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EXTENDED REAL OPERATIONS FOR GENERALIZED TRIANGULAR FUZZY NUMBERS IN FOUR-DIMENSIONAL SPACE

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Abstract. The four-dimensional extension of the extended real number operator requires a novel approach that explicitly incorporates a time-like axis. Determining the appropriate range of the additional variables is therefore of paramount importance. The extended real number operator for four-dimensional triangular fuzzy numbers has been computed, and the resulting structures are visualized by introducing a time axis. We define and compute the extended real number operator for generalized triangular fuzzy numbers in four-dimensional space.

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1. Introduction

Triangular fuzzy numbers are among the most fundamental models in fuzzy theory and have therefore attracted considerable attention in the literature. With the increasing complexity of real-world phenomena, the focus of research has expanded from classical triangular fuzzy numbers to their generalized counterparts. Generalized triangular fuzzy numbers have been studied in diverse contexts, including fuzzy environments [1], Bayesian water quality evaluation [2], medical decision making [3], fuzzy averaging [4], and aggregation operators [5], and have been widely applied in various practical fields [6,7].

Zadeh operators have been employed in various areas of research, including dynamical systems [8], fuzzy controllers [9], and fuzzy operators [10]. In particular, studies on extended real number operations based on Zadeh's extension principle have been actively pursued. Studies of one-dimensional generalized triangular fuzzy numbers [11] have been cited in applications such as chain models [12] and linear systems [13]. Extended real operators for two-dimensional triangular fuzzy numbers have

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also been reported [14], and studies of parameter operators for generalized triangular fuzzy numbers in two dimensions have also been published [15].

A two-dimensional triangular fuzzy number can be visualized as a graph in three-dimensional space. However, since a three-dimensional triangular fuzzy number has a three-dimensional domain, its graphical representation requires a four-dimensional space. To address this challenge, the computation and graphical representation of extended real number operations for three-dimensional triangular fuzzy numbers have been realized within a three-dimensional framework by encoding membership function values through color intensity [16]. This visualization approach has subsequently been cited and applied in fuzzy control strategies [17].

The four-dimensional extension of the extended real number operator requires a novel approach that explicitly incorporates a time-like axis. Determining the appropriate range of the additional variables is therefore of paramount importance. The extended real number operator for four-dimensional triangular fuzzy numbers has been computed, and the resulting structures are visualized by introducing a time axis [18]. In this study, we define and compute the extended real number operator for generalized triangular fuzzy numbers in four-dimensional space. The results obtained for generalized triangular fuzzy numbers in three dimensions [19] serve as a fundamental basis for the proposed four-dimensional extension.

2. 2-Dimensional Generalized Triangular Fuzzy Number

Let A be a fuzzy set on $\mathbb{R}$ with membership function $\mu_A(x)$. We define the α-cut and the α-set of A as follows.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \text{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$, where $\text{cl}(B)$ is the closure of $B \subset \mathbb{R}$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

We define generalized two-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}$, together with parametric operations between pairs of generalized two-dimensional triangular fuzzy sets. To this end, it is necessary to compute operations between α-cuts. In $\mathbb{R}$, α-cuts are intervals, whereas in $\mathbb{R}^2$ they are regions, which renders the conventional methods for computing operations between α-cuts inapplicable. We therefore reinterpret the classical approach from a different perspective and extend it to region-valued α-cuts in $\mathbb{R}^2$.

Definition 2.2. [15] A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2\, b^2\, h^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $0 < h < 1$ is called *the generalized two dimensional triangular fuzzy set* and denoted by $((a,\ x_1,\ h,\ b,\ y_1))^2$.

The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \quad (k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a generalized two-dimensional triangular fuzzy number $A = (a, x_1, h, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$A_\alpha = \left\{(x,y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1}{a(h-\alpha)}\right)^2 + \left(\frac{y - y_1}{b(h-\alpha)}\right)^2 \le 1\right\}.$$

Definition 2.3. A two dimensional fuzzy number A defined on $\mathbb{R}^2$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^2$.

