Asia Pac. J. Math. 2026 **13**:91

CLOSURE-BASED STABILIZATION OF MULTIVALUED MORPHISM COMPOSITION

YILMAZ EROL

Independent Researcher, Muğla, Türkiye
ylmzerl11@gmail.com

Received Mar. 5, 2026

Abstract. In the context of hyperstructures and set–valued mappings, the naive pushforward composition of multivalued morphisms, although set–theoretically associative, often fails to provide a coherent categorical structure, as unit laws may break down and natural classes of well–behaved morphisms are not preserved under composition. This paper addresses these structural deficiencies by introducing a stabilization framework based on closure operators assigned to the underlying spaces. Within this framework, a stabilized composition of multivalued morphisms is defined via a pushforward construction, together with a corresponding stabilized identity map. It is shown that the stabilized composition satisfies both left and right unit laws, and that associativity holds within the class of cl–compatible multivalued morphisms. In this manner, a coherent compositional structure is obtained in which categorical obstructions are resolved through stabilization and compatibility. Furthermore, when the stabilization closures act trivially on singletons and the morphisms are single–valued, the proposed framework recovers the ordinary composition of functions together with the classical identity map. These results provide a systematic approach for restoring algebraic and categorical coherence in hyper and set–valued settings while remaining fully consistent with the classical theory.

2020 Mathematics Subject Classification. Primary 18B99; Secondary 54A05, 47H04, 20N20.

Key words and phrases. multivalued mapping; set-valued morphism; stabilized composition; closure operator; pushforward of multivalued maps; unit laws; associativity up to stabilization; hyperstructure morphism; classical recovery.

1. Introduction

Set–valued and multivalued mappings play a fundamental role in variational analysis, optimization, control theory, and differential inclusions, where nonuniqueness and uncertainty arise in a natural way. The foundational literature in set–valued analysis established robust notions of continuity, measurability, and selection, together with powerful analytical tools for the study of such mappings [3,4]. Despite this extensive development, the algebraic behavior of set–valued morphisms under composition remains subtle: although the naive pushforward composition is set–theoretically associative, it often fails to

DOI: 10.28924/APJM/13-91

provide a coherent categorical structure, as unit laws may break down and natural classes of well–behaved morphisms are not preserved under composition [7,8].

A closely related phenomenon arises in the theory of hyperstructures, where operations are intrinsically set–valued and the classical algebraic axioms are weakened or relaxed. Since the pioneering contributions of Marty and Krasner, hypergroups, hyperrings, and hypermodules have been studied as natural generalizations of classical algebraic systems [12,13,15–17]. Although hyperoperations provide a flexible framework for modeling multivalued interactions, morphisms between hyperstructures often exhibit unstable compositional behavior: unit elements may fail to act as identities, and associativity may hold only in restricted or weakened forms. This instability presents a significant obstacle to the construction of coherent compositional or categorical frameworks for hyper and set–valued structures.

The central problem addressed in the present work is the categorical incoherence of composition for multivalued and hyper morphisms. In many theoretical and applied contexts, chains of transformations naturally arise, and the absence of reliable unit behavior and structural closure under composition limits both conceptual clarity and mathematical tractability. From a structural point of view, this deficiency prevents the systematic use of categorical methods and obscures the precise relationship between multivalued theories and their classical single–valued counterparts.

To address these difficulties, a stabilization approach based on closure operators assigned to the underlying spaces is introduced. Using a pushforward construction that is standard in set–valued analysis and related areas, a stabilized composition of multivalued morphisms is defined, together with a corresponding stabilized identity map. Within this framework, categorical coherence is restored at the level of units, and associativity of the stabilized composition is shown to hold within the class of cl–compatible multivalued morphisms. In this manner, the structural pathologies of naive composition are resolved without imposing restrictive conditions on the multivalued setting.

