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WEAKLY CONNECTED GEODETIC NUMBERS OF GRAPHS

JOSEPHINE C. GALABIN[1,2,*], IMELDA S. ANIVERSARIO[1,2], FERDINAND P. JAMIL[1,2]

[1]Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, Iligan City, Philippines

[2]Center for Mathematical and Theoretical Physical Sciences, Premier Research Institute of Science and Mathematics, MSU-Iligan Institute of Technology, Iligan City, Philippines

[*]Corresponding author: josephine.galabin@g.msuiit.edu.ph

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Abstract. Given two vertices u and v of a connected simple graph $G=(V(G),E(G))$, $I_G[u,v]$ is the closed interval consisting of u, v and all vertices lying on some u-v geodesic in G. For vertex $v\in V(G)$, $N_G[v]$ is the closed neighborhood of v consisting of v and all vertices of G adjacent to v. For $S\subseteq V(G)$, $I_G[S]=\cup_{u,v\in S}I_G[u,v]$ and $N_G[S]=\cup_{v\in S}N_G[v]$. A subset S of $V(G)$ is a geodetic cover of G if $I_G[S]=V(G)$. The minimum cardinality of a geodetic cover is the *geodetic number* of G. Any set $S\subseteq V(G)$ is a *weakly connected geodetic cover* of G if S is a geodetic cover of G and $\langle S\rangle_w$ is connected, where $\langle S\rangle_w=\langle N_G[S],E_w\rangle$ with E_w consisting of edges $uv\in E(G)$ such that $u\in S$ or $v\in S$. The *weakly connected geodetic number* of G is the minimum cardinality of a weakly connected geodetic cover of G. In this paper, we initiate the study of weakly connected geodetic numbers. First, we determine the weakly connected geodetic numbers of some common graphs, obtain bounds on the weakly connected geodetic number of general graphs, and then characterize graphs attaining the extremal and small values of the parameter. Next, we solve a realization problem involving the geodetic number, the weakly connected geodetic number and the order of the graph. Lastly, we characterize the weakly connected geodetic covers of the corona and edge corona of two graphs, obtain necessary and sufficient conditions for the weakly connected geodetic covers of the lexicographic product of two graphs, and determine their corresponding weakly connected geodetic numbers.

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1. Introduction

In 1993, F. Harary, E. Loukakis, and C. Tsouros [16] introduced the concept of geodetic number of a connected graph. The idea revolves around the concept of shortest path between vertices. It focuses on how sets of vertices cover the entire graph through these shortest paths. In network design and

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reliability, the geodetic number tells the minimum number of monitor nodes required so that every node in the network lies on a primary (shortest) data route between monitors.

Following its introduction, the geodetic number has been extensively investigated. In his book "Geodesic Convexity in Graphs (2013)", I. Pelayo [24] listed 84 references in its bibliography on geodetic numbers and its related parameters.

In 2006, D. Mojdeh and N. Rad [20] introduced the connected geodetic number of a graph, a variation of the geodetic number but not part of the bibliography of [24] and which requires that the subgraph induced by the geodetic set is connected. In network design above, this means that the monitor nodes are chosen to be able to reach one another through other monitor nodes in the set without passing through non-monitor nodes. While geodetic number provides best for raw coverage, connected geodetic number provides best for high security. Unfortunately, this model entails higher cost.

The present paper introduces the weakly connected geodetic number. It aims to bridge the gap between geodesic coverage and connectivity in graphs, but with lower cost. Here, we don't require a node to be right next to another but we also don't allow a node to be totally isolated. The weakly connected constraint forces the set to cluster or expand until vertices can see a path to one another through the edges of the graph.

All throughout this paper, we consider only graphs that are simple, finite, and undirected. Given a graph $G = (V(G), E(G))$, we call $V(G)$ the vertex set of G and $E(G)$ its edge set. For $S \subseteq V(G)$, $|S|$ denotes the cardinality of S. In particular, $|V(G)|$ is the order of G. All terminologies used here that are not defined are adapted from [23].

Let G and H be graphs with disjoint vertex sets. The corona $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i-th vertex of G to every vertex of the i-th copy of H. The edge corona $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The lexicographic product $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Let G be a connected graph. For any two distinct vertices $u, v \in V(G)$, a shortest path joining u and v is called a u-v *geodesic*. The length of a u-v geodesic is called the *distance* between u and v, which is denoted by $d_G(u, v)$. The *diameter* of the graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices in G.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A vertex v is an *extreme* *vertex* if the induced subgraph $\langle N_G(v)\rangle$ is a complete graph. The symbol $Ext(G)$ denotes the set of all extreme vertices in G.

For $u, v \in V(G)$, the set $I_G(u,v)$ refers to the set consisting of all vertices lying on any u-v geodesic of G, and $I_G[u,v] = I_G(u,v) \cup \{u,v\}$. For $S \subseteq V(G)$, the *geodetic closure* $I_G[S]$ is defined by $I_G[S] = \bigcup\{I_G[u,v] : u,v \in S\}$. A *geodetic cover* is any set $S \subseteq V(G)$ with $I_G[S] = V(G)$. The minimum cardinality of a geodetic cover of G is the $geodetic\ number$ of G, denoted by $g(G)$. A geodetic cover of cardinality $g(G)$ is a g-*basis* of G. The reader is referred to [4, 6–12, 16, 24] for the introduction, fundamental concepts, extensive studies, and some recent developments on geodetic numbers in graphs.

A subset A of a geodetic cover S of G with $Ext(G) \subseteq A$ is said to satisfy the *geodetic interiority condition* (GIC) *with respect to* S if for every $a \in A \setminus Ext(G)$, there exist $x, y \in S$ for which $a \in I_G(x,y)$. Any $A \subseteq S$ which has the GIC with respect to S is termed as a GIC*-set with respect to* S. A GIC-set with respect to S of maximum cardinality is called *maximum* GIC*-set with respect to* S. More precisely, A is a maximum GIC-set with respect to S if A is the union of all GIC-sets with respect to S. A geodetic cover of G that is a GIC-set with respect to itself is called a GIC*-set of* G. These concepts were introduced by G. Cagaanan and S. Canoy [6] in 2006 in their study of geodetic covers of the lexicographic product $G[K_m]$.