Theorem 2.4. [20] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

Definition 2.5. [15] Let A and B be convex fuzzy sets defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric addition, parametric subtraction, parametric multiplication, and parametric division are fuzzy sets whose α-sets are given as follows:

(1) parametric addition $A(+)_p B$:

$$(A(+)_p B)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(2) parametric subtraction $A(-)_p B$:

$$(A(-)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi. \end{cases}$$

(3) parametric multiplication $A(\cdot)_p B$:

$$(A(\cdot)_p B)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(4) parametric division $A(/)_p B$:

$$(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi),$$
$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \to 0^+} (A(*)_p B)^\alpha$ and $(A(*)_p B)^1 = \lim_{\alpha \to 1^-} (A(*)_p B)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 2.6. [15] *Let $A = ((a_1,\ x_1,\ h_1,\ b_1,\ y_1))^2$ and $B = ((a_2,\ x_2,\ h_2, b_2, y_2))^2$ be two generalized two dimensional triangular fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the following:*

(1) *For $0 < \alpha < h_1$, the α-set of $A(+)_p B$ is*

$$(A(+)_p B)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 - x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\right)^2 = 1\right\}.$$

(2) *For $0 < \alpha < h_1$, the α-set of $A(-)_p B$ is*

$$(A(-)_p B)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\right)^2 = 1\right\}.$$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$*, where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos t + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t,\ 0 < \alpha < h_1,$$

$$y_\alpha(t) = y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t,\ 0 < \alpha < h_1.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$*, where*

$$x_\alpha(t) = \frac{x_1 + a_1(h_1-\alpha)\cos t}{x_2 - a_2(h_2-\alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(h_1-\alpha)\sin t}{y_2 - b_2(h_2-\alpha)\sin t}, \quad 0 < \alpha < h_1.$$

Furthermore, we have

$$(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$$

and

$$(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we get

$$(A(*)_pB)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /$$

Example 2.7. [15] Let $A = ((6,\ 3,\ \frac{1}{2},\ 8,\ 5))^2$ and $B = ((4,\ 2,\ \frac{2}{3},\ 5, 3))^2$. Then by Theorem 2.6, we have the following:

(1) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2\Big|\Big(\frac{3x-15}{17-30\alpha}\Big)^2 + \Big(\frac{3y-24}{22-39\alpha}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2\Big|\Big(\frac{3x-3}{17-30\alpha}\Big)^2 + \Big(\frac{3y-6}{22-39\alpha}\Big)^2 = 1\Big\}.$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + (14-24\alpha)\cos t + 4(1-2\alpha)(2-3\alpha)\cos^2 t, \quad 0<\alpha<\frac{1}{2},$$
$$y_\alpha(t) = 15 + (\frac{86}{3} - 49\alpha)\sin t + 20(1-2\alpha)(\frac{2}{3}-\alpha)\sin^2 t, \quad 0<\alpha<\frac{1}{2}.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{9+9(1-2\alpha)\cos t}{6-4(2-3\alpha)\cos t}, \quad y_\alpha(t) = \frac{15+12(1-2\alpha)\sin t}{9-15(2-3\alpha)\sin t}, \quad 0<\alpha<\frac{1}{2}.$$

3. 3-Dimensional Generalized Triangular Fuzzy Number

We define generalized three-dimensional triangular fuzzy sets on $\mathbb{R}^3$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}^2$. We then define parametric operations between two generalized three-dimensional triangular fuzzy sets. To this end, we must compute operations between α-sets in $\mathbb{R}^3$. While the α-sets in $\mathbb{R}^2$ are regions, the α-sets in $\mathbb{R}^3$ are ellipsoids including their interiors, which renders the existing methods for computing operations between α-sets inapplicable. Therefore, we reinterpret the existing method from a different perspective and extend it to α-sets in $\mathbb{R}^3$ that take the form of solid ellipsoids.

Definition 3.1. [19] A fuzzy set A with a membership function $\mu_A(x,y,z)$ such that

$$\begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}}, & \text{if } b^2c^2(x-x_1)^2 + c^2a^2(y-y_1)^2 + a^2b^2(z-z_1)^2 \le a^2b^2c^2h^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b, c > 0$ and $0 < h < 1$ is called *the generalized three dimensional triangular fuzzy set* and denoted by $((h,\ a,\ x_1,\ b,\ y_1,\ c,\ z_1))^3$.

