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APPLICATIONS OF M-POLAR FUZZY SET THEORY TO BCH-ALGEBRAS

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Abstract. The concepts of m-polar fuzzy subalgebras and m-polar fuzzy closed ideals in BCH-algebras are defined, and their essential features are examined. The connections between m-polar fuzzy subalgebras and m-polar fuzzy closed ideals are analysed comprehensively. Conditions that are both necessary and sufficient for a m-polar fuzzy subalgebra to qualify as a m-polar fuzzy closed ideal are established. Additionally, several characterisations of m-polar fuzzy closed ideals in BCH-algebras are provided. The findings gained generalise certain established ideas from fuzzy, bipolar fuzzy, and m-polar fuzzy algebraic structures, extending them to the more comprehensive framework of BCH-algebras. The results enhance the comprehension of the relationship between multi-polar fuzziness and algebraic structures.

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1. Introduction

Since the seminal work of Zadeh [22], fuzzy set theory has become one of the most influential mathematical tools for modeling uncertainty and vagueness. Its rapid development has led to several important extensions, including bipolar-valued fuzzy sets, interval-valued fuzzy sets, and intuitionistic fuzzy sets, each designed to capture additional layers of uncertainty and duality [13,14,23]. Among these, bipolar-valued fuzzy sets introduced by Lee [13] have attracted significant attention due to their ability to simultaneously represent positive and negative information, a feature particularly useful in logical reasoning and algebraic structures.

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To get past the limits of bipolarity and to model the dimensional preference information Chen *et al.* [8] introduced m-polar sets as a clear generalization of the bipolar fuzzy sets. M-polar fuzzy sets assign each element independent membership degrees. M-polar fuzzy sets give a mathematical structure for modeling the uncertainty. The effectiveness of m-polar fuzzy sets has been demonstrated in various applications, most notably in decision-making, such as the m-polar fuzzy ELECTRE-I method proposed by Akram *et al.* [1]. These developments have motivated extensive research on m-polar fuzzy structures in algebraic systems.

Parallel to these advances in fuzzy set theory, BCH-algebras have been studied as an important generalization of BCK- and BCI-algebras, with strong connections to logic, information theory, and computer science. Several fundamental aspects of BCH-algebras have been investigated, including special subsets [20], endomorphisms [9], and Smarandache BCH-algebras [6]. The algebraic flexibility of BCH-algebras has also been explored through soft set theory [12] and cubic subalgebras [21], highlighting their suitability for further generalizations.

The integration of fuzzy concepts into BCH-algebras has resulted in a rich body of literature. Jun [10] initiated the study of fuzzy closed ideals and fuzzy filters in BCH-algebras, laying a solid foundation for subsequent fuzzy-algebraic investigations. Later, Borumand Saeid *et al.* [7] introduced fuzzy n-fold ideals in BCH-algebras, significantly extending classical ideal theory. The role of bipolar-valued fuzziness in BCH-algebras was further emphasized by Jun and Park [11], who investigated filters based on bipolar-valued fuzzy sets.

In recent years, research interest has shifted toward multi-polar fuzzy algebraic structures, particularly within BCK- and BCI-algebras. Al-Masarwah and Ahmad introduced m-polar fuzzy ideals of BCK/BCI-algebras in [3] and later proposed generalized forms of m-polar fuzzy ideals in [4]. Further developments include generalized m-polar fuzzy positive implicative ideals [5] and bipolar fuzzy implicative ideals [2]. Interval-valued and cubic extensions of m-polar fuzzy structures were investigated in [17, 18], while generalized fuzzy ideals based on interval-valued m-polar fuzzy structures were studied in [16]. Moreover, m-polar fuzzy q-ideals and cubic $p(q,a)$-ideals in BCI-algebras were analyzed in [15, 19], further demonstrating the versatility and depth of the m-polar fuzzy framework.

Although the theory of m-polar fuzzy structures in $\mathbb{BC}\mathfrak{H}$-algebra has made significant contributions, it has not attracted extensive attention. Specifically, the exploration of $\mathbb{MPF}^{Sub}$ and $\mathbb{MPF}^{CI}$ in $\mathbb{BC}\mathfrak{H}$-algebra remains underdeveloped, and their level subsets have not been systematically analyzed. Level subsets are vital in fuzzy algebra because they serve as a critical bridge between fuzzy concepts and traditional algebraic structures, enabling the interpretation of fuzzy findings within a conventional algebraic framework.

Structure of the paper. The remainder of this paper is organized as follows. Section 2 presents the necessary preliminary concepts and basic definitions related to BCH-algebras and m-polar fuzzy sets, which are essential for the development of the subsequent results. In Section 3, we introduce the notions of m-polar fuzzy subalgebras and m-polar fuzzy closed ideals in BCH-algebras and investigate their fundamental properties. This section also provides a detailed characterization of these structures via their level subsets, establishing equivalence results between m-polar fuzzy concepts and classical subalgebras and closed ideals. Finally, Section 4 concludes the paper with a summary of the main results and highlights possible directions for future research.

Novelty and Contributions. Motivated by the above observations, this paper aims to fill this gap by presenting a comprehensive study of $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ and $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ in $\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra. The main contributions of this work are summarized as follows:

- We formally introduce the notion of $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ in the $\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra. We illustrate $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$. We formally introduce the notion of $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$, in the $\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra.
- We investigate the properties. We investigate the relationships, between $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ and $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$.
- We show the sufficient conditions. The necessary and sufficient conditions make an $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ become an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$.
- Characterizations of $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ use the level subsets to link the fuzzy and the classical algebraic structures.
- The results generalize and unify several earlier findings on fuzzy, bipolar fuzzy, interval-valued, cubic, and $\mathbb{M}\mathcal{P}\mathfrak{F}^{I}$ in BCK,BCI and BCH-algebras.