For $S \subseteq V(G)$, the 2*-path closure* of S, denoted by $P_2[S]_G$, is the set $P_2[S]_G = S \cup \{z \in V(G) : z \in I_G(u,v)$ with $d_G(u,v) = 2$ for some $u, v \in S\}$. A set $S \subseteq V(G)$ is a 2*-path closure absorbing set* of G if $P_2[S]_G = V(G)$. The minimum cardinality of a 2-path closure absorbing set of G is the 2-$path\ closure\ absorbing\ number$ of G, denoted by $\rho_2(G)$. A 2-path closure absorbing set of cardinality $\rho_2(G)$ is a ρ_2*-set* of G. In 2006, S. Canoy et al. [8] defined the concept of the 2-path closure absorbing set and used it to characterize the hull sets of $G + K_m$. Recently, a variation of this concept has been explored in [27].

For $S \subseteq V(G)$, $\langle S\rangle$ is the subgraph of G induced by S, i.e., $\langle S\rangle$ is largest subgraph of G with vertex set S. Any geodetic cover S of G for which $\langle S\rangle$ is connected is a *connected geodetic cover* of G. The minimum cardinality of a connected geodetic cover of G is the *connected geodetic number* of G, denoted by $g_c(G)$. A connected geodetic cover of cardinality $g_c(G)$ is a g_c*-basis* of G. The connected geodetic number of a graph, following its introduction by D. Mojdeh and N. Rad [20] in 2006, was further studied by Santhakumaran et al. in 2009 [25, 26].

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set of G is the *domination number* of G, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is a γ*-set* of G. Dominating sets and domination numbers of graphs are among the most extensively studied concepts in graph theory. The reader is referred to [3, 13, 17, 18, 21, 22] for the history, fundamental concepts, and recent developments on domination in graphs as well as its various applications.

The symbol $\langle S\rangle_w = \langle N_G[S], E_w\rangle$ denotes the *weakly induced subgraph* of G with vertex-set $N_G[S]$ and whose edge-set is $E_w = \{uv \in E(G) : u \in S$ or $v \in S\}$. Any dominating set S of G for which $\langle S\rangle_w$ is connected is a weakly connected dominating set of G. The minimum cardinality of a weakly connected dominating set of G is the *weakly connected domination number* of G, denoted by $\gamma_w(G)$. A weakly connected dominating set of cardinality $\gamma_w(G)$ is a γ_w*-set* of G. Motivated by the work of J. Grossman [15], J. Dunbar et al. [14] initiated the study of weakly connected domination in graphs in 1997, and the concept was further studied by K. Alzoubi et al. [2] in 2003 with applications to wireless ad hoc networks.

2. Some known results

Lemma 1. [19] *For a path P_n of n vertices, $\gamma_w(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and for a cycle C_n of n vertices, $\gamma_w(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Lemma 2. [1] *For all n, $\rho_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$ and for $n \geq 4$, $\rho_2(C_n) = \left\lceil \frac{n}{2} \right\rceil$.*

Lemma 3. [5] *Let G be a nontrivial connected graph and S a geodetic cover of $G \circ H$. Then $S \cap V(H_u) \neq \varnothing$ for all $u \in V(G)$.*

Proposition 1. [6] *Let G be a connected graph, $m \geq 2$, S a geodetic cover of G and A a GIC-set with respect to S. Then*

$$C = Ext(G[K_m]) \cup [(A \setminus Ext(G)) \times \{v_0\}] \cup [(S \setminus A) \times V(K_m)] \tag{1}$$

is a geodetic cover of $G[K_m]$ for every $v_0 \in V(K_m)$.

Proposition 2. [6] *Let G be a connected graph and C a geodetic cover of $G[K_m]$. Then C_G is a geodetic cover of G and there exists a GIC-set A with respect to C_G such that*

$$C = Ext(G[K_m]) \cup [(C_G \setminus A) \times V(K_m)] \cup B, \tag{2}$$

where $B = \{(u, v) \in C : u \in A \setminus Ext(G), v \in V(K_m)\}$.

3. The weakly connected geodetic number of a graph

A set S of vertices of a connected graph G is a *weakly connected geodetic cover* of G if S is a geodetic cover of G and $\langle S\rangle_w$ is connected. The minimum cardinality of a weakly connected geodetic cover of G is the *weakly connected geodetic number* of G, denoted by $wgn(G)$. A weakly connected geodetic cover of cardinality $wgn(G)$ is a *weakly connected geodetic basis* of G, denoted by *wgn-basis* of G.

Every weakly connected geodetic cover is a geodetic cover, and every connected geodetic cover is a weakly connected geodetic cover. Thus

$$g(G) \leq wgn(G) \leq g_c(G) \tag{3}$$

for connected graphs G.

For convenience, the $WGN(G)$ refers to the family of all weakly connected geodetic covers of G.

Proposition 3. *Let G be a nontrivial connected graph, and let $S \subseteq V(G)$ be a geodetic cover of G. If $S \in WGN(G)$, then for every $v \in S$ there exists $u \in S \setminus \{v\}$ such that $d_G(u,v) \leq 2$.*

Proof. Since G is nontrivial, $|S| \geq g(G) \geq 2$. Suppose, in the contrary, there exists $v \in S$ for which $d_G(u,v) \geq 3$ for all $u \in S \setminus \{v\}$. Then $N_G[S \setminus \{v\}] \cap N_G[v] = \varnothing$. Consequently, the graph $\langle S \rangle_w$ is disconnected, a contradiction. □

The converse of Proposition 3 is not in general true. For example, let $P_8 = [x_1, x_2, \ldots, x_8]$ be a path on 8 vertices. Observe that $S = \{x_1, x_3, x_6, x_8\}$ is a geodetic cover but not a weakly connected geodetic cover yet satisfies the desired property.

Proposition 4. *For paths P_n and cycles C_n,*

(i) $wgn(P_n) = \lceil \frac{n+1}{2} \rceil$ *for* $n \geq 1$;

(ii) $wgn(C_n) = \lceil \frac{n}{4} \rceil + 1$ *for* $n \geq 4$.