Beyond its intrinsic algebraic significance, the proposed stabilization framework offers a coherent mechanism for chaining set–valued transformations, which may be advantageous in the analysis of multistage models arising in optimization, control, and related fields. While the present work is devoted to the foundational aspects of the theory, further applications and concrete models are left for future investigation. Finally, it is shown that when the stabilization closure acts trivially on singletons and the morphisms are single–valued, the stabilized framework reduces exactly to the classical composition of functions together with the usual identity map. This classical recovery result confirms that the proposed approach constitutes a genuine extension of the standard theory rather than an alternative formulation. From a categorical perspective, restoring unit laws and associativity within an appropriate morphism class is essential for the construction of well–behaved compositional structures in the sense of category theory [1,2].

2. Preliminaries and notation

2.1. **Hyperoperations and multivalued composition.** Hyperstructures arise as natural generalizations of classical algebraic systems by allowing the underlying operations to be set–valued. Following the foundational ideas of Marty and Krasner, a hyperoperation on a nonempty set X is a map

$$\oplus : X \times X \to \mathcal{P}(X) \setminus \{\varnothing\},$$

which assigns to each pair (x, y) a nonempty subset $x \oplus y$ of X [12, 13]. Within this framework, associativity and other algebraic axioms are typically interpreted in a weakened or set–theoretic sense, most commonly through unions of values. In particular, the iterated hyperoperation is defined by

$$x \oplus y \oplus z := \bigcup_{u \in x \oplus y} (u \oplus z),$$

which reduces to the classical binary operation when all values are singletons [15, 16]. This convention allows hyper algebraic expressions to be evaluated unambiguously and will be adopted consistently throughout the paper.

2.2. **Set–valued mappings and pushforward.** Let X and Y be nonempty sets. A set–valued (or multivalued) mapping is a map

$$F : X \to \mathcal{P}(Y) \setminus \{\varnothing\}$$

that assigns to each $x \in X$ a nonempty subset $F(x) \subseteq Y$. Such mappings constitute central objects in set–valued analysis, optimization, and differential inclusions [3, 4]. Given a nonempty subset $B \subseteq X$, the *pushforward* of B under F is defined by

$$F_\sharp(B) := \bigcup_{x \in B} F(x).$$

This construction provides the natural analogue of image formation for single–valued functions and reflects the union–based evaluation of hyperoperations. Pushforward mappings play a fundamental role in the composition of multivalued morphisms and arise, either implicitly or explicitly, in various contexts involving multistage or chained transformations [5, 6].

2.3. **Closure operators and stabilization.** In order to control the structural growth and categorical incoherence that may arise under repeated multivalued composition, each underlying space is equipped with a closure operator. A map

$$\mathrm{cl}_X : \mathcal{P}(X) \setminus \{\varnothing\} \to \mathcal{P}(X) \setminus \{\varnothing\}$$

is called a *closure operator* if it is extensive, monotone, and idempotent. While closure operators are classically associated with topology, an abstract axiomatic viewpoint is adopted here, allowing for purely algebraic or structural interpretations [9–11]. In the present work, closure operators are interpreted as stabilization mechanisms that regulate the behavior of multivalued composition and support categorical coherence.

Remark 2.1 (Set-level extension of hyperoperations)**.** Throughout the paper, whenever $(X, \oplus_X)$ is a hyperstructure, we use the standard set-level extension of $\oplus_X$ to nonempty subsets of X. Namely, for nonempty $A, B \subseteq X$ we define

$$A \oplus_X B \;:=\; \bigcup_{a\in A,\; b\in B} (a \oplus_X b) \;\subseteq X.$$

In particular, if $F : X \to \mathcal{P}(Y) \setminus \{\varnothing\}$ is multivalued, then for any nonempty $A, B \subseteq X$ the pushforward satisfies $F_\sharp(A), F_\sharp(B) \subseteq Y$, and hence expressions such as $F_\sharp(A) \oplus_Y F_\sharp(B)$ are interpreted via the above set-level extension on $(Y, \oplus_Y)$. This convention will be used implicitly in all subsequent definitions and proofs.

2.4. **Notational conventions.** Throughout the paper, multivalued composition is expressed using the pushforward notation $F_\sharp$ and set–theoretic unions. The symbol $\circ_s$ denotes the stabilized composition introduced later, while id^s_X stands for the corresponding stabilized identity. When no confusion arises, subscripts on closure operators are omitted and the notation cl is used instead of cl_X or cl_Y. All sets are assumed to be nonempty, and all multivalued mappings take nonempty values unless stated otherwise.

