(\sigma(t))| \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r}|\mu^\alpha f_2(t,y)|\Bigg\} - \frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\leq |y(\sigma(t))|\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r} \\
&\quad + |f_2(t,y)|\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r\binom{\alpha}{r} - \frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

Applying (6) and (8) we have

$$^{C}\Delta_{+}^{\alpha}\mathcal{V}_2 = \frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_2(t;y)| - \frac{|y_0|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$$

$$^{C}\Delta_{+}^{\alpha}\mathcal{V}_2 \leq \frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_2(t;y)|.$$

As $t \to \infty$, $\frac{|y(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \to 0$, then,

$$\begin{aligned}
^{C}\Delta_{+}^{\alpha}\mathcal{V}_2 &\leq -|f_2(t;x_2)| \\
&= -\big[| - x_2^2\cos^2 x_1 - 3x_2 - x_1^2\sin^2 x_2|\big] \\
&\leq -[| - 3x_2|] - \big[| - x_2^2\cos^2 x_1 - x_1^2\sin^2 x_2|\big] \\
&\leq -[| - 3x_2|] \\
&\leq -3|x_2|.
\end{aligned}$$

Therefore,

$$^{C}\Delta_{+}^{\alpha}\mathcal{V}_2 \leq 0\mathcal{V}_1 - 3\mathcal{V}_2. \tag{20}$$

Combining (19) and (20), we have that

$$^{C}\Delta_{+}^{\alpha}\mathcal{V} \leq \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{pmatrix} = g(t,\mathcal{V}). \tag{21}$$

Now, consider the comparison system

$$^{C}\Delta_{+}^{\alpha}\chi = g(t,\chi) = A\chi, \tag{22}$$

where $A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$.

The vectorial inequality (21) and all other conditions of Theorem 3.1 are satisfied if A has eigen values with negative real parts, since the eigen values of A are both -3 then (18) is uniformly stable.

5. Applications

Application I: Library Circulation Dynamics. Consider a university library that monitors two daily metrics which we will define as:

$x_1(t)$: deviation (in hundreds) of the number of book checkouts from a desired equilibrium and $x_2(t)$: deviation (in hundreds) of the number of patron visits from a desired equilibrium.

Positive x_i means higher than target, negative means lower. The library administration applies a feedback policy that depends on the current deviations, but due to delays in acquisition, staffing adjustments, and user habit persistence, the system exhibits memory effects which can be modelled by

a Caputo fractional derivative of order $\alpha \in (0,1)$, with:
Linear correction: $-3x_i$ (immediate proportional response), saturation terms: $-x_i^3$ prevents excessive correction when deviations are large and interaction: an excess in checkouts may reduce visits (because users borrow online instead of coming in) and vice versa. These are modelled by $-x_2^2x_1$ and $-x_1^2x_2$ respectively.
So, our system is:

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} x_1(t) &= -3x_1 - x_1^3 - x_2^2 x_1, \\ {}^{C}\Delta_{+}^{\alpha} x_2(t) &= -3x_2 - x_2^3 - x_1^2 x_2, \end{aligned} \qquad t \geq t_0, \tag{23}$$

with initial conditions $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}$. The origin is an equilibrium ($f_1(0,0) = f_2(0,0) = 0$). Define the vector Lyapunov function $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)^T$ by

$$\mathcal{L}_1(t, x_1, x_2) = |x_1|, \qquad \mathcal{L}_2(t, x_1, x_2) = |x_2|.$$

The associated norm is $\|x\| = \sqrt{x_1^2 + x_2^2}$. The sum

$$\mathcal{L}_0(t,x) = \mathcal{L}_1 + \mathcal{L}_2 = |x_1| + |x_2|$$

satisfies $\|x\| \leq \mathcal{L}_0 \leq \sqrt{2}\|x\|$, so taking $b(r) = r$, $a(r) = \sqrt{2}r$ we have $b(\|x\|) \leq \mathcal{L}_0 \leq a(\|x\|)$ with $a, b \in \mathcal{K}$.

We choose the two measures:

$$m_0(t,x) = \mathcal{L}_0(t,x) = |x_1| + |x_2|, \qquad m(t,x) = \|x\|.$$

Then m_0 is uniformly finer than m because $m \leq m_0$. Also $\mathcal{L}$ is m_0-decrescent since each $\mathcal{L}_i \leq m_0$. We compute ${}^{C}\Delta_{+}^{\alpha}\mathcal{L}_1$ following the same steps as in the illustrative example.