Proof. Let $P_n = [x_1, x_2, \ldots, x_n]$. Write $n = 2k$ when n is even, and write $n = 2k+1$ when n is odd. In any case, define $S = \{x_1, x_3, \ldots, x_{2k-1}, x_n\}$. Then $S \in WGN(P_n)$ so that $wgn(P_n) \leq |S| = \lceil \frac{n+1}{2} \rceil$. It is easy to check for the other inequality when $n = 1, 2, 3$. Suppose $n \geq 4$. Let $S \in WGN(P_n)$ with $|S| < \lceil \frac{n+1}{2} \rceil$. Then there exists $v \in S$ for which $d_{P_n}(u,v) \geq 3$ for all $u \in S \setminus \{v\}$, contrary to Proposition 3. Thus, $|S| \geq \lceil \frac{n+1}{2} \rceil$ for all $S \in WGN(P_n)$. Hence, $wgn(P_n) \geq \lceil \frac{n+1}{2} \rceil$.

For $n \geq 4$, let $C_n = [x_1, x_2, \ldots, x_n, x_1]$. If n is even, say $n = 2k$, then $S \in WGN(C_n)$ for all wgn-basis S of the path $P_{k+1} = [x_1, x_2, \ldots, x_{k+1}]$. Thus, together with (i), $wgn(C_n) \leq \lceil \frac{k+2}{2} \rceil = 1 + \lceil \frac{n}{4} \rceil$. Similarly, if n is odd and $n = 2k+1$, then $S \in WGN(C_n)$ for all wgn-basis S of $P_{k+2} = [x_1, x_2, \ldots, x_{k+2}]$, yielding $wgn(C_n) \leq \lceil \frac{k+3}{2} \rceil = 1 + \lceil \frac{n}{4} \rceil$. Now, if $S \in WGN(C_n)$ such that $|S| < 1 + \lceil \frac{n}{4} \rceil$, then there exists $v \in S$ for which $d_{C_n}(u,v) \geq 3$ for all $u \in S \setminus \{v\}$, contradicting Proposition 3. □

Theorem 1. *Let G be a connected graph of order $n \geq 2$. Then*

(i) $wgn(G) = n$ *if and only if* $G = K_n$;

(ii) $wgn(G) = 2$ *if and only if either* $G = K_2$ *or* $G = \overline{K_2} + H$, *where H is any graph of order* $n \geq 1$;

(iii) $g(G) = wgn(G)$ *whenever* $diam(G) \leq 2$;

(iv) $wgn(G) = g_c(G)$ *if and only if G has a wgn-basis S for which $\langle S \rangle$ is connected;*

(v) $wgn(G) = 3$ *if and only if exactly one of the following holds:*

  (a) $G = K_3$;

  (b) *G is noncomplete with $n \geq 4$, $G \neq \overline{K_2} + H$ for any graph H and G has a geodetic cover $\{x_1, x_2, x_3\}$ such that for each i, there exists $j \neq i$ for which $d_G(x_i, x_j) \leq 2$.*

Proof. Suppose that $wgn(G) = n$. If $G \neq K_n$, then choose $u, v \in V(G)$ for which $d_G(u,v) = 2$. Let $[u, z, v]$ be a geodesic in G. Since $S = (V(G) \setminus \{z\}) \in WGN(G)$, $wgn(G) \leq |S| = n - 1$, a contradiction. Thus, $G = K_n$. The converse is clear.

Next, suppose that $wgn(G) = 2$. If $G = K_2$, then we are done by (i). Suppose that $G \neq K_2$. Let $S = \{u, v\}$ be a wgn-basis of G. Then by Proposition 3, $d_G(u, v) \leq 2$. Since G is of order $n \geq 3$, $d_G(u, v) = 2$ and $\{u, v\}$ is a geodetic cover of G. Put $H = \langle V(G) \setminus S\rangle$. Then $G = \overline{K_2} + H$. Conversely, the conclusion immediately follows from (i) if $G = K_2$. Suppose that $G = \overline{K_2} + H$ for some graph H. Let $S = V(\overline{K_2}) = \{u, v\}$. Since G is nontrivial and $S \in WGN(G)$, $2 \leq wgn(G) \leq |S| = 2$.

Now, suppose $diam(G) \leq 2$, and let $S \subseteq V(G)$ be a g-basis of G. Since G is nontrivial, $|S| \geq 2$. Let $x, y \in N_G[S]$ with $x \neq y$. Suppose $x, y \in S$. If $xy \in E(G)$, then $[x, y]$ is an x-y path in $\langle S\rangle_w$. Suppose $xy \notin E(G)$. Then $d_G(x, y) = 2$ by assumption. Pick $z \in N_G(x) \cap N_G(y) \neq \varnothing$. Then $[x, z, y]$ is an x-y path in $\langle S\rangle_w$. Suppose that $x, y \notin S$. There exist $u, v \in S$ for which $x \in N_G(u)$ and $y \in N_G(v)$. If $u = v$, then $[x, u, y]$ is an x-y path in $\langle S\rangle_w$. Suppose $u \neq v$. If $uv \in E(G)$, then $[x, u, v, y]$ is an x-y path in $\langle S\rangle_w$. If $uv \notin E(G)$, then pick $z \in N_G(u) \cap N_G(v)$. Then $[x, u, z, v, y]$ is an x-y path in $\langle S\rangle_w$. Lastly, suppose $x \in S$ and $y \notin S$, and let $u \in S$ for which $y \in N_G(u)$. If $xu \in E(G)$, then $[x, u, y]$ is a path in $\langle S\rangle_w$. Suppose $xu \notin E(G)$. Then pick $z \in N_G(x) \cap N_G(u)$. Then $[x, z, u, y]$ is an x-y path in $\langle S\rangle_w$. This shows that $\langle S\rangle_w$ is connected, implying that $S \in WGN(G)$. Therefore, $wgn(G) \leq |S| = g(G)$. Inequality 3 above completes the desired equality.

Next, if G has a wgn-basis S for which $\langle S\rangle$ is connected, then $g_c(G) \leq |S| = wgn(G)$. Inequality 3 yields the desired equality. An example is illustrated in Figure 1 where $wgn(G) = g_c(G) = 4$.