The results presented in this paper enrich the theory of m-polar fuzzy algebra and provide a solid foundation for future research on generalized fuzzy structures in BCH-algebras and related logical systems.

Table 1 lists this paper's acronyms.

Table 1. List of Acronyms

Acronyms	**Meaning**
$\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra	BCH-algebra
$\mathbb{M}\mathcal{P}\mathfrak{F}^{Set}$	m-polar fuzzy set
$\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$	m-polar fuzzy subalgebra
$\mathbb{M}\mathcal{P}\mathfrak{F}^{I}$	m-polar fuzzy ideal
$\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$	m-polar fuzzy closed ideal

2. Preliminaries

Let $K(\tau)$ be the collection of all algebras of type $\tau = (2, \vec{\theta})$. We call a system $(\mathfrak{L}; *, \vec{\theta}) \in K(\tau)$ a $\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra if it fulfills the following axioms:

$$(\forall \varpi \in \mathfrak{L})(\varpi * \varpi = \vec{\theta}), \tag{2.1}$$

$$(\forall \varpi, \zeta \in \mathfrak{L})(\varpi * \zeta = \vec{\theta}) \ \& \ \zeta * \varpi = \vec{\theta}) \ \Rightarrow \ \varpi = \zeta), \tag{2.2}$$

$$(\forall \varpi, \zeta, \xi \in \mathfrak{L})((\varpi * \zeta) * \xi = (\varpi * \xi) * \zeta). \tag{2.3}$$

A BCH-algebra $\mathfrak{L}$ is characterized by the following axioms:

$$(\forall \varpi \in \mathfrak{L})(\varpi * \vec{\theta}) = \varpi), \tag{2.4}$$

$$(\forall \varpi \in \mathfrak{L})(\varpi * \vec{\theta}) = \vec{\theta}) \ \Rightarrow \ \varpi = \vec{\theta}), \tag{2.5}$$

$$(\forall \varpi, \zeta \in \mathfrak{L})(\vec{\theta} * (\varpi * \zeta) = (\vec{\theta} * \varpi) * (\vec{\theta} * \zeta)), \tag{2.6}$$

$$(\forall \varpi \in \mathfrak{L})(\vec{\theta} * (\vec{\theta} * (\vec{\theta} * \varpi)) = \vec{\theta} * \varpi). \tag{2.7}$$

A subset $\mathfrak{Q} \subseteq \mathfrak{L}$ is called a *subalgebra* of the $\mathbb{B}\mathcal{C}\mathfrak{H}$-algebra $\mathfrak{L}$ whenever $\varpi * \zeta \in \mathfrak{Q}$ for all $\varpi, \zeta \in \mathfrak{Q}$. A subset I of a BCH-algebra $\mathfrak{L}$ is called a *closed ideal* of $\mathfrak{L}$ if it fulfills the following requirements:

$$(\forall \varpi \in \mathfrak{L})(\varpi \in \mathfrak{W} \ \Rightarrow \ \vec{\theta} * \varpi \in \mathfrak{W}), \tag{2.8}$$

$$(\forall \varpi \in \mathfrak{W})(\forall \zeta \in \mathfrak{L})(\zeta * \varpi \in \mathfrak{W} \ \Rightarrow \ \zeta \in \mathfrak{W}). \tag{2.9}$$

An $\mathbb{MPF}^{\mathrm{Set}}$ $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ defined on a nonempty set $\mathfrak{L}$ (see [8]) is a mapping $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast} : \mathfrak{L} \to [0,1]^m$. The membership value corresponding to any element $\varpi \in \mathfrak{L}$ is represented by

$$\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi) = \left((\pi_1 \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi), (\pi_2 \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi), \cdots, (\pi_m \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi) \right),$$

where $\pi_i : [0,1]^m \to [0,1]$ is the i-th projection for each $i = 1, 2, \cdots, m$.

For a given $\mathbb{MPF}^{\mathrm{Set}}$ set on the set $\mathfrak{L}$, we define the set

$$U(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}; \hat{t}) := \{\varpi \in \mathfrak{L} \mid \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi) \geq \hat{t}\}, \tag{2.10}$$

that is,

$$U(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}; \hat{t}) := \{\varpi \in \mathfrak{L} \mid (\pi_1 \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi) \geq t_1, (\pi_2 \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi) \geq t_2, \cdots, (\pi_m \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi) \geq t_m\}, \tag{2.11}$$

which is referred to as an *m-polar level set* of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$.

An $\mathbb{MPF}^{\mathrm{Set}}$ point on a set $\mathfrak{L}$ is defined as an $\mathbb{MPF}^{\mathrm{Set}}$ set $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ on $\mathfrak{L}$ of the form

$$\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta) = \begin{cases} \hat{r} = (r_1, r_2, \cdots, r_m) \in (0,1]^m & \text{if } \zeta = \varpi, \\ \hat{0} = (0, 0, \cdots, 0) & \text{if } \zeta \neq \varpi, \end{cases} \tag{2.12}$$

and it is denoted by $\varpi_{\hat{r}}$. We say that ϖ is the *support* of $\varpi_{\hat{r}}$ and $\hat{r}$ is the *value* of $\varpi_{\hat{r}}$.

We say that an $\mathbb{MPF}^{\mathrm{Set}}$ point $\varpi_{\hat{r}}$ is contained in an $\mathbb{MPF}^{\mathrm{Set}}$ set $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$, denoted by $\varpi_{\hat{r}} \in \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$, if $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi) \geq \hat{r}$, that is, $(\pi_i \circ \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast})(\varpi) \geq r_i$ for all $i = 1, 2, \cdots, m$.

3. m-Polar Fuzzy Subalgebras and Closed Ideals

Unless explicitly mentioned otherwise, $\mathfrak{L}$ will always represent a $\mathbb{BCH}$-algebra in what follows.