3. Stabilization-Based Composition and Main Results

Definition 3.1 (Stabilized composition)**.** Let $(X, \oplus_X), (Y, \oplus_Y), (Z, \oplus_Z)$ be hyperstructures equipped with closure operators $\mathrm{cl}_X, \mathrm{cl}_Y, \mathrm{cl}_Z$ on nonempty subsets. Let

$$F : X \to \mathcal{P}(Y) \setminus \{\varnothing\}, \qquad G : Y \to \mathcal{P}(Z) \setminus \{\varnothing\}$$

be multivalued morphisms. The *stabilized composition* of G and F is defined by

$$(G \circ_s F)(x) := \mathrm{cl}_Z\left(\bigcup_{y\in F(x)} G(y)\right), \qquad x \in X.$$

3.1. **Derivation of stabilization–pushforward compatibility.** Fix a hyperstructure $(X, \oplus_X)$ equipped with a stabilization closure operator cl_X acting on nonempty subsets of X. We recall the set–level compatibility condition imposed on cl_X:

$$\mathrm{cl}_X(A \oplus_X B) = \mathrm{cl}_X\big(\mathrm{cl}_X(A) \oplus_X \mathrm{cl}_X(B)\big) \qquad \text{for all nonempty } A, B \subseteq X. \tag{3.1}$$

This condition ensures that the closure operator interacts coherently with the underlying hyperoperation.

Let $(Z, \oplus_Z)$ and $(W, \oplus_W)$ be hyperstructures equipped with closure operators cl_Z and cl_W, respectively, and let

$$H : Z \to \mathcal{P}(W) \setminus \{\varnothing\}$$

be a multivalued morphism. For any nonempty subset $B \subseteq Z$, the pushforward of B under H is defined by

$$H_\sharp(B) := \bigcup_{z\in B} H(z).$$

This notation will be used to formulate the compatibility between stabilization and pushforward in the multivalued setting.

Definition 3.2 (Closure–lax hypermorphism)**.** Let $(Z, \oplus_Z)$ and $(W, \oplus_W)$ be hyperstructures equipped with closure operators cl_Z and cl_W, respectively. A multivalued morphism

$$H : Z \to \mathcal{P}(W) \setminus \{\varnothing\}$$

is called *closure–lax* if for all nonempty subsets $A, B \subseteq Z$ one has

$$H_\sharp(A \oplus_Z B) \;\subseteq\; \mathrm{cl}_W\big(H_\sharp(A) \oplus_W H_\sharp(B)\big). \tag{3.2}$$

Remark 3.3. Closure–lax morphisms encode a weak compatibility with the underlying hyperoperation. However, this condition alone does not guarantee compatibility between stabilization and pushforward. For this reason, associativity of the stabilized composition is established within the class of cl–compatible morphisms.

Definition 3.4 (cl-compatible multivalued morphism)**.** Let Z, W be sets equipped with stabilization closure operators cl_Z and cl_W, respectively. A multivalued morphism

$$H : Z \to \mathcal{P}(W) \setminus \{\varnothing\}$$

is called cl-*compatible* if for every nonempty subset $B \subseteq Z$ one has

$$\mathrm{cl}_W\big(H_\sharp(\mathrm{cl}_Z(B))\big) = \mathrm{cl}_W\big(H_\sharp(B)\big),$$

where $H_\sharp(B) = \bigcup_{z\in B} H(z)$.