Note that $\mu_A(x,y)$ is a cone in $\mathbb{R}^2$, but we can not know the shape of $\mu_A(x,y,z)$ in $\mathbb{R}^3$. The α-cut A_α of a generalized three dimensional triangular fuzzy number $A = ((h,\ a, x_1, b, y_1, c, z_1))^3$ is the following set

$$A_\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x-x_1}{a(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1}{b(h-\alpha)}\Big)^2 + \Big(\frac{z-z_1}{c(h-\alpha)}\Big)^2 \le 1\Big\}$$

Definition 3.2. A three dimensional fuzzy number A defined on $\mathbb{R}^3$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 3.3. [19] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s)$, $f_2^\alpha(s,t)$, and $f_3^\alpha(s,t)(0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2})$ such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}.$$

Definition 3.4. [19] Let A and B are two continuous convex fuzzy sets defined on $\mathbb{R}^3$ and

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$
$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy sets A and B are fuzzy numbers whose α-sets are given as follows:

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$
$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s)\cdot g_1^\alpha(s),\ \ f_2^\alpha(s,t)\cdot g_2^\alpha(s,t),\ \ f_3^\alpha(s,t)\cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$
$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 3.5. [19] *Let $A = ((h_1, a_1, x_1, b_1, y_1, c_1, z_1))^3$ and $B = ((h_2, a_2, x_2, b_2, y_2, c_2, z_2))^3$ be two generalized three dimensional triangular fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the following:*

(1) *For $0 < \alpha < h_1$, the α-set of $A(+)_pB$ is*

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 - x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 \\ + \Big(\frac{y - y_1 - y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(2) *For $0 < \alpha < h_1$, the α-set of $A(-)_pB$ is*

$$(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 \\ + \Big(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(3) *For $0 < \alpha < h_1$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where*

$$x_\alpha(s) = x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos s + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 s,$$
$$y_\alpha(s,t) = y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin s\cos t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\cos^2 t,$$
$$z_\alpha(s,t) = z_1z_2 + (z_1c_2(h_2-\alpha) + z_2c_1(h_1-\alpha))\sin s\sin t + c_1c_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 t.$$

(4) *For $0 < \alpha < h_1$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$, where*

$$x_\alpha(s) = \frac{x_1 + a_1(h_1-\alpha)\cos s}{x_2 - a_2(h_2-\alpha)\cos s}, \quad y_\alpha(s.t) = \frac{y_1 + b_1(h_1-\alpha)\sin s\cos t}{y_2 - b_2(h_2-\alpha)\sin s\cos t},$$
$$z_\alpha(s.t) = \frac{z_1 + c_1(h_1-\alpha)\sin s\sin t}{z_2 - c_2(h_2-\alpha)\sin s\sin t}.$$

Furthermore, we have

$$(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /,$$

$$(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /.$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we get

$$(A(*)_p B)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /$$

4. 4-Dimensional Generalized Triangular Fuzzy Number

We define generalized four-dimensional triangular fuzzy sets on $\mathbb{R}^4$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}^3$. We then define parametric operations between two generalized four-dimensional triangular fuzzy sets. To this end, we must compute operations between α-sets in $\mathbb{R}^4$. While the α-sets in $\mathbb{R}^3$ are solid ellipsoids, the α-sets in $\mathbb{R}^4$ are four-dimensional convex subsets containing their interiors. Consequently, existing methods for computing operations between α-sets cannot be directly applied. Therefore, we reinterpret the existing approach from a different perspective and extend it to operations between α-sets in $\mathbb{R}^4$.

Definition 4.1. A fuzzy set A with a membership function $\mu_A(x,y,z,t) =$

$$\begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} + \frac{(t-t_1)^2}{d^2}}, & \text{if } b^2c^2d^2(x-x_1)^2 \\ \qquad +c^2d^2a^2(y-y_1)^2 + d^2a^2b^2(z-z_1)^2 + a^2b^2c^2(t-t_1)^2 \le a^2b^2c^2d^2h^2, & \\ 0, \quad \text{otherwise}, & \end{cases}$$

where $a, b, c, d > 0$, and $0 < h < 1$ is called the *generalized 4-dimensional triangular fuzzy number* and denoted by $((h,\ a,\ x_1,\ b,\ y_1,\ c,\ z_1,\ d,\ t_1))^4$.

The α-cut A_α of a generalized 4-dimensional triangular fuzzy number $A = ((h, a, x_1, b, y_1, c, z_1, d, t_1))^4$ is the following set

$$A_\alpha = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1}{a(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1}{b(h-\alpha)}\Big)^2 + \Big(\frac{z-z_1}{c(h-\alpha)}\Big)^2 + \Big(\frac{t-t_1}{d(h-\alpha)}\Big)^2 \le 1\Big\}.$$

Definition 4.2. A 4-dimensional fuzzy number A defined on $\mathbb{R}^4$ is called a *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^4$.