Definition 3.1. An $\mathbb{MPF}^{\mathrm{Set}}$ $\widehat{\mathcal{GP}}_{\circledast}$ defined on $\mathfrak{L}$ is called a $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$ whenever the following assertions hold:

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \tag{3.1}$$

for all $\varpi, \zeta \in \mathfrak{L}$.

that is,

$$\big(\forall\, \varpi, \zeta \in \mathfrak{L},\ i \in \{1, 2, \ldots, m\}\big)\ (\pi_i \circ \widehat{\mathcal{GP}}_{\circledast})(\varpi * \zeta) \geq \pi_i \circ \widehat{\mathcal{GP}}_{\circledast}(\varpi) \wedge \pi_i \circ \widehat{\mathcal{GP}}_{\circledast}(\zeta). \tag{3.2}$$

Theorem 3.2. *Let $\widehat{\mathcal{GP}}_{\circledast}$ be an $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$. Then the following statements hold.*

(i) $(\forall \alpha \in [0, 1])$ $(\widehat{\mathcal{GP}}_{\circledast} \neq \emptyset \Rightarrow \widehat{\mathcal{GP}}_{\circledast}$ is a subalgebra of $\mathfrak{L})$.

Proof. (i) Let $\alpha \in [0, 1]$ be such that $\widehat{\mathcal{GP}}_{\circledast} \neq \emptyset$. If $x, y \in \widehat{\mathcal{GP}}_{\circledast}$, then $\widehat{\mathcal{GP}}_{\circledast}(\varpi) \geq \alpha$ and $\widehat{\mathcal{GP}}_{\circledast}(\zeta) \geq \alpha$. It follows from (3.2) that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \geq \alpha$$

so that $\varpi * \zeta \in \widehat{\mathcal{GP}}_{\circledast}$. Therefore $\widehat{\mathcal{GP}}_{\circledast}$ is a subalgebra of $\mathfrak{L}$. □

Corollary 3.3. *If $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$, then the sets $\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta})$ is a subalgebras of $\mathfrak{L}$.*

Proof. Straightforward. □

Necessary and sufficient conditions are provided for an $\mathbb{MPF}^{\mathrm{Set}}$ to qualify as an $\mathbb{MP}\mathfrak{F}^{Sub}$.

Theorem 3.4. *Suppose $\widehat{\mathcal{GP}}_{\circledast}$ is an $\mathbb{MPF}^{\mathrm{Set}}$ set on $\mathfrak{L}$ given by*

$$\widehat{\mathcal{GP}}_{\circledast} : \mathfrak{L} \to [\vec{\theta}, 1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } \vec{\theta} * (\vec{\theta} * \varpi) = \varpi, \\ \alpha_2 & \text{otherwise}, \end{cases}$$

for all $\varpi \in \mathfrak{L}$ where $\alpha_1 > \alpha_2$ in $[\vec{\theta}, 1]$. Then $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$.

Proof. Let $\varpi \in \mathfrak{L}$ be such that $\vec{\theta} * (\vec{\theta} * \varpi) \neq \varpi$. Then $\widehat{\mathcal{GP}}_{\circledast}(\varpi) = \alpha_2$. It follows that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \alpha_2 = \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\},$$

Similarly, if $\zeta \in \mathfrak{L}$ does not satisfy the equality $\vec{\theta} * (\vec{\theta} * \zeta) = \zeta$, then we have

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \alpha_2 = \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\}.$$

Let $\varpi, \zeta \in \mathfrak{L}$ be such that $\vec{\theta} * (\vec{\theta} * \varpi) = \varpi$ and $\vec{\theta} * (\vec{\theta} * \zeta) = \zeta$. Then

$$(\varpi * \zeta) * (\vec{\theta} * (\vec{\theta} * (\varpi * \zeta))) \quad = \quad ((\vec{\theta} * (\vec{\theta} * \varpi)) * \zeta) * (\vec{\theta} * (\vec{\theta} * (\varpi * \zeta)))$$

$$
\begin{aligned}
&= ((\vec{\theta} * (\vec{\theta} * \varpi)) * (\vec{\theta} * (\vec{\theta} * (\varpi * \zeta)))) * \zeta \\
&= ((\vec{\theta} * (\vec{\theta} * (\vec{\theta} * (\varpi * \zeta)))) * (\vec{\theta} * \varpi)) * \zeta \\
&= ((\vec{\theta} * (\varpi * \zeta)) * (\vec{\theta} * \varpi)) * \zeta \\
&= ((\vec{\theta} * (\vec{\theta} * \varpi)) * (\varpi * \zeta)) * \zeta \\
&= (\varpi * (\varpi * \zeta)) * \zeta \;=\; \vec{\theta}.
\end{aligned}
$$

Since $(\vec{\theta} * (\vec{\theta} * (\varpi * \zeta))) * (\varpi * \zeta) = \vec{\theta})$ by (2.1) and (2.3), it follows from (2.2) that $\vec{\theta} * (\vec{\theta} * (\varpi * \zeta)) = \varpi * \zeta$. Hence $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta) = \alpha_1 = \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta)\}$. Therefore $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$. □

Theorem 3.5. *Assume that $\mathfrak{P}$ is a nonempty subset of $\mathfrak{L}$, and let $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ denote an $\mathbb{MPF}^{\text{Set}}$ on $\mathfrak{L}$ defined as*

$$
\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast} : \mathfrak{L} \to [\vec{\theta}, 1], \quad \varpi \mapsto \begin{cases} \alpha_1 & \text{if } \varpi * \vec{\varrho_1} = (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \varpi),\ a \in \mathfrak{P} \\ \alpha_2 & \text{otherwise,} \end{cases}
$$

for all $\varpi \in \mathfrak{L}$ where $\alpha_1 > \alpha_2$ in $[\vec{\theta}, 1]$. Then $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$.