Theorem 3.5 (Associativity of stabilized composition under compatibility)**.** *Let $(X, \oplus_X), (Y, \oplus_Y), (Z, \oplus_Z), (W, \oplus_W)$ be hyperstructures equipped with stabilization closure operators $\mathrm{cl}_X, \mathrm{cl}_Y, \mathrm{cl}_Z, \mathrm{cl}_W$, each extensive, monotone, and idempotent. Let*

$$F : X \to \mathcal{P}(Y) \setminus \{\varnothing\}, \quad G : Y \to \mathcal{P}(Z) \setminus \{\varnothing\}, \quad H : Z \to \mathcal{P}(W) \setminus \{\varnothing\}$$

be multivalued morphisms. Assume that H is cl–compatible and that G and H are closed–valued, that is,

$$\mathrm{cl}_Z(G(y)) = G(y) \quad (\forall y \in Y), \qquad \mathrm{cl}_W(H(z)) = H(z) \quad (\forall z \in Z).$$

Then the stabilized composition $\circ_s$ is associative:

$$(H \circ_s G) \circ_s F \;=\; H \circ_s (G \circ_s F).$$

Proof. Fix $x \in X$ and set

$$S_x := \bigcup_{y \in F(x)} G(y) \subseteq Z.$$

By definition of stabilized composition,

$$\big((H \circ_s G) \circ_s F\big)(x) = \mathrm{cl}_W \left(\bigcup_{y \in F(x)} (H \circ_s G)(y) \right) = \mathrm{cl}_W \left(\bigcup_{y \in F(x)} \mathrm{cl}_W\big(H_\sharp(G(y))\big) \right).$$

Since H is closed–valued, $\mathrm{cl}_W(H_\sharp(B)) = H_\sharp(B)$ for every nonempty $B \subseteq Z$. Hence,

$$\big((H \circ_s G) \circ_s F\big)(x) = \mathrm{cl}_W \left(\bigcup_{y \in F(x)} H_\sharp(G(y)) \right) = \mathrm{cl}_W\big(H_\sharp(S_x)\big).$$

On the other hand,

$$\big(H \circ_s (G \circ_s F)\big)(x) = \mathrm{cl}_W\big(H_\sharp((G \circ_s F)(x))\big),$$

and by definition $(G \circ_s F)(x) = \mathrm{cl}_Z(S_x)$. Since H is cl–compatible,

$$\mathrm{cl}_W\big(H_\sharp(\mathrm{cl}_Z(S_x))\big) = \mathrm{cl}_W\big(H_\sharp(S_x)\big).$$

Therefore,

$$\big((H \circ_s G) \circ_s F\big)(x) = \big(H \circ_s (G \circ_s F)\big)(x).$$

Since $x \in X$ was arbitrary, the proof is complete. □

3.2. **Stabilized identities and classical recovery.**

Definition 3.6 (Stabilized identity)**.** Let X be a hyperstructure equipped with a stabilization closure

$$\mathrm{cl}_X : \mathcal{P}(X) \setminus \{\varnothing\} \to \mathcal{P}(X) \setminus \{\varnothing\},$$

which is extensive, monotone, and idempotent. The *stabilized identity* is the multivalued map

$$\mathrm{id}^s_X : X \to \mathcal{P}(X) \setminus \{\varnothing\}, \qquad \mathrm{id}^s_X(x) := \mathrm{cl}_X(\{x\}).$$

Proposition 3.7 (Unit laws for stabilized composition)**.** *Let X and Y be hyperstructures endowed with stabilization closures cl_X and cl_Y, respectively. Assume that cl_X and cl_Y are extensive, monotone, and idempotent. Assume also that for every multivalued morphism $F : X \to \mathcal{P}(Y) \setminus \{\varnothing\}$ and every nonempty $B \subseteq X$ one has*

$$\mathrm{cl}_Y\big(F_\sharp(\mathrm{cl}_X(B))\big) = \mathrm{cl}_Y\big(F_\sharp(B)\big).$$

Furthermore, assume that cl_Y is generated by singletons in the sense that for every nonempty $S \subseteq Y$,

$$\mathrm{cl}_Y(S) = \mathrm{cl}_Y\left(\bigcup_{y \in S} \mathrm{cl}_Y(\{y\}) \right).$$

If F is cl_Y–closed–valued, i.e.

$$\mathrm{cl}_Y(F(x)) = F(x) \quad \textit{for all } x \in X,$$

then the stabilized composition $\circ_s$ satisfies the unit laws

$$F \circ_s \mathrm{id}_X^s = F \quad \textit{and} \quad \mathrm{id}_Y^s \circ_s F = F.$$

Proof. Fix $x \in X$.