Theorem 4.3. [18] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^4$ and $A^\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)$, and $f_4^\alpha(s,p,q)$ $(0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2})$, such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q),\ f_4^\alpha(s,p,q)) \in \mathbb{R}^4 |\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

Proof. Let $\alpha \in (0,1)$ be fixed. Since A is a convex fuzzy number defined on $\mathbb{R}^4$, the α-cut A_α is a convex subset in $\mathbb{R}^4$. Therefore, the set

$$A^\alpha = \{(x,y,z,t) \in \mathbb{R}^4 | \mu_A(x,y,z,t) = \alpha\}$$

is a subset in $\mathbb{R}^4$. Let

$$A_3^\alpha = \{(x,y,z) \in \mathbb{R}^3 | (x,y,z,t) \in A_\alpha\}, \quad A_2^\alpha = \{(x,y) \in \mathbb{R}^2 | (x,y,z,t) \in A_\alpha\}$$

and $\bar{A_3^\alpha}, \bar{A_2^\alpha}, \bar{A^\alpha}$ are the boundaries of $A_3^\alpha, A_2^\alpha, A^\alpha$, respectively. The upper surface of A^α is the graph of a continuous concave function $k_2(x,y,z)$, and the lower surface of A^α is also the graph of a continuous convex function $k_1(x,y,z)$ defined on A_3^α. And the upper surface of A_3^α is the graph of a continuous concave function $h_2(x,y)$, and the lower surface of A_3^α is also the graph of a continuous convex function $h_1(x,y)$ defined on A_2^α. Let

$$l = \inf\{x | (x,y) \in \bar{A_2^\alpha}\} \quad \text{and} \quad m = \sup\{x | (x,y) \in \bar{A_2^\alpha}\}.$$

The upper boundary of $\bar{A_2^\alpha}$ is the graph of some continuous concave function $g_2(x)$ defined on $[l,m]$, and the lower boundary of $\bar{A_2^\alpha}$ is also the graph of some continuous convex function $g_1(x)$ defined on $[l,m]$ (see [13]). Define

$$f_1^\alpha(s) = \frac{1}{2}(l-m)(\cos s - 1) + l, \quad \text{if} \quad s \in [0,\pi].$$

Then, $f_1^\alpha(s)$ moves from l to m if $0 \le s \le \pi$. Define

$$f_2^\alpha(s,p) = \frac{1}{2}(g_2(f_1^\alpha(s)) - g_1(f_1^\alpha(s)))(\sin p - 1) + g_2(f_1^\alpha(s)), 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}.$$

Then, $f_2^\alpha(s,p)$ moves from $g_1(f_1^\alpha(s))$ to $g_2(f_1^\alpha(s))$ if $0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}$. We then have

$$A_2^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p)) \in \mathbb{R}^2 | 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}\}.$$

If we define

$$\begin{aligned} f_3^\alpha(s,p,q) = {} & \frac{1}{2}(h_2(f_1^\alpha(s), f_2^\alpha(s,p)) - h_1(f_1^\alpha(s), f_2^\alpha(s,p)))(\sin q - 1) \\ & + h_2(f_1^\alpha(s), f_2^\alpha(s,p)), 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \end{aligned}$$

then $f_3^\alpha(s,p,q)$ moves from $h_1(f_1^\alpha(s), f_2^\alpha(s,p))$ to $h_2(f_1^\alpha(s), f_2^\alpha(s,p))$ if $0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}$. We then have

$$A_3^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

If we define

$$f_4^\alpha(s,p,q) = \begin{cases} k_1(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)), & 0 \le s \le \pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \\ k_2(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q)), & \pi \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}, \end{cases}$$

then we have

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q), f_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

The proof is now complete. □

Remark 4.4. We proved that Theorem 4.3 is satisfied in the case that A is a continuous convex fuzzy number. If A is a piecewise continuous convex fuzzy number, we can prove similarly.