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathcal{L}_1(t,x) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \Bigg\{ \mathcal{L}_1(\sigma(t), x(\sigma(t))) \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} \left[\mathcal{L}_1(\sigma(t) - r\mu, x(\sigma(t)) - \mu^\alpha f_1(t, x(t))\right] \Bigg\} \\ &\quad - \frac{\mathcal{L}_1(t_0, x_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \left\{ |x_1(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} [|x_1(\sigma(t)) - \mu^\alpha f_1(t,x)|] \right\} \\ &\quad - \frac{|x_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \left\{ |x_1(\sigma(t))| + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} [|x_1(\sigma(t))| + |\mu^\alpha f_1(t,x)|] \right\} \end{aligned}$$

$$
\begin{aligned}
& -\frac{|x_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
\leq \quad & |x_1(\sigma(t))| \limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} \\
& + |f_1(t,x)| \limsup_{\mu \to 0^+} \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} - \frac{|x_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Using the known identities (6) and (8):

$$
\limsup_{\mu \to 0^+} \frac{1}{\mu^\alpha} \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad \limsup_{\mu \to 0^+} \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r \binom{\alpha}{r} = -1.
$$

Thus

$$
{}^C\Delta_+^\alpha \mathcal{L}_1 \leq \frac{|x_1(\sigma(t))|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} - |f_1(t,x)| - \frac{|x_{10}|(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
$$

As $t \to \infty$, the terms with $(t-t_0)^{-\alpha}$ vanish. So that

$$
{}^C\Delta_+^\alpha \mathcal{L}_1 \leq -|f_1(t,x)|.
$$

Now we bound $|f_1|$ from below. From (23),

$$
f_1 = -3x_1 - x_1^3 - x_2^2 x_1.
$$

Using the reverse triangle inequality,

$$
|f_1| \geq 3|x_1| - |x_1|^3 - x_2^2|x_1|.
$$

For small deviations, say $|x_1|, |x_2| \leq 1$, we have $|x_1|^3 \leq |x_1|$ and $x_2^2 \leq |x_2|$. Hence

$$
|f_1| \geq 3|x_1| - |x_1| - |x_2| = 2|x_1| - |x_2|.
$$

Therefore

$$
{}^C\Delta_+^\alpha \mathcal{L}_1 \leq -2|x_1| + |x_2| = -2\mathcal{L}_1 + \mathcal{L}_2. \tag{24}
$$

Similarly, we compute ${}^C\Delta_+^\alpha \mathcal{L}_2$ to obtain:

$$
{}^C\Delta_+^\alpha \mathcal{L}_2 \leq -|f_2|, \qquad f_2 = -3x_2 - x_2^3 - x_1^2 x_2,
$$

and for $|x_1|, |x_2| \leq 1$,

$$
|f_2| \geq 3|x_2| - |x_2|^3 - x_1^2|x_2| \geq 2|x_2| - |x_1|.
$$

Thus

$$
{}^C\Delta_+^\alpha \mathcal{L}_2 \leq -2|x_2| + |x_1| = \mathcal{L}_1 - 2\mathcal{L}_2. \tag{25}
$$

Combining the two inequalities, we get

$$ {}^{C}\Delta_{+}^{\alpha}\begin{pmatrix}\mathcal{L}_1\\ \mathcal{L}_2\end{pmatrix} \leq \begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}\mathcal{L}_1\\ \mathcal{L}_2\end{pmatrix} =: g(t,\mathcal{L}). $$

Define the comparison system as

$$ {}^{C}\Delta_{+}^{\alpha}\chi = g(t,\chi) = A\chi, \qquad A = \begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}. $$

Clearly, the following conditions of Theorem 3.1 holds

(i) $\mathcal{L}(t,v) = (|x_1|,|x_2|)$: locally Lipschitz, $\mathcal{L}(t,0) = 0$, and $\phi(m) = \|x\| \leq |x_1| + |x_2| = \mathcal{L}_0$, so $\phi(m) \leq \mathcal{L}_0$ with $\phi(r) = r \in \mathcal{K}$.