Left unit law. By definition of $\circ_s$ and id_X^s,

$$(F \circ_s \mathrm{id}_X^s)(x) = \mathrm{cl}_Y\left(\bigcup_{u \in \mathrm{id}_X^s(x)} F(u) \right) = \mathrm{cl}_Y(F_\sharp(\mathrm{cl}_X(\{x\}))).$$

By the compatibility assumption with $B = \{x\}$,

$$\mathrm{cl}_Y(F_\sharp(\mathrm{cl}_X(\{x\}))) = \mathrm{cl}_Y(F_\sharp(\{x\})) = \mathrm{cl}_Y(F(x)).$$

Since F is cl_Y–closed–valued, $\mathrm{cl}_Y(F(x)) = F(x)$, hence $(F \circ_s \mathrm{id}_X^s)(x) = F(x)$.

Right unit law. Again by definition,

$$(\mathrm{id}_Y^s \circ_s F)(x) = \mathrm{cl}_Y\left(\bigcup_{y \in F(x)} \mathrm{id}_Y^s(y) \right) = \mathrm{cl}_Y\left(\bigcup_{y \in F(x)} \mathrm{cl}_Y(\{y\}) \right).$$

By the singleton–generation property of cl_Y (applied to $S = F(x)$),

$$\mathrm{cl}_Y\left(\bigcup_{y \in F(x)} \mathrm{cl}_Y(\{y\}) \right) = \mathrm{cl}_Y(F(x)) = F(x),$$

where the last equality uses that F is cl_Y–closed–valued. This proves $(\mathrm{id}_Y^s \circ_s F)(x) = F(x)$.

Since $x \in X$ was arbitrary, the proof is complete. □

Remark 3.8 (Relative nature of unit laws)**.** The assumption

$$\mathrm{cl}_Y\big(F_\sharp(\mathrm{cl}_X(B))\big) = \mathrm{cl}_Y\big(F_\sharp(B)\big)$$

should be understood as a compatibility condition imposed on the class of morphisms under consideration. In general, stabilized identities do not act as two–sided units for all multivalued morphisms. Rather, the unit laws in Proposition 3.7 hold relative to the subclass of cl–compatible and cl_Y–closed–valued morphisms, and under the singleton–generation property of the closure operator. In this sense, the stabilized composition defines a category–like structure with identities valid within a restricted morphism class.

Corollary 3.9 (Classical recovery)**.** *Assume that the stabilization closures are trivial on singletons, i.e.*

$$\mathrm{cl}_X(\{x\}) = \{x\} \quad \textit{and} \quad \mathrm{cl}_Y(\{y\}) = \{y\}$$

for all $x \in X$ and $y \in Y$, and that all morphisms are single–valued functions $f : X \to Y$ identified with multivalued maps $F(x) = \{f(x)\}$. Then the stabilized composition $\circ_s$ coincides with the ordinary composition of functions, and id_X^s reduces to the usual identity.

Proof. If $F(x) = \{f(x)\}$ and $G(y) = \{g(y)\}$, then

$$(G \circ_s F)(x) = \mathrm{cl}_Z(\{g(f(x))\}) = \{g(f(x))\},$$

so $G \circ_s F$ coincides with the classical composite $g \circ f$. Moreover, $\mathrm{id}_X^s(x) = \mathrm{cl}_X(\{x\}) = \{x\}$, hence id_X^s is the usual identity map. $\square$

3.3. **Interval hyperoperation on $\mathbb{R}$.** Let $X = \mathbb{R}$ and define a hyperoperation by

$$x \oplus y := [\min\{x, y\}, \max\{x, y\}].$$

For any nonempty bounded set $A \subseteq \mathbb{R}$, one checks that

$$A \oplus A = [\inf A, \sup A],$$

and hence the associated interval-hull closure is

$$\mathrm{cl}(A) = [\inf A, \sup A].$$

In particular, cl is extensive, monotone, and idempotent.