Definition 4.5. Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^4$, and

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q), f_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$
$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,p), g_3^\alpha(s,p,q), g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B are fuzzy numbers whose α-sets are given as follows:

(1) parametric addition $A(+)_pB$: $(A(+)_pB)^\alpha =$

$$\{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,p) + g_2^\alpha(s,p), f_3^\alpha(s,p,q) + g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) + g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_pB$: $(A(-)_pB)^\alpha =$

$$\{(f_1^\alpha(s) - g_1^\alpha(s),\ f_2^\alpha(s,p) - g_2^\alpha(s,p), f_3^\alpha(s,p,q) - g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) - g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(3) parametric multiplication $A(\cdot)_pB$: $(A(\cdot)_pB)^\alpha =$

$$\{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ f_2^\alpha(s,p) \cdot g_2^\alpha(s,p), f_3^\alpha(s,p,q) \cdot g_3^\alpha(s,p,q), f_4^\alpha(s,p,q) \cdot g_4^\alpha(s,p,q)) \in \mathbb{R}^4 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$$

(4) parametric division $A(/)_pB$: $(A(/)_pB)^\alpha =$

$$\left\{\Big(\frac{f_1^\alpha(s)}{g_1^\alpha(s)}, \frac{f_2^\alpha(s,p)}{g_2^\alpha(s,p)}, \frac{f_3^\alpha(s,p,q)}{g_3^\alpha(s,p,q)}, \frac{f_4^\alpha(s,p,q)}{g_3^\alpha(s,p,q)}\Big) \in \mathbb{R}^4 \Big| \right. \left. 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\right\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 4.6. *Let* $A = ((h_1,\ a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1,\ d_1,\ t_1))^4$ *and* $B = ((h_2\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2,\ d_2,\ t_2))^4$ *be two 4-dimensional generalized triangular fuzzy numbers. If* $0 < h_1 < h_2 < 1$*, then we have the following:*

(1) *For* $0 < \alpha < h_1$*, the* α*-set of* $A(+)_pB$ *is*

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x - x_1 - x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 + \Big(\frac{t - t_1 - t_2}{d_1(h_1-\alpha) + d_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(2) *For* $0 < \alpha < h_1$*, the* α*-set of* $A(-)_pB$ *is*

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 + \Big(\frac{t - t_1 + t_2}{d_1(h_1-\alpha) + d_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

(3) *For* $0 < \alpha < h_1$*,* $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$*, where*

$$\begin{aligned} x_\alpha(s) &= x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos s \\ &\quad + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 s, \\ y_\alpha(s,p) &= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin s\cos p \\ &\quad + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\cos^2 p, \\ z_\alpha(s,p,q) &= z_1z_2 + (z_1c_2(h_2-\alpha) + z_2c_1(h_1-\alpha))\sin s\sin p\cos q \\ &\quad + c_1c_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 p\cos^2 q, \\ t_\alpha(s,p,q) &= t_1t_2 + (t_1d_2(h_2-\alpha) + t_2d_1(h_1-\alpha))\sin s\sin p\sin q \\ &\quad + d_1d_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 p\sin^2 q. \end{aligned}$$

(4) *For* $0 < \alpha < h_1$*,* $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$*, where*

$$x_\alpha(s) = \frac{x_1 + a_1(h_1-\alpha)\cos s}{x_2 - a_2(h_2-\alpha)\cos s}, \quad y_\alpha(s.p) = \frac{y_1 + b_1(h_1-\alpha)\sin s\cos p}{y_2 - b_2(h_2-\alpha)\sin s\cos p},$$

$$z_\alpha(s.p,q) = \frac{z_1 + c_1(h_1-\alpha)\sin s\sin p\cos q}{z_2 - c_2(h_2-\alpha)\sin s\sin p\cos q}, \quad t_\alpha(s.p,q) = \frac{t_1 + d_1(h_1-\alpha)\sin s\sin p\sin q}{t_2 - d_2(h_2-\alpha)\sin s\sin p\sin q},$$

Furthermore, we have

$$(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$$

and

$$(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we get

$$(A(*)_pB)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /$$

Proof. Since A and B are continuous convex fuzzy sets defined on $\mathbb{R}^4$, by Theorem 4.3, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,p), g_1^\alpha(s), g_2^\alpha(s,p),\ f_i^\alpha(s,p,q)$, and $g_i^\alpha(s,p,q)$ $(i=3,4)(0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2})$, such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,p), f_3^\alpha(s,p,q),\ f_4^\alpha(s,p,q)) \in \mathbb{R}^4 |\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\},$$
$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,p), g_3^\alpha(s,p,q),\ g_4^\alpha(s,p,q)) \in \mathbb{R}^4 |\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}.$$