Proof. Consider $\varpi, \zeta \in \mathfrak{L}$. Suppose there exists $a \in \mathfrak{P}$ for which either $\varpi * \vec{\varrho_1} \neq (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \varpi)$ or $\zeta * \vec{\varrho_1} \neq (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \zeta)$, then either $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi) = \alpha_2$ or $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta) = \alpha_2$. It follows that $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta) \geq \alpha_2 = \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta)\}$. Assume that ϖ and ζ satisfy the equalities: $\varpi * \vec{\varrho_1} = (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \varpi)$ and $\zeta * \vec{\varrho_1} = (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \zeta)$ for all $\vec{\varrho_1} \in \mathfrak{P}$. Then

$$
\begin{aligned}
(\varpi * \zeta) * \vec{\varrho_1} &= (\varpi * \vec{\varrho_1}) * \zeta \;=\; ((\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * \varpi)) * \zeta \;=\; ((\vec{\theta} * (\vec{\theta} * \varpi)) * \vec{\varrho_1}) * \zeta \\
&= ((\vec{\theta} * (\vec{\theta} * \varpi)) * \zeta) * \vec{\varrho_1} \;=\; ((\vec{\theta} * \zeta) * (\vec{\theta} * \varpi)) * \vec{\varrho_1} \\
&= ((\vec{\theta} * (\vec{\theta} * (\vec{\theta} * \zeta))) * (\vec{\theta} * (\vec{\theta} * (\vec{\theta} * \varpi)))) * \vec{\varrho_1} \\
&= ((\vec{\theta} * (\vec{\theta} * (\vec{\theta} * (\vec{\theta} * \varpi)))) * (\vec{\theta} * (\vec{\theta} * \zeta))) * \vec{\varrho_1} \\
&= ((\vec{\theta} * (\vec{\theta} * \varpi)) * (\vec{\theta} * (\vec{\theta} * \zeta))) * \vec{\varrho_1} \\
&= (\vec{\theta} * ((\vec{\theta} * \varpi) * (\vec{\theta} * \zeta))) * \vec{\varrho_1} \;=\; (\vec{\theta} * (\vec{\theta} * (\varpi * \zeta))) * \vec{\varrho_1} \\
&= (\vec{\theta} * \vec{\varrho_1}) * (\vec{\theta} * (\varpi * \zeta)).
\end{aligned}
$$

Hence $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta) = \alpha_1 = \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta)\}$. Therefore $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$. □

Definition 3.6. A $\mathbb{MPF}^{\text{Set}}$ $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ in $\mathfrak{L}$ is called a $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$ if it satisfies the following assertions:

$$
\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta} * \varpi) \geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi), \tag{3.3}
$$

$$
\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta) \geq \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta * \varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi)\} \tag{3.4}
$$

for all $\varpi, \zeta \in \mathfrak{L}$.

Proposition 3.7. *For any* $\mathbb{MPF}^{CI}$ $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ *of* $\mathfrak{L}$*, the following assertion holds.*

$$(\forall \varpi \in \mathfrak{L})\,(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta}) \geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi). \tag{3.5}$$

Proof. Straightforward. □

Proposition 3.8. *For any* $\mathbb{MPF}^{Sub}$ $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ *of* $\mathfrak{L}$*, the following assertion holds.*

$$(\forall \varpi \in \mathfrak{L})\,(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta} * \varpi) \geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi). \tag{3.6}$$

Proof. Let $\varpi \in \mathfrak{L}$. Using (3.2), we have

$$\begin{aligned} \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta} * \varpi) &\geq \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta}), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi)\} = \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi)\} \\ &\geq \min\{\min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi)\}, \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi)\} = \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi). \end{aligned}$$

This completes the proof. □

Corollary 3.9. *Any* $\mathbb{MPF}^{Sub}$ *fulfilling* (3.4) *is an* $\mathbb{MPF}^{CI}$.

Proposition 3.10. *Assume that* $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ *is an* $\mathbb{MPF}^{CI}$ *of* $\mathfrak{L}$ *such that:*

$$(\forall \varpi \in \mathfrak{L})\,(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi) \geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta} * \varpi). \tag{3.7}$$

Then we have

$$(\forall \varpi, \zeta \in \mathfrak{L})\,(\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta * \varpi) \geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta). \tag{3.8}$$

Proof. Let $\varpi, \zeta \in \mathfrak{L}$. Then

$$\begin{aligned} \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\zeta * \varpi) &\geq \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta} * (\zeta * \varpi)) \geq \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}((\vec{\theta} * (\zeta * \varpi)) * (\varpi * \zeta)), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(((\vec{\theta} * \zeta) * (\vec{\theta} * \varpi)) * (\varpi * \zeta)), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(((\vec{\theta} * \zeta) * (\varpi * \zeta)) * (\vec{\theta} * \zeta)), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(((\vec{\theta} * (\varpi * \zeta)) * \zeta) * (\vec{\theta} * \varpi)), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}((((\vec{\theta} * \varpi) * (\vec{\theta} * \zeta)) * (\vec{\theta} * \varpi)) * \zeta), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}((\vec{\theta} * (\vec{\theta} * \zeta)) * \zeta), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} \\ &= \min\{\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\vec{\theta}), \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta)\} = \widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}(\varpi * \zeta), \end{aligned}$$

Hence, the proof is complete. □

Theorem 3.11. *Every* $\mathbb{MPF}^{CI}$ *is a* $\mathbb{MPF}^{Sub}$.

Proof. Let $\widehat{\mathcal{GP}}_{\circledast}$ be a $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. For every $\varpi, \zeta \in \mathfrak{L}$, we have

$$\begin{aligned}\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) &\geq \min\{\widehat{\mathcal{GP}}_{\circledast}((\varpi * \zeta) * \varpi), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\} = \min\{\widehat{\mathcal{GP}}_{\circledast}((\varpi * \varpi) * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\} \\ &= \min\{\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\} \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\zeta), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\}.\end{aligned}$$

Therefore, $\widehat{\mathcal{GP}}_{\circledast}$ forms an $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\widehat{\mathcal{GP}}_{\circledast}$. □

In general, the converse of Theorem 3.11 fails to be true, as shown in the example below.