Statement (v) is clear from Statement (ii) and Proposition 3. The set $S = \{x_1, x_2, x_3\}$ in Statement (v) (b) implies that $\langle S\rangle \in \{\overline{K_3}, P_2 \cup P_1\}$. □

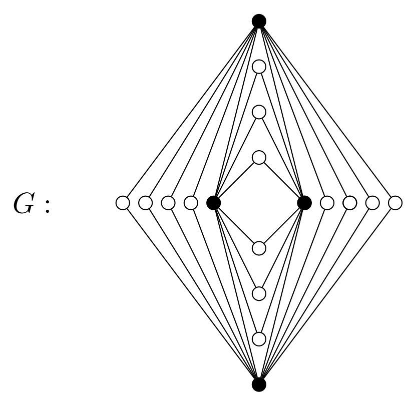

FIGURE 1. A graph G with $wgn(G) = g_c(G) = 4$

The following is an immediate consequence of Theorem 1(iii).

Corollary 1. *For any graphs G and H, $wgn(G + H) = g(G + H)$.*

Interested reader may refer to [8] for the geodetic number of the join of graphs.

It is worth noting that equality $g(G) = wgn(G)$ does not necessarily imply that $diam(G) \leq 2$. Note that $g(C_7) = wgn(C_7) = 3$, but $diam(C_7) = 3$.

Theorem 2. *Let G be a connected graph of order n. Then $wgn(G) = n - 1$ if and only if one of the following holds:*

(i) *There exists $v \in V(G)$ such that $Ext(G) = V(G) \setminus \{v\}$ and $uv \in E(G)$ for all $u \in V(G) \setminus \{v\}$;*

(ii) *There exists $uv \in E(G)$ such that $Ext(G) = V(G) \setminus \{u, v\}$ and $|N_G(z) \cap \{u, v\}| = 1$ for every $z \in V(G) \setminus \{u, v\}$.*

Proof. First, note that Statement (i) implies that $G = K_1 + (\cup_j K_{m_j})$, where $j \geq 2$ and $\sum_j m_j = n - 1$. On the other hand, Statement (ii) implies that G is the graph obtained by combining the graphs $\langle\{v\}\rangle + (\cup_j K_{m_j})$ and $\langle\{u\}\rangle + (\cup_k K_{m_k})$ by the edge uv, where $\sum_j m_j + \sum_k m_k = n - 2$.

Clearly, if (i) or (ii) holds for G, then $wgn(G) = n-1$. Conversely, suppose that $wgn(G) = n-1$, and suppose that G does not satisfy (i). Let $v \in V(G)$ such that $S = V(G) \setminus \{v\}$ is a wgn-basis of G. Hence, $v \notin Ext(G)$. By assumption, there exists $p \in V(G) \setminus N_G(v)$. Since G is connected, we may choose p so that $d_G(p, v) = 2$. Let $u \in N_G(v) \cap N_G(p)$. Then $u \notin Ext(G)$ and $uv \in E(G)$. Let $c \in V(G) \setminus \{u, v\}$. Suppose $c \notin Ext(G)$ and $[a, c, b]$ is a geodesic in G. If $vc \notin E(G)$ then $S \setminus \{c\} \in WGN(G)$, a contradiction. Thus, $vc \in E(G)$ so that $V(G) \setminus \{u, c\} \in WGN(G)$, a contradiction. Since c is arbitrary, $V(G) \setminus \{u, v\} = Ext(G)$. Let $z \in V(G) \setminus \{u, v\}$. Let $P = [z = x_1, x_2, \ldots, x_{k-1}, x_k = v]$ be a z-v geodesic in G. Suppose P does not contain u. Since $x_{k-1} \in Ext(G)$, $x_{k-2}v \in E(G)$. Continuing in this manner, $zv \in E(G)$. Suppose P contains u, i.e., $x_{k-1} = u$. Then $zu \in E(G)$. If $zv, zu \in E(G)$, then $V(G) \setminus \{u, v\} \in WGN(G)$, a contradiction. Thus, $|N_G(z) \cap \{u, v\}| = 1$.

□

4. Realization Results

Theorem 3. *For every pair of positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $g(G) = a$ and $wgn(G) = b$. Further, if $a < b$, then it is realizable that $|V(G)| \geq 2b - a$.*

Proof. If $a = b$, then take $G = K_a$. For this graph G, $g(G) = a = b = wgn(G)$. Assume $a < b$. First suppose that $b = a + 1$. Consider the graph $G = G_1$ in Figure 2 in case where $|V(G)| = 2b - a$. On the other hand, in the case where $|V(G)| > 2b - a$, take $G = G_2$, where $l \geq b - a$, obtained from G_1 by adding the paths $[u, y_j, z_2]$, $j = 1, 2, \ldots, l$. In any case, $S = \{x_1, x_2, \ldots, x_{a-1}, z_2\}$ and $S \cup \{u\}$ are a g-basis and a wgn-basis, respectively, of G. Thus, $g(G) = a$, $wgn(G) = a + 1 = b$ and $|V(G)| \geq 2b - a$.

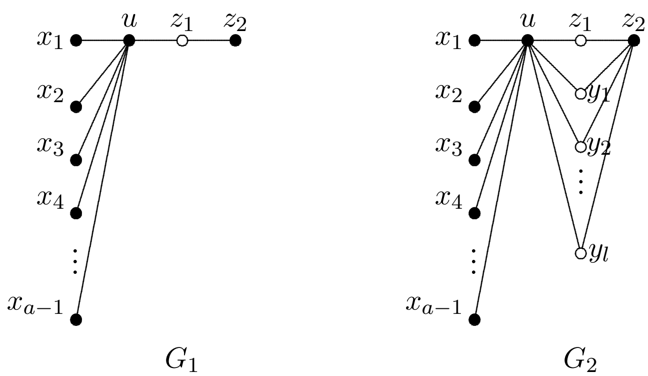

FIGURE 2. Examples of graphs G with $wgn(G) = g(G) + 1$

Now, suppose $b = a+k$, where $k \geq 2$. If $|V(G)| = 2b-a$, then consider the graph $G = G_1$ as in Figure 3 obtained from $P_{2k+2} = [z_1, z_2, \ldots, z_{2k+1}, z_{2k+2}]$ by adding the pendant edges z_2x_j, $j = 1, 2, \ldots, a-2$. Then $g(G) = |Ext(G)| = (a-1) + 1 = a$ and

$$wgn(G) = (a-2) + wgn(P_{2k+2}) = (a-2) + \lceil \frac{2k+3}{2} \rceil = a + k = b.$$

In this case, $|V(G)| = (a-2) + (2k+2) = 2b - a$.