Since $A = (h_1,\ a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1,\ d_1,\ t_1)^4$ and $B = (h_2\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2,\ d_2,\ t_2)^4$, we have for $0 \le \alpha \le h_1$,

$$f_1^\alpha(s) = x_1 + a_1(h_1-\alpha)\cos s, \quad f_2^\alpha(s,p) = y_1 + b_1(h_1-\alpha)\sin s\cos p,$$
$$f_3^\alpha(s,p,q) = z_1 + c_1(h_1-\alpha)\sin s\sin p\cos q,$$
$$f_4^\alpha(s,p,q) = t_1 + d_1(h_1-\alpha)\sin s\sin p\sin q,$$

and for $0 \le \alpha \le h_2$,

$$g_1^\alpha(s) = x_2 + a_2(h_2-\alpha)\cos s, \quad g_2^\alpha(s,p) = y_2 + b_2(h_2-\alpha)\sin s\cos p,$$
$$g_3^\alpha(s,p,q) = z_2 + c_2(h_2-\alpha)\sin s\sin p\cos q$$
$$g_4^\alpha(s,p,q) = t_2 + d_2(h_2-\alpha)\sin s\sin p\sin q$$

(1) If $0 < \alpha < h_1$, since

$$f_1^\alpha(s) + g_1^\alpha(s) = x_1 + x_2 + (a_1(h_1-\alpha) + a_2(h_2-\alpha))\cos s,$$
$$f_2^\alpha(s,p) + g_2^\alpha(s,p) = y_1 + y_2 + (b_1(h_1-\alpha) + b_2(h_2-\alpha))\sin s\cos p,$$
$$f_3^\alpha(s,p,q) + g_3^\alpha(s,p,q) = z_1 + z_2 + (c_1(h_1-\alpha) + c_2(h_2-\alpha))\sin s\sin p\cos q,$$
$$f_4^\alpha(s,p,q) + g_4^\alpha(s,p,q) = t_1 + t_2 + (d_1(h_1-\alpha) + d_2(h_2-\alpha))\sin s\sin p\sin q,$$

we have the α-set of $A(+)_pB$ is

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1-x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z-z_1-z_2}{c_1(h_1-\alpha)+c_2(h_2-\alpha)}\Big)^2 + \Big(\frac{t-t_1-t_2}{d_1(h_1-\alpha)+d_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have the 0-set and h_1-set of $A(+)_pB$ are

$$(A(+)_pB)^0 = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1-x_2}{a_1h_1+a_2h_2}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_1h_1+b_2h_2}\Big)^2 + \Big(\frac{z-z_1-z_2}{c_1h_1+c_2h_2}\Big)^2 + \Big(\frac{t-t_1-t_2}{d_1h_1+d_2h_2}\Big)^2 = 1\Big\}.$$

and

$$(A(+)_pB)^{h_1} = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1-x_2}{a_2(h_2-h_1)}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_2(h_2-h_1)}\Big)^2 + \Big(\frac{z-z_1-z_2}{c_2(h_2-h_1)}\Big)^2 + \Big(\frac{t-t_1-t_2}{d_2(h_2-h_1)}\Big)^2 = 1\Big\}.$$

respectively.

For $h_1 < \alpha \le h_2$, we have $(A(+)_pB)^\alpha = \emptyset$.

(2) If $0 < \alpha < h_1$, since

$$f_1^\alpha(s) - g_1^\alpha(s+\pi) = x_1 - x_2 + (a_1(h_1-\alpha) + a_2(h_2-\alpha))\cos s$$

$$f_2^\alpha(s,p) - g_2^\alpha(s+\pi,p) = y_1 - y_2 + (b_1(h_1-\alpha) + b_2(h_2-\alpha))\sin s\cos p,$$

$$f_3^\alpha(s,p,q) - g_3^\alpha(s+\pi,p,q) = z_1 - z_2 + (c_1(h_1-\alpha) + c_2(h_2-\alpha))\sin s\sin p\cos q,$$

$$f_4^\alpha(s,p,q) - g_4^\alpha(s+\pi,p,q) = t_1 - t_2 + (d_1(h_1-\alpha) + d_2(h_2-\alpha))\sin s\sin p\sin q,$$

we have the α-set of $A(-)_pB$ is

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1+x_2}{a_1(h_1-\alpha)+a_2(h_2-\alpha)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1(h_1-\alpha)+b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z-z_1+z_2}{c_1(h_1-\alpha)+c_2(h_2-\alpha)}\Big)^2 + \Big(\frac{t-t_1+t_2}{d_1(h_1-\alpha)+d_2(h_2-\alpha)}\Big)^2 = 1\Big\}.$$