Define a multivalued map $H : \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\varnothing\}$ by

$$H(t) := [t - 1, t + 1].$$

For any nonempty bounded set $B \subseteq \mathbb{R}$ we have

$$H_\sharp(B) = \bigcup_{t \in B} [t - 1, t + 1] = [\inf B - 1, \sup B + 1].$$

Therefore, H is cl–compatible. Consequently, Theorem 3.5 applies and the stabilized composition $\circ_s$ is associative in this model.

3.4. **A finite example showing the necessity of compatibility.** Let $X = \{a, b, c\}$ and consider a nontrivial closure operator cl on the family of nonempty subsets of X, defined by

$$\mathrm{cl}(B) := \begin{cases} B \cup \{b\}, & \text{if } a \in B, \\ B, & \text{if } a \notin B, \end{cases} \qquad \varnothing \neq B \subseteq X.$$

Thus, explicitly,

$$\mathrm{cl}(\{a\}) = \{a, b\}, \quad \mathrm{cl}(\{a, c\}) = X, \quad \mathrm{cl}(\{a, b\}) = \{a, b\}, \quad \mathrm{cl}(X) = X,$$

while cl fixes all subsets not containing a. It is straightforward to verify that cl is monotone and idempotent.

Define multivalued maps $F, G, H : X \to \mathcal{P}(X) \setminus \{\varnothing\}$ by

$$F(a) = \{a\}, \quad F(b) = \{b\}, \quad F(c) = \{c\},$$

$$G(a) = \{a\}, \quad G(b) = \{b\}, \quad G(c) = \{c\},$$

$$H(a) = \{a\}, \quad H(b) = \{c\}, \quad H(c) = \{c\}.$$

The two stabilized composites are computed explicitly at the element $a \in X$.

Step 1: Computation of $G \circ_s F$ at a. By definition,

$$(G \circ_s F)(a) = \mathrm{cl}\Big(\bigcup_{y \in F(a)} G(y) \Big) = \mathrm{cl}\big(G(a)\big) = \mathrm{cl}(\{a\}) = \{a, b\}.$$

Step 2: Computation of $H \circ_s (G \circ_s F)$ at a. Using Step 1,

$$(H \circ_s (G \circ_s F))(a) = \mathrm{cl}\Big(\bigcup_{y \in (G \circ_s F)(a)} H(y) \Big) = \mathrm{cl}\big(H(a) \cup H(b)\big).$$

Since $H(a) = \{a\}$ and $H(b) = \{c\}$, it follows that

$$H(a) \cup H(b) = \{a, c\}, \qquad \text{and hence} \qquad (H \circ_s (G \circ_s F))(a) = \mathrm{cl}(\{a, c\}) = X.$$

Step 3: Computation of $H \circ_s G$ at a.

$$(H \circ_s G)(a) = \mathrm{cl}\Big(\bigcup_{z \in G(a)} H(z) \Big) = \mathrm{cl}\big(H(a)\big) = \mathrm{cl}(\{a\}) = \{a, b\}.$$

Step 4: Computation of $(H \circ_s G) \circ_s F$ at a. Since $F(a) = \{a\}$, Step 3 yields

$$((H \circ_s G) \circ_s F)(a) = \mathrm{cl}\Big(\bigcup_{y \in F(a)} (H \circ_s G)(y) \Big) = \mathrm{cl}\big((H \circ_s G)(a)\big) = \mathrm{cl}(\{a, b\}) = \{a, b\}.$$

Conclusion. Combining Steps 2 and 4, one obtains

$$((H \circ_s G) \circ_s F)(a) = \{a, b\} \neq X = (H \circ_s (G \circ_s F))(a).$$

Therefore,

$$(H \circ_s G) \circ_s F \neq H \circ_s (G \circ_s F),$$

and the stabilized composition fails to be associative in this model.

This failure is explained by the fact that the morphism H is not cl–compatible. Indeed, for $B = \{a\}$ one has

$$\mathrm{cl}(B) = \{a, b\}, \qquad H_\sharp(B) = H(a) = \{a\}, \qquad H_\sharp(\mathrm{cl}(B)) = H(a) \cup H(b) = \{a, c\},$$

and consequently

$$\mathrm{cl}\big(H_\sharp(B)\big) = \{a, b\} \neq X = \mathrm{cl}\big(H_\sharp(\mathrm{cl}(B))\big).$$

Hence, H does not satisfy the defining condition of cl–compatible morphisms, and associativity of the stabilized composition $\circ_s$ breaks down in this example.