Suppose $|V(G)| > 2b - a$. Then consider the graph $G = G_2$ as in Figure 3, where $l \geq 1$, obtained from graph G_1 by adding the paths $[z_3, y_j, z_5]$, $j = 1, 2, \ldots, l$. □

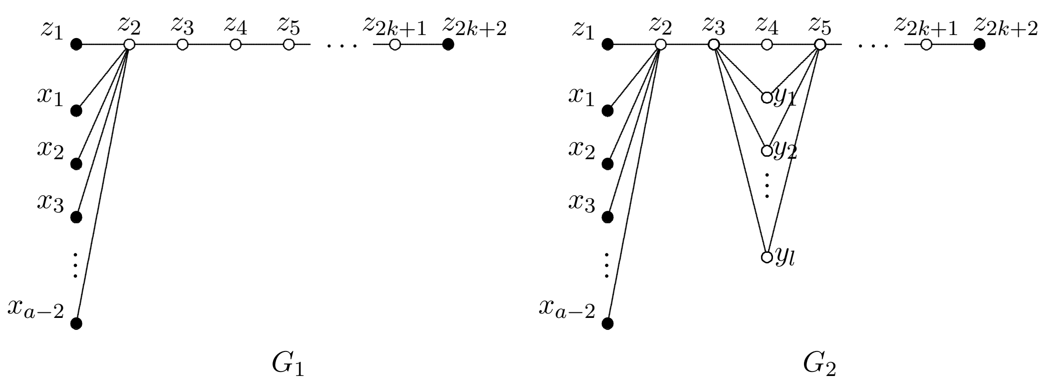

FIGURE 3. Examples of graphs G with $wgn(G) = g(G) + k$, where $k \geq 2$

Corollary 2. *The difference $wgn(G) - g(G)$ can be made arbitrarily large.*

Proposition 5. *For every set of positive integers a, b, c with $3 \leq a < b \leq c$ and $b \leq 2a - 2$, there exists a connected graph G for which $wgn(G) = a$, $g_c(G) = b$ and $|V(G)| = c$.*

Proof. Let $r = c-b$. Suppose that $b = a+1$. If $r = 0$, then we take $G = K_{1,a}$. If $r \geq 1$, then we consider the graph $G = G_1$ as in Figure 4. The sets $\{w, z, x_1, x_2, \ldots, x_{a-2}\}$ and $\{u, w, z, x_1, x_2, \ldots, x_{a-2}\}$ are a wgn-basis and a g_c-basis, respectively, of G. In this case, $wgn(G) = a$, $g_c(G) = a+1 = b$ and $|V(G)| = b+r = c$.

Suppose $b = a + k$, where $2 \leq k \leq a - 2$. Consider the graph $G = G_2$ as in Figure 4. If $r = 0$, then remove the vertices y_1, y_2, …, y_r. Put $S = \{x_1, x_2, \ldots, x_{a-(k+1)}\}$. Then $S \cup \{z_{2j-1} : j = 1, 2, \ldots, k+1\}$ and $S \cup V(P_{2k+1})$ are a wgn-basis and a g_c-basis of G, respectively, where $P_{2k+1} = [z_1, z_2, \ldots, z_{2k+1}]$. In any case, $wgn(G) = (a - k - 1) + \lceil \frac{(2k+1)+1}{2} \rceil = a$, $g_c(G) = a - (k+1) + (2k+1) = a + k = b$ and $|V(G)| = b + r = c$. □

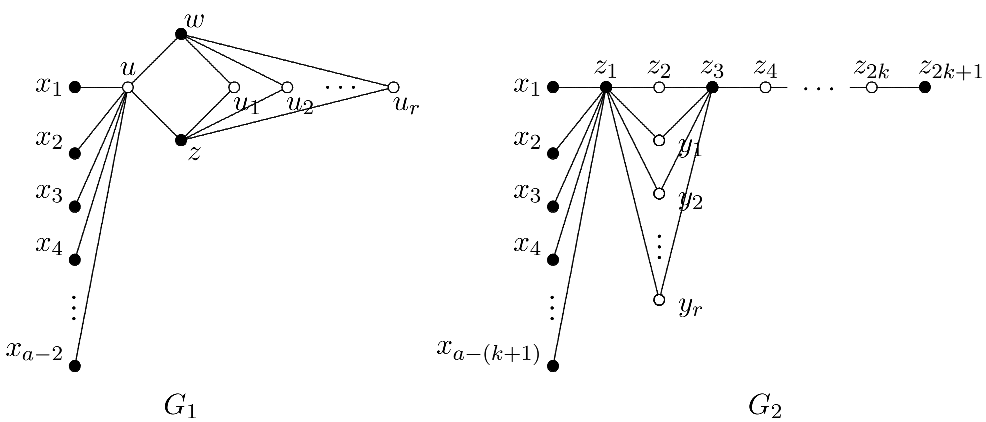

FIGURE 4. A graph G with $g_c(G) = wgn(G) + k$, where $k \geq 1$

Corollary 3. *The difference $g_c(G) - wgn(G)$ can be made arbitrarily large.*

Corollary 4. *For every pair of positive integers a and b with $2 \leq a < b$, there exists a connected graph G for which $g(G) = a$, $wgn(G) = b$ and $g_c(G) = 2b - a$.*

Proof. Write $b = a + k$, where $k \geq 1$. Take the graph G resulting from revising the graph G_2 in Figure 4 by having $S = \{x_1, x_2, \ldots, x_{a-1}\}$ and with $r \geq 1$. Then $g(G) = (a-1)+1 = a$, $wgn(G) = (a-1)+(k+1) = b$ and $g_c(G) = (a-1) + (2k+1) = b + k = b + (b - a) = 2b - a$. □

5. In the corona of graphs

A weakly connected geodetic cover need not be a weakly connected dominating set. Likewise, a weakly connected dominating set need not be a weakly connected geodetic cover.