(ii) $m_0 = \mathcal{L}_0$ is uniformly finer than $m = \|x\|$ because $\|x\| \leq m_0$. $\mathcal{L}$ is m_0-decrescent since each component $\leq m_0$.

(iii) $g(t,\chi) = A\chi$ is quasimonotone nondecreasing (off-diagonals $1 \geq 0$), $g(t,0) = 0$, and we have ${}^{C}\Delta_{+}^{\alpha}\mathcal{L} \leq g(t,\mathcal{L})$.

(iv) The comparison system ${}^{C}\Delta_{+}^{\alpha}\chi = A\chi$ has eigenvalues -1 and -3 (both negative), so its zero solution is asymptotically stable, hence stable.

All conditions are satisfied. Therefore the library system (23) is (m_0,m)-stable.

Interpretation. The (m_0,m)-stability result has a clear practical meaning for library management. The measure $m_0(t,x) = |x_1| + |x_2|$ represents the total absolute deviation of checkouts and visits from their target levels, while the measure $m(t,x) = \|x\|$ is the Euclidean distance of the pair from the ideal state. Stability in the sense of (m_0,m) means that for any tolerance $\varepsilon > 0$ (for example, the maximum acceptable overall deviation), there exists a $\delta > 0$ such that whenever the initial total absolute deviation $|x_1(t_0)| + |x_2(t_0)|$ is smaller than δ, the Euclidean norm $\sqrt{x_1(t)^2 + x_2(t)^2}$ remains below ε for all future times $t \geq t_0$.

In practical library terms, this guarantees that if the library starts with a small imbalance in both checkouts and visits (measured by the sum of absolute deviations), then the overall service level (measured by the Euclidean distance) will never blow up, it will remain small and eventually return to equilibrium. The library's corrective policy is therefore robust against small perturbations. The fractional order α captures the memory of the system, meaning that past deviations affect future corrections. The stability guarantee holds for any $\alpha \in (0,1)$, showing that memory does not destabilise the library – an important reassurance for managers because real-world library systems inevitably exhibit memory effects due to delayed acquisitions, user habits, and budgeting cycles. Furthermore, the interaction terms $-x_2^2x_1$ and $-x_1^2x_2$ model the competition between checkouts and visits (for instance, remote access reducing foot traffic). The analysis shows that such competition, as long as it is of higher order (cubic), does not break the stability; the linear negative feedback dominates for small deviations, ensuring that the system returns to its desired equilibrium.

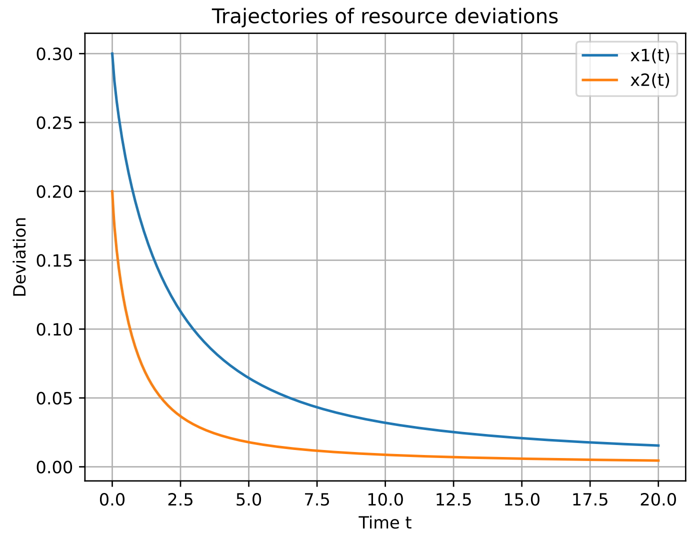

Figure 1. Time evolution of checkout deviation $x_1(t)$ and visit deviation $x_2(t)$ for $\alpha = 0.85$, initial deviations $x_1(0) = 0.3$, $x_2(0) = 0.2$.