Example 3.12. Let $\mathfrak{L} = \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}, \vec{\varrho_3}, \vec{\varrho_4}\}$ be a $\mathbb{BCH}$-algebra whose Cayley table is shown below

$*$	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\varrho_4}$
$\vec{\varrho_1}$	$\vec{\varrho_1}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_4}$
$\vec{\varrho_2}$	$\vec{\varrho_2}$	$\vec{\varrho_2}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\varrho_4}$
$\vec{\varrho_3}$	$\vec{\varrho_3}$	$\vec{\varrho_3}$	$\vec{\varrho_3}$	$\vec{\theta}$	$\vec{\varrho_4}$
$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\theta}$

Let $\widehat{\mathcal{GP}}_{\circledast}$ be a $\mathbb{MPF}^{\mathrm{Set}}$ in $\mathfrak{L}$ defined by

	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\widehat{\mathcal{GP}}_{\circledast}$	0.7	0.2	0.2	0.2	0.7

Then $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$, but it is not a m-polar fuzzy closed ideal of $\mathfrak{L}$ because

$$\widehat{\mathcal{GP}}_{\circledast}(\vec{\varrho_3}) = 0.2 < \min\{\widehat{\mathcal{GP}}_{\circledast}(\vec{\varrho_3} * \vec{\varrho_4}), \widehat{\mathcal{GP}}_{\circledast}(\vec{\varrho_4})\}.$$

Conditions are established for a m-polar fuzzy set to form an $\mathbb{MP}\mathfrak{F}^{CI}$.

Theorem 3.13. *Let $\widehat{\mathcal{GP}}_{\circledast}$ be a $\mathbb{MPF}^{\mathrm{Set}}$ in $\mathfrak{L}$ defined by*

$$\widehat{\mathcal{GP}}_{\circledast} : \mathfrak{L} \to [\vec{\theta}, 1], \quad \varpi \mapsto \begin{cases} \alpha_1 & \text{if } \vec{\theta} * \varpi = \vec{\theta}, \\ \alpha_2 & \text{otherwise}, \end{cases}$$

for every $\varpi \in \mathfrak{L}$ with $\alpha_1 > \alpha_2$ in $[\vec{\theta}, 1]$. Consequently, $\widehat{\mathcal{GP}}_{\circledast}$ is an $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$.

Proof. Let $\varpi \in \mathfrak{L}$ be such that $\vec{\theta} * \varpi \neq 0$. Then $\widehat{\mathcal{GP}}_{\circledast}(x) = \alpha_2 \leq \widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \varpi)$, $\mu^N_{\Phi}(x) = \beta_2 \geq \mu^N_{\Phi}(\vec{\theta} * \varpi)$, $\widehat{\mathcal{GP}}_{\circledast}(y) \geq \alpha_2 = \min\{\widehat{\mathcal{GP}}_{\circledast}(\zeta * \varpi), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\}$, for all $\zeta \in \mathfrak{L}$. Assume that $\varpi \in \mathfrak{L}$ satisfies the equality $\vec{\theta} * \varpi = \vec{\theta}$. Then $\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \varpi) = \widehat{\mathcal{GP}}_{\circledast}(\vec{\theta}) = \alpha_1 = \widehat{\mathcal{GP}}_{\circledast}(\varpi)$. Let $\zeta \in \mathfrak{L}$ satisfy the equality $\vec{\theta} * \zeta = \vec{\theta}$. Then $\widehat{\mathcal{GP}}_{\circledast}(\zeta) = \alpha_1 \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\zeta * \varpi), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\}$. If $\zeta \in \mathfrak{L}$ does not satisfy $\vec{\theta} * \zeta = \vec{\theta}$, then $\zeta * \varpi$ also does not satisfy the equality $\vec{\theta} * (\zeta * \varpi) = 0$. In fact, if $\vec{\theta} * (\zeta * \varpi) = \vec{\theta}$ then

$$\vec{\theta} * \zeta = ((\zeta * \varpi) * (\zeta * \varpi)) * \zeta = ((\zeta * \varpi) * \zeta) * (\zeta * \varpi)$$

$$\begin{aligned} &= ((\zeta * \zeta) * \varpi) * (\zeta * \varpi) = (\vec{\theta} * \varpi) * (\zeta * \varpi) \\ &= \vec{\theta} * (\zeta * \varpi) = 0, \end{aligned}$$

which is a contradiction. Hence we have $\widehat{\mathcal{GP}}_{\circledast}(\zeta) = \alpha_2 = \min\{\widehat{\mathcal{GP}}_{\circledast}(\zeta * \varpi), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\}$. Therefore $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. □

We provide a condition for a $\mathbb{MP}\mathfrak{F}^{Sub}$ to be a $\mathbb{MP}\mathfrak{F}^{CI}$.

Theorem 3.14. *Let $\widehat{\mathcal{GP}}_{\circledast}$ be a $\mathbb{MP}\mathfrak{F}^{Sub}$ of $\mathfrak{L}$ that satisfies the following assertions:*

$$(\forall \varpi, \zeta \in \mathfrak{L})\, (\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \widehat{\mathcal{GP}}_{\circledast}(\zeta * \varpi). \tag{3.9}$$

Consequently, $\widehat{\mathcal{GP}}_{\circledast}$ forms an $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$.