Theorem 4. **(vertex corona)** *Let G be a nontrivial connected graph, and H any graph. Let $S \subseteq V(G \circ H)$. Then $S \in WGN(G \circ H)$ if and only if*

$$S = A \cup \left(\cup_{v \in V(G)} S_v\right), \tag{4}$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for all $v \in V(G)$ satisfying each of the following:

- (i) *A is a weakly connected dominating set of G;*
- (ii) *S_v is a 2-path closure absorbing set of H^v for each $v \in V(G)$.*

Proof. First, assume that $S \in WGN(G \circ H)$. Define $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. By Lemma 3, $S_v \neq \varnothing$ for all $v \in V(G)$. Moreover, in view of Proposition 3, since $d_{G\circ H}(x,y) \geq 3$ for all $x \in V(H^v)$ and for all $y \in V(H^u)$ whenever u and v are distinct vertices of G, $A \neq \varnothing$. Let $v \in V(G) \setminus A$. Suppose $uv \notin E(G)$ for all $u \in A$. Then $uv \notin E_w$ for all $u \in A$. This means that $\langle S_v \rangle_w$ is a component of $\langle S \rangle_w$. Since G is nontrivial, it follows that $\langle S \rangle_w$ is not connected, a contradiction. Since v is arbitrary, A is a dominating set of G. Moreover, if $\langle A \rangle_w$ is disconnected, then $\langle S \rangle_w$ is disconnected, a contradiction. This shows (i). To show (ii), let $v \in V(G)$ and $q \in V(H^v) \setminus S_v$. There exist $x, y \in S$ for which $q \in I_{G\circ H}(x,y)$. Necessarily, $x, y \in V(H^v)$. Since $d_{G\circ H}(x,y) = 2$, $d_{H^v}(x,y) = 2$ and $q \in I_{H^v}(x,y)$. Since q is arbitrary, S_v is a 2-path closure absorbing set of H^v.

Conversely, suppose that S is as given in Equation 4 with A and all S_v satisfying (i) and (ii), respectively. Let $u \in V(G \circ H) \setminus S$. Let $v \in V(G)$ for which $u \in V(H^v + v)$. Suppose $u = v$. Since A is a dominating set of G and $u \notin A$, there exists $y \in A \cap N_G(u)$. Pick any $x \in S_u$. Then $u \in I_{G\circ H}(x,y)$. Now, suppose $u \in V(H^v)$. Since $u \notin S_v$, then condition (ii) implies that there exist $x, y \in S_v$ such that $d_{H^v}(x,y) = 2$ and $u \in I_{H^v}(x,y)$. This means that $d_{G\circ H}(x,y) = 2$ and $u \in I_{G\circ H}(x,y)$. In any case, $u \in I_{G\circ H}[S]$. Since u is arbitrary, S is a geodetic cover of $G \circ H$. Finally, for each $v \in V(G)$, $vx \in E_w$ for all $x \in S_v$. Thus, if $\langle S \rangle_w$ is disconnected, then necessarily, $\langle A \rangle_w$ is disconnected, a contradiction to (i). Hence, $\langle S \rangle_w$ is connected. Accordingly, $S \in WGN(G \circ H)$. □

Corollary 5. *Let G be a nontrivial connected graph of order n, and let H be any graph. Then*

$$wgn(G \circ H) = \gamma_w(G) + n\rho_2(H).$$

Example 1. Let n and m be positive integers. Then

(i) $wgn(P_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor + n \left\lceil \frac{m+1}{2} \right\rceil$ for $n \geq 2$;

(ii) $wgn(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor + n \left\lceil \frac{m+1}{2} \right\rceil$ for $n \geq 3$;

(iii) $wgn(C_n \circ C_m) = \left\lfloor \frac{n}{2} \right\rfloor + n \left\lceil \frac{m}{2} \right\rceil$ for $n \geq 3$, $m \geq 4$;

(iv) $wgn(P_n \circ C_m) = \left\lfloor \frac{n}{2} \right\rfloor + n \left\lceil \frac{m}{2} \right\rceil$ for $n \geq 2$, $m \geq 4$;

(v) $wgn(K_n \circ K_m) = 1 + nm$ for $n \geq 2$.

Theorem 5. (edge corona) *Let G be a nontrivial connected graph, and H any graph. Let $S \subseteq V(G \diamond H)$. Then $S \in WGN(G \diamond H)$ if and only if*

$$S = A \cup \left(\cup_{uv \in E(G)} S_{uv} \right), \tag{5}$$

where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ for all $uv \in E(G)$ satisfying each of the following:

(i) *For every $v \in Ext(G) \setminus A$, there exists $u \in N_G(v)$ such that $\langle S_{uv} \rangle$ is not complete;*

(ii) *S_{uv} is a 2-path closure absorbing set of H^{uv} for all $uv \in E(G)$.*

Proof. Assume $S \in WGN(G \diamond H)$. Define $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Since $V(H^{uv}) \cap I_{G\diamond H}(u,v) = \varnothing$, $S_{uv} \neq \varnothing$ for all $uv \in E(G)$. Let $v \in Ext(G) \setminus A$, and let $x, y \in S$ such that $v \in I_{G\diamond H}(x,y)$. Since $v \in Ext(G)$, there exists $u \in N_G(v)$ such that $x, y \in S_{uv}$. Necessarily, $xy \notin E(H^{uv})$. Thus, $\langle S_{uv} \rangle$ is not complete, showing (i). Let $uv \in E(G)$, and let $x \in V(H^{uv}) \setminus S_{uv}$. There exist $a, b \in S$ such that $x \in I_{G\diamond H}(a,b)$. This means $x \in I_{H^{uv}}(a,b)$ and $d_{H^{uv}}(a,b) = d_{G\diamond H}(a,b) = 2$. Since x is arbitrary, S_{uv} is a 2-path closure absorbing set of H^{uv}. This proves (ii).