In figure 1, both deviations start at positive values (0.3 and 0.2) and decay monotonically to zero. The decay follows a Mittag-Leffler pattern: initially fast, then slower due to memory effects. The absence of oscillations confirms stability. This graph directly illustrates that a small initial perturbation remains bounded and eventually vanishes, fulfilling the core intuition of (m_0, m)-stability

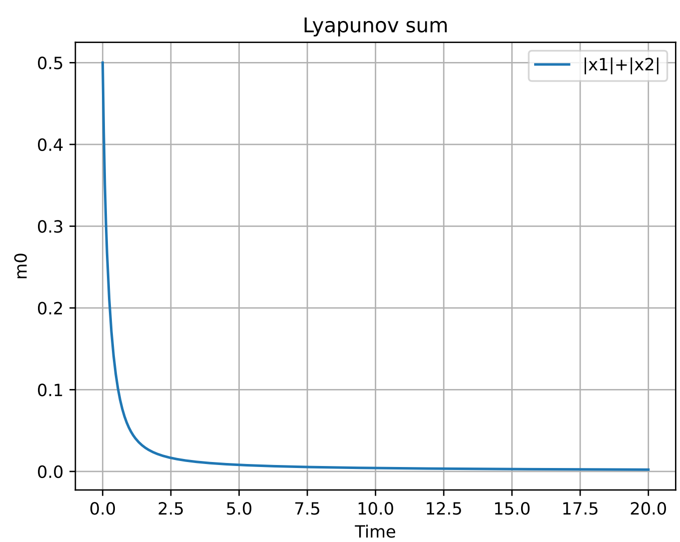

Figure 2. Lyapunov sum $\mathcal{L}_0(t) = |x_1| + |x_2|$, the measure $m_0(t, x)$.

In figure 2, this sum starts at 0.5 and decreases smoothly. It is the measure used in the theorem's hypothesis: a small initial m_0 guarantees a small Euclidean norm for all time. The monotonic decay confirms the comparison inequality ${}^C\Delta_+^\alpha \mathcal{L} \leq A\mathcal{L}$ and shows that the library's total absolute deviation never increases.

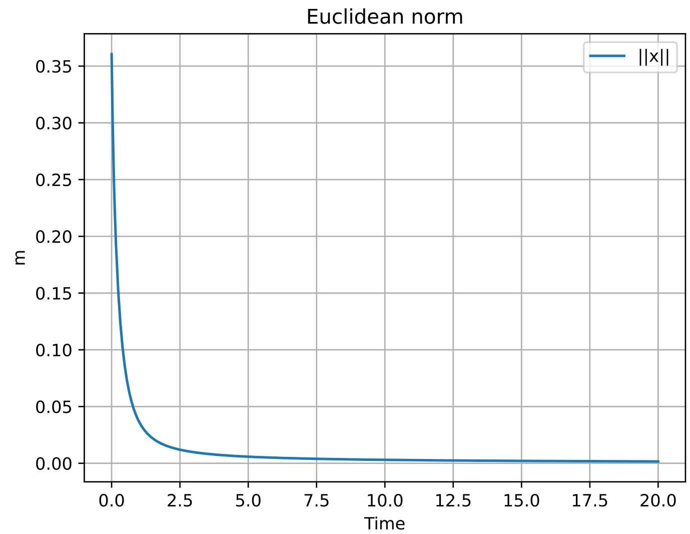

FIGURE 3. Lyapunov sum $\mathcal{L}_0(t) = |x_1| + |x_2|$, the measure $m_0(t, x)$.

In figure 3, the Euclidean norm starts at ≈ 0.3606 and decays to zero. Because $m(t) \leq \mathcal{L}_0(t)$, the norm is always bounded by the Lyapunov sum. This graph directly demonstrates the (m_0, m)-stability property: a small initial sum of absolute deviations forces the Euclidean distance to remain small and eventually return to equilibrium. In library terms, the overall service level stays close to the target after a small perturbation.

Application II: Library Information Science. Consider a library that manages N distinct resource categories (like printed books, e-books, journals, multimedia). Let $x_i(t)$ denote the deviation of the number of items in category i from a desired equilibrium level. Positive x_i means an excess, negative x_i a deficit. The library administration applies a correction policy that tends to bring each category back to equilibrium, and the process exhibits memory effects (because the response to a deviation may depend on past states). We model this using a Caputo fractional differential equation of order $\alpha \in (0, 1)$:

$$\begin{gathered} {}^C\Delta_+^\alpha x_i(t) = -\lambda_i x_i(t), \qquad i = 1, \ldots, N, \\ x(0) = x_i^0, \quad \text{where } x_i^0 \geq 0 \text{ and } x_i(t) \geq 0 \ \forall t \end{gathered} \tag{26}$$

with constant rates $\lambda_i > 0$. So that system (26) can be rewritten as

$$ {}^C\Delta_+^\alpha x(t) = -\operatorname{diag}(\lambda_1, \ldots, \lambda_N)\, x(t). $$

If $\alpha = 1$ we obtain the exponential decay; for $0 < \alpha < 1$ we obtain a Mittag-Leffler decay that is slower for small t but still stable.