Furthermore, we have the 0-set and h_1-set of $A(-)_pB$ are

$$(A(-)_pB)^0 = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1+x_2}{a_1h_1+a_2h_2}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_1h_1+b_2h_2}\Big)^2 + \Big(\frac{z-z_1+z_2}{c_1h_1+c_2h_2}\Big)^2 + \Big(\frac{t-t_1+t_2}{d_1h_1+d_2h_2}\Big)^2 = 1\Big\}.$$

and

$$(A(-)_pB)^{h_1} = \Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{x-x_1+x_2}{a_2(h_2-h_1)}\Big)^2 + \Big(\frac{y-y_1+y_2}{b_2(h_2-h_1)}\Big)^2 + \Big(\frac{z-z_1+z_2}{c_2(h_2-h_1)}\Big)^2 + \Big(\frac{t-t_1+t_2}{d_2(h_2-h_1)}\Big)^2 = 1\Big\}.$$

respectively.

For $h_1 < \alpha \le h_2$, we have $(A(-)_pB)^\alpha = \emptyset$.

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$. From the definitions of $f_1^\alpha(s)$, $f_2^\alpha(s,p)$, $g_1^\alpha(s)$, $g_2^\alpha(s,p)$, $f_i^\alpha(s,p,q)$, and $g_i^\alpha(s,p,q)$ $(i=3,4)(0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2})$, we have for $0 < \alpha < h_1$,

$$
\begin{aligned}
x_\alpha(s) &= x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos s + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 s,\\
y_\alpha(s,p) &= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin s\cos p + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\cos^2 p,\\
z_\alpha(s,p,q) &= z_1z_2 + (z_1c_2(h_2-\alpha) + z_2c_1(h_1-\alpha))\sin s\sin p\cos q\\
&\quad + c_1c_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 p\cos^2 q,\\
t_\alpha(s,p,q) &= t_1t_2 + (t_1d_2(h_2-\alpha) + t_2d_1(h_1-\alpha))\sin s\sin p\sin q\\
&\quad + d_1d_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\sin^2 p\sin^2 q.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
x_0(s) &= x_1x_2 + (x_1a_2h_2 + x_2a_1h_1)\cos s + a_1a_2h_1h_2\cos^2 s,\\
x_{h_1}(s) &= x_1x_2 + x_1a_2(h_2-h_1)\cos s,\\
y_0(s,p) &= y_1y_2 + (y_1b_2h_2 + y_2b_1h_1)\sin s\cos p + b_1b_2h_1h_2\sin^2 s\cos^2 p,\\
y_{h_1}(s,p) &= y_1y_2 + y_1b_2(h_2-h_1)\sin s\cos p,\\
z_0(s,p,q) &= z_1z_2 + (z_1c_2h_2 + z_2c_1h_1)\sin s\sin p\cos q + c_1c_2h_1h_2\sin^2 s\sin^2 p\cos^2 q,\\
z_{h_1}(s,p,q) &= z_1z_2 + z_1c_2(h_2-h_1)\sin s\sin p\cos q,\\
t_0(s,p,q) &= t_1t_2 + (t_1d_2h_2 + t_2d_1h_1)\sin s\sin p\sin q + d_1d_2h_1h_2\sin^2 s\sin^2 p\sin^2 q,\\
t_{h_1}(s,p,q) &= t_1t_2 + t_1d_2(h_2-h_1)\sin s\sin p\sin q,
\end{aligned}
$$

and

$$(A(\cdot)_pB)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$. Similarly, we have if $0 < \alpha < h_1$,

$$x_\alpha(s) = \frac{x_1 + a_1(h_1-\alpha)\cos s}{x_2 - a_2(h_2-\alpha)\cos s}, \quad y_\alpha(s.p) = \frac{y_1 + b_1(h_1-\alpha)\sin s\cos p}{y_2 - b_2(h_2-\alpha)\sin s\cos p},$$

$$z_\alpha(s.p,q) = \frac{z_1 + c_1(h_1-\alpha)\sin s\sin p\cos q}{z_2 - c_2(h_2-\alpha)\sin s\sin p\cos q}, \quad t_\alpha(s.p,q) = \frac{t_1 + d_1(h_1-\alpha)\sin s\sin p\sin q}{t_2 - d_2(h_2-\alpha)\sin s\sin p\sin q}.$$