Proof. By Proposition 3.8, we have $\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \varpi) \geq \widehat{\mathcal{GP}}_{\circledast}(\varpi)$ for all $\varpi \in \mathfrak{L}$. Using (2.4), (2.1), (2.3), (3.2) and (3.9), we have

$$\begin{aligned} \widehat{\mathcal{GP}}_{\circledast}(\zeta) &= \widehat{\mathcal{GP}}_{\circledast}(\zeta * \vec{\theta}) \geq \widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \zeta) = \widehat{\mathcal{GP}}_{\circledast}((\varpi * \varpi) * \zeta) = \widehat{\mathcal{GP}}_{\circledast}((\varpi * \zeta) * \varpi) \\ &\geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\} \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\zeta * \varpi), \widehat{\mathcal{GP}}_{\circledast}(\varpi)\} \end{aligned}$$

for all $\varpi, \zeta \in \mathfrak{L}$. Hence $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. □

Theorem 3.15. *Let $\widehat{\mathcal{GP}}_{\circledast}$ be a $\mathbb{MPF}^{\mathrm{Set}}$ in $\mathfrak{L}$. Then $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$ if and only if it satisfies the following assertions:*

$$(\forall \alpha \in [\vec{\theta}, 1])(\widetilde{\mathcal{G}} \neq \emptyset \Rightarrow \widetilde{\mathfrak{P}} \text{ is a closed ideal of } \mathfrak{L}). \tag{3.10}$$

Proof. Assume that $\widehat{\mathcal{GP}}_{\circledast}$ is a $\mathbb{MP}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. Let $(\beta, \alpha) \in [-1, \vec{\theta}] \times [\vec{\theta}, 1]$ be such that $\widetilde{\mathcal{G}} \neq \emptyset$. Obviously, $\vec{\theta} \in \widetilde{\mathfrak{P}}$. If $\varpi \in \widetilde{\mathfrak{P}}$, then $\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \varpi) \geq \widehat{\mathcal{GP}}_{\circledast}(\varpi) \geq \alpha$ by (3.3). Hence $\vec{\theta} * \varpi \in \widetilde{\mathfrak{P}}$. Let $\varpi, \zeta \in \mathfrak{L}$ be such that $\varpi * \zeta \in \widetilde{\mathfrak{P}}$. Then $\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \alpha$, $\widehat{\mathcal{GP}}_{\circledast}(\zeta) \geq \alpha$. It follows from (3.4) that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi) \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \geq \alpha$$

which implies that $\varpi \in \widetilde{\mathfrak{P}}$. Hence, $\widetilde{\mathfrak{P}}$ forms a closed ideal of $\mathfrak{L}$.Conversely, suppose the condition (3.10) is valid. For any $\varpi \in \mathfrak{L}$, let $\widehat{\mathcal{GP}}_{\circledast}(\varpi) = \alpha$. Then $\varpi \in \widetilde{\mathfrak{P}}$, and so $\widetilde{\mathfrak{P}}$ is non-empty. Since $\widetilde{\mathfrak{P}}$ is closed ideals of $\mathfrak{L}$, $\vec{\theta} * \varpi \in \widetilde{\mathfrak{P}}$. Hence $\widehat{\mathcal{GP}}_{\circledast}(\vec{\theta} * \varpi) \geq \alpha = \widehat{\mathcal{GP}}_{\circledast}(\varpi)$. If there exist $\varpi', \zeta' \in \mathfrak{L}$ such that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi') < \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi' * \zeta'), \widehat{\mathcal{GP}}_{\circledast}(\zeta')\},$$

then by taking

$$\alpha_0 := \tfrac{1}{2}(\widehat{\mathcal{GP}}_{\circledast}(\varpi') + \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi' * \zeta'), \widehat{\mathcal{GP}}_{\circledast}(\zeta')\}),$$

we have

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi') < \alpha_0 < \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi' * \zeta'), \widehat{\mathcal{GP}}_{\circledast}(\zeta')\}.$$

Hence $\varpi' \notin \widetilde{\mathfrak{P}}[\alpha_0]$, $\varpi' * \zeta' \in \widetilde{\mathfrak{P}}[\alpha_0]$, $\zeta' \in \widetilde{\mathfrak{P}}[\alpha_0]$. This contradiction shows that $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. □

Corollary 3.16. *Let $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ be an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. Then, for any nonempty positive α-cut and nonempty negative β-cut of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$, their intersection forms a closed ideal of $\mathfrak{L}$ for all $(\beta, \alpha) \in [-1, \vec{\theta}] \times [\vec{\theta}, 1]$. In particular, the nonempty δ-cut of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a closed ideal of $\mathfrak{L}$ for all $\delta \in [\vec{\theta}, 1]$.*

The following example shows that there exists $(\beta, \alpha) \in [-1, \vec{\theta}] \times [\vec{\theta}, 1]$ such that if $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$, then the union of a nonempty positive α-cut and a nonempty negative β-cut of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is not a closed ideal of $\mathfrak{L}$ in general.

Example 3.17. Let $\mathfrak{L} = \vec{\theta}, 1, 2, \vec{\varrho_1}, \vec{\varrho_2}$ be a BCH-algebra whose Cayley table is shown below:

$*$	$\vec{\theta}$	1	2	$\vec{\varrho_1}$	$\vec{\varrho_2}$
$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_1}$
1	1	0	1	$\vec{\varrho_2}$	$\vec{\varrho_1}$
2	2	2	0	$\vec{\varrho_1}$	$\vec{\varrho_1}$
$\vec{\varrho_1}$	$\vec{\varrho_1}$	$\vec{\varrho_1}$	$\vec{\varrho_1}$	0	$\vec{\theta}$
$\vec{\varrho_2}$	$\vec{\varrho_2}$	$\vec{\varrho_1}$	$\vec{\varrho_2}$	1	$\vec{\theta}$

Let $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ be a $\mathbb{MPF}^{\text{Set}}$ in $\mathfrak{L}$ defined by

	$\vec{\theta}$	1	2	$\vec{\varrho_1}$	$\vec{\varrho_2}$
$\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$	0.9	0.8	0.7	0.4	0.4