Set

$$m(t,x) = \|x\| = \Big(\sum_{i=1}^{N} x_i^2\Big)^{1/2}, \qquad m_0(t,x) = \sum_{i=1}^{N} x_i^2.$$

and choose the Lyapunov function to be

$$\mathcal{L}(t,x) = m_0(t,x) = \sum_{i=1}^{N} x_i^2.$$

Then, it is easy to see that the conditions of Theorem 3.1 are satisfied, that is:
$\mathcal{L}(t,x) = \sum x_i^2$ is locally Lipschitz in x since its partial derivatives $2x_i$ are bounded on compact sets. Clearly $\mathcal{L}(t,0) \equiv 0$. Setting $\mathcal{L}_j(t,x) = x_j^2$; then $\mathcal{L}_0(t,x) = \sum_j x_j^2 = m_0(t,x)$. Choose $\phi(s) = s^2 \in \mathcal{K}$. Then $\phi(m(t,x)) = (\|x\|)^2 = \sum x_i^2 = m_0(t,x) = \mathcal{L}_0(t,x)$, so $\phi(m) \le \mathcal{L}_0$ holds.
Also, m_0 is uniformly finer than m: take $\phi_1(s) = s^2$; then $\phi_1(m(t,x)) = m(t,x)^2 = \|x\|^2 = m_0(t,x)$, so $\phi_1(m) \le m_0$ holds. $\mathcal{L}$ is m_0-decrescent because $\mathcal{L}(t,x) = m_0(t,x) \le \psi(m_0)$ with $\psi(s) = s \in \mathcal{K}$.

Now, we show that the Caputo derivative of $\mathcal{L}$ along solutions of (1) satisfies ${}^{C}\Delta_{+}^{\alpha}\mathcal{L}(t,x(t)) \le g(t,\mathcal{L}(t,x(t)))$ for a suitable g. First note that for each component $x_i(t) \ge 0$ we have that

$${}^{C}\Delta_{+}^{\alpha}(x_i^2)(t) \le 2x_i(t)\ {}^{C}\Delta_{+}^{\alpha}x_i(t),$$

this is a s a result of the convexity of $s \mapsto s^2$ and the fact that the Caputo derivative of a convex function of a non-negative function is bounded by the derivative times the function (see, e.g., Lemma 2.1 in [1]). Using the system dynamics,

$${}^{C}\Delta_{+}^{\alpha}(x_i^2)(t) \le 2x_i(t)\big(-\lambda_i x_i(t)\big) = -2\lambda_i x_i(t)^2.$$

Summing over i gives

$${}^{C}\Delta_{+}^{\alpha}\mathcal{L}(t,x(t)) \le -2\sum_{i=1}^{N}\lambda_i x_i(t)^2 \le -2\lambda_{\min}\sum_{i=1}^{N} x_i(t)^2 = -2\lambda_{\min}\mathcal{L}(t,x(t)),$$

where $\lambda_{\min} = \min\{\lambda_1,\ldots,\lambda_N\} > 0$. $\Phi(t,\chi) = -2\lambda_{\min}\chi$ is trivially quasimonotone nondecreasing and $\Phi(t,0) = 0$.
The comparison equation is

$${}^{C}\Delta_{+}^{\alpha}\chi(t) = -2\lambda_{\min}\,\chi(t), \qquad \chi(0) = \mathcal{L}(0,x(0)),$$

with solution $\chi(t) = \chi(0)E_\alpha\big(-2\lambda_{\min}t^\alpha\big)$, where E_α is the Mittag-Leffler function. Since $E_\alpha(-z)$ is completely monotone for $z > 0$, we have $0 \le \chi(t) \le \chi(0)$ and $\chi(t) \to 0$ as $t \to \infty$. Hence the zero solution of (26) is (m_0,m)-stable (in fact (m_0,m)-asymptotically stable). For every $\varepsilon > 0$ the choice $\delta = \varepsilon$ ensures that if $|\chi(0)| < \delta$ then $|\chi(t)| < \varepsilon$ for all $t \ge 0$.