Furthermore, we have

$$
\begin{aligned}
x_0(s) &= \frac{x_1 + a_1h_1\cos s}{x_2 - a_2h_2\cos s}, \quad x_{h_1}(s) = \frac{x_1}{x_2 - a_2(h_2-h_1)\cos s},\\
y_0(s,p) &= \frac{y_1 + b_1h_1\sin s\cos p}{y_2 - b_2h_2\sin s\cos p}, \quad y_{h_1}(s,p) = \frac{y_1}{y_2 - b_2(h_2-h_1)\sin s\cos p},\\
z_0(s,p,q) &= \frac{z_1 + c_1h_1\sin s\sin p\cos q}{z_2 - c_2h_2\sin s\sin p\cos q}, \quad z_{h_1}(s,p,q) = \frac{z_1}{z_2 - c_2(h_2-h_1)\sin s\sin p\cos q},\\
t_0(s,p,q) &= \frac{t_1 + d_1h_1\sin s\sin p\sin q}{t_2 - d_2h_2\sin s\sin p\sin q}, \quad t_{h_1}(s,p,q) = \frac{t_1}{t_2 - d_2(h_2-h_1)\sin s\sin p\sin q},
\end{aligned}
$$

and

$$(A(/)_p B)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

The proof is complete. □

Example 4.7. Let $A = ((\frac{1}{2}, 6, 3, 8, 5, 4, 7, 2, 5))^4$ and $B = ((\frac{2}{3}, 4, 2, 5, 3, 6, 4, 3, 2))^4$. Then by Theorem 4.6, we have the following:

(1) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(+)_p B$ is

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{3x-15}{17-30\alpha}\Big)^2 + \Big(\frac{3y-24}{22-39\alpha}\Big)^2 + \Big(\frac{z-11}{6-10\alpha}\Big)^2 + \Big(\frac{t-7}{3-5\alpha}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(-)_p B$ is

$$\Big\{(x,y,z,t) \in \mathbb{R}^4 \Big| \Big(\frac{3x-3}{17-30\alpha}\Big)^2 + \Big(\frac{3y-6}{22-39\alpha}\Big)^2 + \Big(\frac{z-3}{6-10\alpha}\Big)^2 + \Big(\frac{t-3}{3-5\alpha}\Big)^2 = 1\Big\}.$$

(3) For $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = 6 + (14 - 24\alpha)\cos s + 4(1-2\alpha)(2-3\alpha)\cos^2 s,$$

$$y_\alpha(s,p) = 15 + (\frac{86}{3} - 49\alpha)\sin s \cos p + 20(1-2\alpha)(\frac{2}{3} - \alpha)\sin^2 s \cos^2 p,$$

$$z_\alpha(s,p,q) = 28 + (36 - 58\alpha)\sin s \sin p \cos q + 4(1-2\alpha)(2-3\alpha)\sin^2 s \sin^2 p \cos^2 q,$$

$$t_\alpha(s,p,q) = 10 + (12 - 19\alpha))\sin s \sin p \sin q + (1-2\alpha)(2-3\alpha)\sin^2 s \sin^2 p \sin^2 q.$$

(4) For $0 < \alpha < \frac{1}{2}$, $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,p), z_\alpha(s,p,q), t_\alpha(s,p,q)) \in \mathbb{R}^4 \mid 0 \le s \le 2\pi, -\frac{\pi}{2} \le p \le \frac{\pi}{2}, -\frac{\pi}{2} \le q \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = \frac{9 + 9(1-2\alpha)\cos s}{6 - 4(2-3\alpha)\cos s}, \quad y_\alpha(s,p) = \frac{15 + 12(1-2\alpha)\sin s \cos p}{9 - 15(2-3\alpha)\sin s \cos p},$$

$$z_\alpha(s,p,q) = \frac{7 + 2(1-2\alpha)\sin s \sin p \cos q}{4 - 2(2-3\alpha)\sin s \sin p \cos q}, \quad t_\alpha(s.p,q) = \frac{5 + (1-2\alpha)\sin s \sin p \sin q}{2 - (2-3\alpha)\sin s \sin p \sin q}.$$

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