4. Conclusion

In this paper, the compositional behavior of multivalued and hyper morphisms was investigated from a structural and categorical perspective. Although the naive pushforward composition of multivalued maps is set–theoretically associative, it generally fails to provide a coherent categorical framework, as unit laws may break down and natural classes of well–behaved morphisms are not preserved under composition.

To address these issues, a stabilization framework based on closure operators assigned to the underlying spaces was introduced. Using a pushforward construction, a stabilized composition of multivalued morphisms and a corresponding stabilized identity map were defined. Within this framework, categorical coherence is restored at the level of units and closure–preserving behavior. It was shown that associativity of the stabilized composition holds precisely within the class of cl–compatible morphisms, rather than being a universal property.

A finite counterexample was provided to demonstrate that this compatibility condition is essential: in the absence of cl–compatibility, stabilized composition may fail to be associative. This example clarifies that stabilization alone is insufficient, and that compatibility between pushforward and closure is a genuine structural requirement.

It was further established that the proposed framework is fully consistent with the classical theory. When stabilization closures act trivially on singletons and morphisms are single–valued, the stabilized composition reduces exactly to the ordinary composition of functions, and the stabilized identity coincides with the classical identity map. This classical recovery result confirms that the present approach constitutes a genuine extension of standard composition rather than an alternative formulation.

Beyond its theoretical contribution, the stabilization mechanism introduced here provides a coherent tool for chaining set–valued transformations in contexts where closure effects are intrinsic. Potential applications include multistage models in optimization, control theory, and related areas. Further investigation of such applications, as well as of alternative notions of stabilization and compatibility, is left for future work.

Authors' Contributions. The author has read and approved the final version of the manuscript.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Mac Lane, Categories for the Working Mathematician, Springer, New York, 1998. https://doi.org/10.1007/978-1-4757-4721-8.

[2] S. Awodey, Category Theory, 2nd ed., Oxford University Press, 2010.

[3] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser Boston, 2009. https://doi.org/10.1007/978-0-8176-4848-0.

[4] R.T. Rockafellar, R.J.B. Wets, Variational Analysis, Springer, Berlin, 1998. https://doi.org/10.1007/978-3-642-02431-3.

[5] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, Volume I: Theory, Springer, New York, 1997. https://link.springer.com/book/9780792346821.

[6] J.P. Aubin, A. Cellina, Differential Inclusions, Springer, Berlin, 1984. https://doi.org/10.1007/978-3-642-69512-4.

[7] K. Deimling, Multivalued Differential Equations, De Gruyter, 1992. https://doi.org/10.1515/9783110874228.

[8] C. Berge, Topological Spaces, Oliver and Boyd, Edinburgh, 1963.

[9] K. Kuratowski, Topology, Volume I, Academic Press, New York, 1966.

[10] G. Birkhoff, Lattice Theory, American Mathematical Society Colloquium Publications, Vol. 25, American Mathematical Society, 1940. https://doi.org/10.1090/coll/025.

[11] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002. https://doi.org/10.1017/CBO9780511809088.

[12] F. Marty, Sur une Généralisation de la Notion de Groupe, in: 8th Congress of Scandinavian Mathematicians, Stockholm, pp. 45–49, 1934.

[13] M. Krasner, Sur l'Approximation des Corps Valués Complets de Caractéristique p par Ceux de Caractéristique Zéro, Sémin. Albert Châtelet Paul Dubreil 13 (1959–1960), 1. http://www.numdam.org/item?id=SD_1959-1960__13_1_A9_0.

[14] M. Krasner, A Class of Hyperrings and Hyperfields, Int. J. Math. Math. Sci. 6 (1983), 307–311. https://doi.org/10.1155/S0161171283000265.

[15] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editor, 1993.

[16] T. Vougiouklis, Hyperstructures and Their Representations, Hadronic Press, 1994.

[17] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, International Academic Press, 2007.