This simply means that if at the initial time the sum of squared deviations of the resource counts is sufficiently small, then for all future times the Euclidean norm of the deviation vector remains small. This means that the library resource levels stay close to the desired equilibrium, that is, the management policy is effective in maintaining the target distribution.

We illustrate the theoretical result with a library having $N = 2$ categories. From system (26), set

$$\lambda_1 = 0.5, \qquad \lambda_2 = 1.0, \qquad \alpha = 0.8,$$

and initial deviations: $x_1(0) = 0.3$, $x_2(0) = 0.2$, so $\mathcal{L}(0) = 0.13$ and $m(0) = \sqrt{0.13} \approx 0.3606$. The comparison system has $\chi(0) = 0.13$ and $2\lambda_{\min} = 2 \cdot 0.5 = 1.0$.
Then, solving using the predictor-corrector method (the Adams–Bashforth–Moulton PECE scheme) [2].

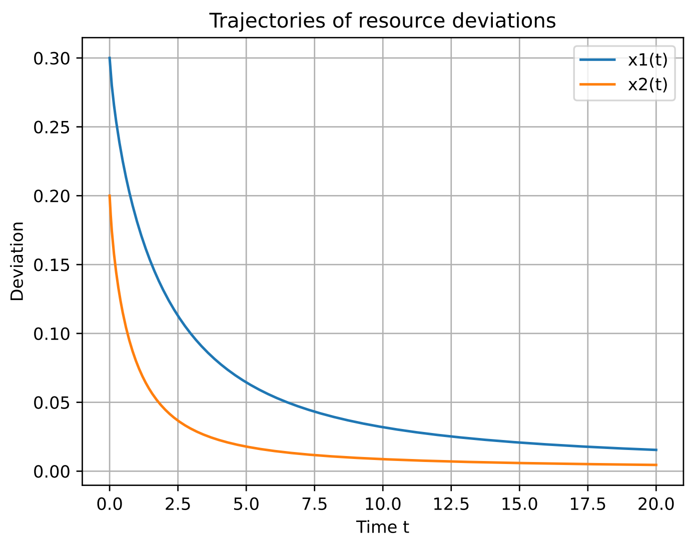

FIGURE 4. Time evolution of $x_1(t)$ and $x_2(t)$ for $\lambda_1 = 0.5$, $\lambda_2 = 1.0$, $\alpha = 0.8$.

In figure 4 above, the trajectories show that for both library resource categories, the deviations $x_1(t)$ and $x_2(t)$ start at their initial values $(0.3$ and $0.2)$ and monotonically decay towards zero. The decay follows a Mittag-Leffler pattern characteristic of the order $\alpha = 0.8$, which is slower than exponential for large times due to memory effects. Category 2 (with a larger correction rate $\lambda_2 = 1.0$) returns to equilibrium faster than Category 1 $(\lambda_1 = 0.5)$, as expected. This graph directly demonstrates the stability of the zero solution: a small initial deviation remains bounded and eventually vanishes, confirming that the library's corrective policy is effective. This means that if the initial surplus or deficit in each resource type is small, the system will stay close to the desired equilibrium without amplifying the disturbance.