Suppose that S is as given in Equation 5, where A and S_{uv} satisfy conditions (i) and (ii). Let $z \in V(G \diamond H) \setminus S$, and let $u, v \in V(G)$ such that $z \in V(H^{uv}) \cup \{u, v\}$. First, consider $z = u$ or $z = v$. WLOG, assume $z = v$. Suppose that $z \notin Ext(G)$. There exist $a, b \in V(G)$ such that $[a, v, b]$ is a geodesic in G. Pick $x \in S_{av}$ and $y \in S_{bv}$. Then $x, y \in S$ and $v \in I_{G\diamond H}(x,y)$. Suppose $z \in Ext(G)$. Then, by (i), there exist $x, y \in S_{uv}$ such that $xy \notin E(H^{uv})$. This means that $d_{G\diamond H}(x,y) = 2$ and $v \in I_{G\diamond H}(x,y)$. Next, suppose that $z \in V(H^{uv})$. By (ii), there exist $x, y \in S_{uv}$ such that $d_{H^{uv}}(x,y) = 2$ and $z \in I_{H^{uv}}(x,y)$. This means that $x, y \in S$, $d_{G\diamond H}(x,y) = 2$ and $z \in I_{G\diamond H}(x,y)$. Therefore, S is a geodetic cover of $G \diamond H$. Finally, for each $uv \in E(G)$, $xu \in E_w$ and $xv \in E_w$ for every $x \in S_{uv}$, showing that $\langle S_{uv} \rangle_w$ is connected. Now, suppose $\langle S \rangle_w$ is disconnected. Then there exists at least one edge, say $uv \in E(G)$, that forms the disconnection of $\langle S \rangle_w$. Pick $x \in S_{uv}$. Then $xv \in E_w$ and $xu \in E_w$ so that $[u, x, v]$ is a geodesic in $\langle S \rangle_w$, a contradiction. Hence, $S \in WGN(G \diamond H)$. $\square$

Corollary 6. *Let G be a nontrivial connected graph of size m, and H be any graph of order n.*

(i) *If H is noncomplete, then $wgn(G \diamond H) = m\rho_2(H)$.*

(ii) *If H is complete, then $wgn(G \diamond H) = mn + |Ext(G)|$.*

Example 2. Let n and m be positive integers. Then

(i) $wgn(P_n \diamond P_m) = (n-1)\left\lceil \frac{m+1}{2} \right\rceil$ for $n \geq 2$;

(ii) $wgn(P_n \diamond C_m) = (n-1)\left\lceil \frac{m}{2} \right\rceil$ for $n \geq 2$ and $m \geq 4$;

(iii) $wgn(C_n \diamond C_m) = n\left\lceil \frac{m}{2} \right\rceil$ for $n \geq 3$ and $m \geq 4$;

(iv) $wgn(C_n \diamond P_m) = n\left\lceil \frac{m+1}{2} \right\rceil$ for $n \geq 3$;

(v) $wgn(P_n \diamond K_m) = (n-1)m + 2$ for $n \geq 2$;

(vi) $wgn(C_n \diamond K_m) = nm$ for $n \geq 3$.

6. In the lexicographic product of graphs

For $C \subseteq V(G[H])$, it is customary to define

$$C_G = \{x \in V(G) : (x,y) \in C \text{ for some } y \in V(H)\},$$

called the *G-projection* of C. For $S \subseteq V(G[H])$ and $x \in V(G)$, we define

$$S_x = \{y \in V(H) : (x,y) \in S\}.$$

It should be noted that $x \in Ext(G)$ if and only if $(x, y) \in Ext(G[K_m])$ for all $y \in V(K_m)$.

Proposition 6. *Let G be a connected graph, $m \geq 2$, $S \in WGN(G)$ and A a GIC-set with respect to S. Then*

$$C = Ext(G[K_m]) \cup [(A \setminus Ext(G)) \times \{v_0\}] \cup [(S \setminus A) \times V(K_m)] \tag{6}$$

is a weakly connected geodetic cover of $G[K_m]$ for every $v_0 \in V(K_m)$.

Proof. Let $S \in WGN(G)$, A a GIC-set with respect to S, and let $v_0 \in V(K_m)$. Since S is a geodetic cover of G, Proposition 1 implies that

$$C = Ext(G[K_m]) \cup [(A \setminus Ext(G)) \times \{v_0\}] \cup [(S \setminus A) \times V(K_m)] \tag{7}$$

is a geodetic cover of $G[K_m]$. To show connectedness of $\langle C \rangle_w$, let $(x, y), (u, v)$ be distinct vertices in $N_{G[K_m]}[C]$. It is enough to consider that $(x, y), (u, v) \in C$. Suppose $x = u$. Then $x \in (S \setminus A) \cup Ext(G)$ and y and v are distinct vertices in K_m. Thus, $(x, y)(u, v) \in E_w$. Suppose that $x \neq u$. Since $S \in WGN(G)$ and $x, u \in S$, G has a x-u path $[x = x_1, x_2, \ldots, x_k = u]$, where $\{x_1, x_2, \ldots, x_k\} \subseteq N_G[S]$. Choose $y_1, y_2, \ldots, y_k \in V(K_m)$, where $y_j = v_0$ whenever $x_j \in A \setminus Ext(G)$. Then $[(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)]$ is a (x, y)-(u, v) path in $\langle C \rangle_w$. Thus, $\langle C \rangle_w$ is connected, showing $C \in WGN(G[K_m])$. □

Proposition 7. *Let G be a connected graph and $C \in WGN(G[K_m])$. Then $C_G \in WGN(G)$ and there exists a GIC-set A with respect to C_G such that*

$$C = Ext(G[K_m]) \cup [(C_G \setminus A) \times V(K_m)] \cup B, \tag{8}$$

where $B = \{(u, v) \in C : u \in A \setminus Ext(G), v \in V(K_m)\}$.