Then

$$\widetilde{\mathfrak{P}} = \begin{cases} \emptyset & \text{if } 0.9 < \alpha \leq 1, \\ \{\vec{\theta}\} & \text{if } 0.8 < \alpha \leq 0.9, \\ \{0, 1\} & \text{if } 0.7 < \alpha \leq 0.8, \\ \{0, 1, 2\} & \text{if } 0.4 < \alpha \leq 0.7, \\ \mathfrak{L} & \text{if } 0 \leq \alpha \leq 0.4, \end{cases}$$

It follows from Theorem 3.15 that $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. But $\Phi^P_{0.7} \cup \Phi^N_{-0.5} = \{\vec{\theta}, 1, 2\} \cup \{\vec{\theta}, 2, \vec{\varrho_1}\} = \{\vec{\theta}, 1, 2, \vec{\varrho_1}\}$ is not a closed ideal of $\mathfrak{L}$ since $\vec{\varrho_2} * \vec{\varrho_1} = 1 \in \{\vec{\theta}, 1, 2, \vec{\varrho_1}\}$, but $\vec{\varrho_2} \notin \{\vec{\theta}, 1, 2, \vec{\varrho_1}\}$.

The example below demonstrates the existence of $\delta \in [\vec{\theta}, 1]$ for which, although $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ is an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$, the union of a nonempty positive δ-cut and a nonempty negative $(-\delta)$-cut of $\widehat{\mathcal{G}^{\mathcal{P}}}_{\circledast}$ does not form a closed ideal of $\mathfrak{L}$ in general.

Example 3.18. Consider a BCH-algebra $\mathfrak{L} = \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}, \vec{\varrho_3}, \vec{\varrho_4}\}$ with the following Cayley table:

$*$	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\vec{\theta}$	$\vec{\theta}$	$\vec{\theta}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\vec{\varrho_1}$	$\vec{\varrho_1}$	$\vec{\theta}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\vec{\varrho_2}$	$\vec{\varrho_2}$	$\vec{\varrho_2}$	$\vec{\theta}$	$\vec{\varrho_4}$	$\vec{\varrho_3}$
$\vec{\varrho_3}$	$\vec{\varrho_3}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$	$\vec{\theta}$	$\vec{\varrho_2}$
$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\varrho_4}$	$\vec{\varrho_3}$	$\vec{\varrho_2}$	$\vec{\theta}$

Let $\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$ be a $\mathbb{MPF}^{\mathrm{Set}}$ in $\mathfrak{L}$ defined by

	$\vec{\theta}$	$\vec{\varrho_1}$	$\vec{\varrho_2}$	$\vec{\varrho_3}$	$\vec{\varrho_4}$
$\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$	0.7	0.5	0.4	0.2	0.2

Then

$$\widetilde{\mathfrak{P}} = \begin{cases} \emptyset & \text{if } 0.7 < \alpha \leq 1, \\ \{\vec{\theta}\} & \text{if } 0.5 < \alpha \leq 0.7, \\ \{\vec{\theta}, \vec{\varrho_1}\} & \text{if } 0.4 < \alpha \leq 0.5, \\ \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}\} & \text{if } 0.2 < \alpha \leq 0.4, \\ \mathfrak{L} & \text{if } \vec{\theta} \leq \alpha \leq 0.2, \end{cases}$$

It follows from Theorem 3.15 that $\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. But $\Phi^P_{0.35} \cup \Phi^N_{-0.35} = \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}\} \cup \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_3}\} = \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}, \vec{\varrho_3}\}$ is not a closed ideal of $\mathfrak{L}$ since $\vec{\varrho_4} * \vec{\varrho_2} = \vec{\varrho_3} \in \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}, \vec{\varrho_3}\}$, but $d \notin \{\vec{\theta}, \vec{\varrho_1}, \vec{\varrho_2}, \vec{\varrho_3}\}$.

A condition is established ensuring that the union of a nonempty positive δ-cut and a nonempty negative $(-\delta)$-cut of $\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$ forms a closed ideal of $\mathfrak{L}$.

Theorem 3.19. *Suppose $\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$ is a $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$ satisfying*

$$(\forall \varpi \in \mathfrak{L})\, (\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}(\varpi)}), \tag{3.11}$$

then, for every $\delta \in [\vec{\theta}, 1]$, the union of a nonempty positive δ-cut of $\widehat{\mathcal{G}^{\mathcal{P}}_{\circledast}}$ forms a closed ideal of $\mathfrak{L}$.

Proof. Take $\delta \in [\vec{\theta}, 1]$. Since both $\widetilde{\Phi_{\circledast\delta}}^P$ and $\widetilde{\Phi_{\circledast-\delta}}^N$ are nonempty, Theorem 3.15 implies that they are closed ideals of $\mathfrak{L}$. Therefore, for every $\varpi \in \widetilde{\Phi_{\circledast\delta}}^P \cup \widetilde{\Phi_{\circledast-\delta}}^N$, we have $\vec{\theta} * \varpi \in \widetilde{\Phi_{\circledast\delta}}^P \cup \widetilde{\Phi_{\circledast-\delta}}^N$.

Now let $\varpi, \zeta \in \mathfrak{L}$ satisfy $\varpi * \zeta \in \widetilde{\Phi_{\circledast\delta}}^P \cup \widetilde{\Phi_{\circledast-\delta}}^N$ and $\zeta \in \widetilde{\Phi_{\circledast\delta}}^P \cup \widetilde{\Phi_{\circledast-\delta}}^N$. The discussion proceeds by examining the following four cases:

(i) $\varpi * \zeta \in \widetilde{\Phi_{\circledast\delta}}^P$ and $\zeta \in \widetilde{\Phi_{\circledast\delta}}^P$.
(ii) $\varpi * \zeta \in \widetilde{\Phi_{\circledast\delta}}^P$ and $\zeta \in \widetilde{\Phi_{\circledast-\delta}}^N$.
(iii) $\varpi * \zeta \in \widetilde{\Phi_{\circledast-\delta}}^N$ and $\zeta \in \widetilde{\Phi_{\circledast\delta}}^P$.