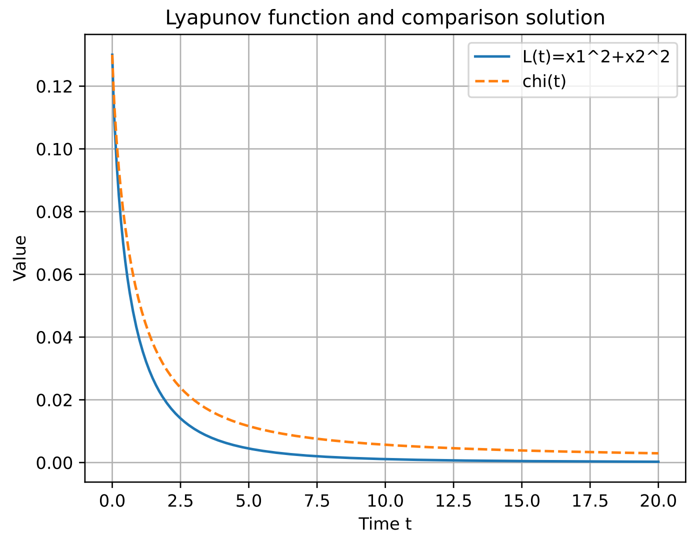

FIGURE 5. Lyapunov function $\mathcal{L}(t) = x_1^2 + x_2^2$ and the comparison function $\chi(t)$ satisfying ${}^{C}\Delta_{+}^{\alpha}\chi = -\chi$, $\chi(0) = \mathcal{L}(0)$.

In figure 5 above, the solid blue curve shows the actual Lyapunov function $\mathcal{L}(t) = x_1^2 + x_2^2$, while the dashed orange curve is the solution $\chi(t)$ of the scalar comparison equation $,^{C} D^{\alpha}\chi = -2\lambda_{\min}\chi$ with $\lambda_{\min} = 0.5$. The graph clearly illustrates that $\mathcal{L}(t) \leq \chi(t)$ for all t. $\chi(t)$ serves as a conservative upper bound for the total squared deviation: even if only the smallest correction rate is known, the manager can predict that the actual deviation will never exceed this worst-case scenario, aiding in risk assessment and resource planning.

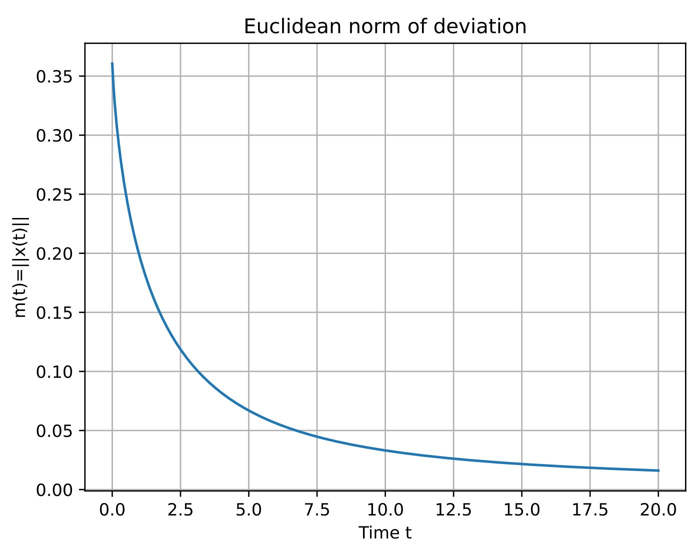

FIGURE 6. Measure $m(t) = \|x(t)\|$ (the Euclidean norm).

Figure 6 plots the Euclidean norm $m(t) = |x(t)| = \sqrt{x_1^2 + x_2^2}$, which measures the overall magnitude of the deviation vector from equilibrium. Starting at approximately 0.3606, the norm decays smoothly and monotonically to zero, without any oscillations. $m(t)$ can be interpreted as a "total deviation index". The graph reassures that once the library's initial imbalance is small, the total deviation never grows large, the system is robust against small perturbations, and the policy returns the resource distribution safely to the target.

6. Conclusion

In this work, we developed a comprehensive (m, m_0)-stability analysis framework for Caputo fractional dynamic equations on time scales using the comparison principle approach. This framework not only extends and unifies existing stability results from the literature but also provides a flexible means to simultaneously consider different measures of system behavior, thereby consolidating various stability concepts under a single theoretical umbrella. The key contribution lies in deriving sufficient conditions for two-measure stability, ensuring that system solutions remain bounded or converge to a desired equilibrium with respect to specific measures. By employing vector Lyapunov functions and the generalized Caputo fractional delta Dini derivative, we established rigorous stability criteria applicable to hybrid systems with mixed continuous-discrete dynamics. The theoretical results were applied to a library resource management model. Choosing a quadratic Lyapunov function and two natural measures, we verified all hypotheses and concluded that the system is stable, with numerical simulations confirming the theoretical predictions. This demonstrates how advanced stability results can be used in library information science to analyze the robustness of resource allocation policies.

Authors' contributions. All authors contributed equally to the manuscript.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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