Proof. Let $C \in WGN(G[K_m])$. Since C is a geodetic cover of $G[K_m]$, Proposition 2 implies that C_G is a geodetic cover of G and there exists a GIC-set A with respect to C_G such that

$$C = Ext(G[K_m]) \cup [(C_G \setminus A) \times V(K_m)] \cup B, \tag{9}$$

where $B = \{(u, v) \in C : u \in A \setminus Ext(G), v \in V(K_m)\}$. Suppose that $\langle C_G \rangle_w$ is not connected, and let u and v be distinct vertices in $N_G[C_G]$ such that $\langle C_G \rangle_w$ has no path joining them. We may assume that $u, v \in C_G$. Let $x, y \in V(K_m)$ such that $(u, x), (v, y) \in C$. Since $\langle C \rangle_w$ is connected, $G[K_m]$ has a (u, x)-(v, y) path $P = [(u, x) = (x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) = (v, y)]$ in $\langle C \rangle_w$. Let $x_{n_1}, x_{n_2}, \ldots, x_{n_r}$ be the distinct vertices in P_G with $n_1 < n_2 < \cdots < n_r$. Then, $[x_{n_1}, x_{n_2}, \ldots, x_{n_r}]$ is a u-v path in $\langle C_G \rangle_w$, a contradiction. Thus, $\langle C_G \rangle_w$ is connected so that $C_G \in WGN(G)$. □

The following follows from Proposition 6 and Proposition 7.

Corollary 7. *Let G be a connected nontrivial graph and $m \geq 2$. Then*

$$wgn(G[K_m]) = (m-1)|Ext(G)| + \min\{m|S| - (m-1)|S^+| : S \in WGN(G)\}, \tag{10}$$

where S^+ is the maximum GIC-set with respect to S.

Corollary 8. *Let G be a connected nontrivial graph and $m \geq 2$. If G has a wgn-basis that is a GIC-set in G, then*

$$wgn(G[K_m]) = (m-1)|Ext(G)| + wgn(G). \tag{11}$$

Moreover, if G has no extreme vertices, $wgn(G[K_m]) = wgn(G)$.

Finally, we end this investigation of the weakly connected geodetic cover in the composition $K_m[G]$.

Proposition 8. *Let G be a connected noncomplete graph and $m \geq 2$. Then*

$$2 \leq wgn(K_m[G]) \leq \min\{wgn(G), 4\}. \tag{12}$$

More precisely,

(i) $wgn(K_m[G]) = 2$ *if and only if* $wgn(G) = 2$;

(ii) $wgn(K_m[G]) = 3$ *if and only if* $wgn(G) = 3$; *and*

(iii) $wgn(K_m[G]) = 4$ *if and only if* $wgn(G) \geq 4$.

Proof. The left-hand side of the Inequality 12 is clear. Now, since $\{z\} \times S \in WGN(K_m[G])$ for each $z \in V(K_m)$ and for each $S \in WGN(G)$, $wgn(K_m[G]) \leq wgn(G)$. Pick distinct $u, v \in V(K_m)$ and $x, y \in V(G)$ for which $d_G(x, y) \geq 2$. Then $C = \{(u, x), (u, y), (v, x), (v, y)\} \in WGN(K_m[G])$. Thus, $wgn(K_m[G]) \leq |C| = 4$. Therefore, the right-side of Inequality 12 holds.

Suppose that $wgn(K_m[G]) = 2$. By Theorem 1(ii), since $K_m[G]$ is not complete, $K_m[G]$ is isomorphic to $\overline{K_2} + H$ for some graph H. Thus, there exist $(x, y), (u, v) \in V(K_m[G])$ such that $d_{K_m[G]}((x, y), (u, v)) = 2$ and $I_{K_m[G]}[(x, y), (u, v)] = V(K_m[G])$. Necessarily, $x = u$ and $d_G(v, y) \geq 2$. Put $S = \{v, y\}$ and let $z \in V(G) \setminus S$. In particular, $(u, z) \in I_{K_m[G]}((u, y), (u, v))$, showing that $z \in I_G(S)$ and $d_G(v, y) = 2$. This means that S is a geodetic cover of G and $\langle S \rangle_w$ is connected. Therefore, $S \in WGN(G)$ so that $wgn(G) = |S| = 2$. The converse is clear.

Suppose that $wgn(K_m[G]) = 3$. By Statement (i) and Theorem 1(v), $K_m[G] \neq \overline{K_2} + H$ for any graph H. Let $C = \{(a, b), (x, y), (u, v)\}$ be a wgn-basis of $K_m[G]$. Suppose $a \neq x$. If $u \neq a$ and $u \neq x$, then $\langle C \rangle$ is complete, which is impossible. If $u = a$, then $\{(a, b), (x, y)\} \in WGN(K_m[G])$, a contradiction. Thus, $C = \{(t, x), (t, y), (t, z)\}$. Put $S = \{x, y, z\}$. Then S is a geodetic cover of G. By Proposition 3, each vertex in C is of distance at most 2 from another vertex in C. Thus, each vertex in S is of distance at most 2 from another vertex in S. By Theorem 1(v), $wgn(G) = 3$. Conversely, suppose $wgn(G) = 3$. Then $G \neq \overline{K_2} + H$ for any graph H and G has a geodetic cover S such that each vertex in S is of distance at most 2 from another vertex in S. This means that $K_m[G]$ is not isomorphic to $\overline{K_2} + H$ for any graph H and $C = \{z\} \times S$, for any $z \in V(K_m)$, is a geodetic cover of $K_m[G]$ satisfying the property that each vertex in C is of distance at most 2 from another vertex in C. By Theorem 1(v), $wgn(K_m[G]) = 3$.

Statement (iii) follows immediately from statements (i) and (ii). □

7. Conclusion

In this paper, we investigated the weakly connected geodetic number of a graph. Several bounds for this parameter were established, and graphs attaining extremal and small values were characterized. We also determined the weakly connected geodetic numbers of some common graphs and described weakly connected geodetic covers in the corona, edge corona, and lexicographic product of graphs together with their corresponding values. The realizability results have further demonstrated that the difference between the weakly connected geodetic number and the geodetic number can be made arbitrarily large, as well as the difference between the connected geodetic number and weakly connected geodetic number.

These results provide further insights into the interaction between geodesic covering and connectivity in graphs. Future work may include determining the weakly connected geodetic number for other graph operations and graph classes, as well as studying the computational aspects of determining $wgn(G)$. Such investigations may lead to a deeper understanding of the interplay between geodesic convexity and connectivity in graphs.

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