(iv) $\varpi * \zeta \in \widetilde{\Phi}^{N}_{\circledast -\delta}$ and $\zeta \in \widetilde{\Phi}^{N}_{\circledast -\delta}$.

From Case (i), it follows that $\varpi \in \widetilde{\Phi}^{P}_{\circledast \delta} \subseteq \widetilde{\Phi}^{P}_{\circledast \delta} \cup \widetilde{\Phi}^{N}_{\circledast -\delta}$. From Case (iv), it follows that $\varpi \in \widetilde{\Phi}^{N}_{\circledast -\delta} \subseteq \widetilde{\Phi}^{P}_{\circledast \delta} \cup \widetilde{\Phi}^{N}_{\circledast -\delta}$. For the case (ii), we have $\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta) \geq \delta$ and $\mu^{N}_{\Phi}(\zeta) \leq -\delta$. It follows from (3.4) and (3.11) that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi) \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta), -\mu^{N}_{\Phi}(\zeta)\} \geq \delta$$

so that $\varpi \in \widetilde{\Phi}^{P}_{\circledast \delta} \subseteq \widetilde{\Phi}^{P}_{\circledast \delta} \cup \widetilde{\Phi}^{N}_{\circledast -\delta}$. From Case (iii), it follows that $\mu^{N}_{\Phi}(\varpi * \zeta) \leq -\delta$ and $\widehat{\mathcal{GP}}_{\circledast}(\zeta) \geq \delta$. It follows from (3.4) and (3.11) that

$$\widehat{\mathcal{GP}}_{\circledast}(\varpi) \geq \min\{\widehat{\mathcal{GP}}_{\circledast}(\varpi * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \geq \min\{-\mu^{N}_{\Phi}(\varpi * \zeta), \widehat{\mathcal{GP}}_{\circledast}(\zeta)\} \geq \delta$$

so that $\varpi \in \widetilde{\Phi}^{P}_{\circledast \delta} \subseteq \widetilde{\Phi}^{P}_{\circledast \delta} \cup \widetilde{\Phi}^{N}_{\circledast -\delta}$. Hence $\widetilde{\Phi}^{P}_{\circledast \delta} \cup \widetilde{\Phi}^{N}_{\circledast -\delta}$ is a closed ideal of $\mathfrak{L}$. □

Theorem 3.20. *Assume that $\mathfrak{P}$ is a nonempty subset of $\mathfrak{L}$, and let $\widehat{\mathcal{GP}}_{\circledast}$ denote an $\mathbb{MPF}^{\mathrm{Set}}$ in $\mathfrak{L}$ given by*

$$\widehat{\mathcal{GP}}_{\circledast} : \mathfrak{L} \to [\vec{\theta}, 1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } \varpi \in \mathfrak{P} \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\mu^{N}_{\Phi} : X \to [-1, \vec{\theta}], \quad x \mapsto \begin{cases} \beta_1 & \text{if } \varpi \in \mathfrak{P}, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $\varpi \in \mathfrak{L}$ where $\alpha_1 > \alpha_2$ in $[\vec{\theta}, 1]$ and $\beta_1 < \beta_2$ in $[-1, \vec{\theta}]$. Then $\widehat{\mathcal{GP}}_{\circledast}$ is an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$ precisely when $\mathfrak{P}$ is a closed ideal of $\mathfrak{L}$. Moreover, in this situation, $\widetilde{\Phi}^{P}_{\circledast \alpha_1} = \mathfrak{P} = \widetilde{\Phi}^{N}_{\circledast \beta_1}$.

Proof. Suppose $\mathfrak{P}$ is a closed ideal of $\mathfrak{L}$. Obviously, $\widetilde{\Phi}^{P}_{\circledast \alpha_1} = \mathfrak{P} = \widetilde{\Phi}^{N}_{\circledast \beta_1}$ and $\widetilde{\Phi}^{P}_{\circledast \alpha_2} = \mathfrak{L} = \widetilde{\Phi}^{N}_{\circledast \beta_2}$, both of which are closed ideals of $\mathfrak{L}$. By applying Theorem 3.15, we conclude that $\widehat{\mathcal{GP}}_{\circledast}$ is an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. Conversely, suppose that $\widehat{\mathcal{GP}}_{\circledast}$ is an $\mathbb{M}\mathcal{P}\mathfrak{F}^{CI}$ of $\mathfrak{L}$. Then it is evident that $\mathfrak{P}$ is a closed ideal of $\mathfrak{L}$. □

4. Conclusion

This paper has presented a unified framework for the study of m-polar fuzzy subalgebras and m-polar fuzzy closed ideals in BCH-algebras. The introduced concepts and their fundamental properties clarify the interplay between multi-polar fuzzy structures and BCH-algebraic operations. By employing the technique of level subsets, exact characterizations of m-polar fuzzy subalgebras and m-polar fuzzy closed ideals have been obtained.

Necessary and sufficient conditions have been established under which an m-polar fuzzy subalgebra becomes an m-polar fuzzy closed ideal, thereby linking fuzzy extensions with their classical counterparts. The obtained results generalize and unify several existing notions of fuzzy, bipolar fuzzy, interval-valued fuzzy, cubic fuzzy, and m-polar fuzzy ideals previously investigated in BCK- and BCI-algebras, and naturally extend them to the broader setting of BCH-algebras.

Future Work

The present study opens several directions for further research. One possible extension is the investigation of homomorphisms and quotient structures associated with m-polar fuzzy subalgebras and m-polar fuzzy closed ideals in BCH-algebras. Another promising direction involves the introduction and analysis of other generalized fuzzy ideal concepts, such as intuitionistic or hesitant m-polar fuzzy ideals, within the same algebraic framework. Moreover, exploring applications of m-polar fuzzy algebraic structures in logical reasoning, information processing, and multi-criteria decision-making under uncertainty remains an interesting topic for future work.

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