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GENERALIZED AND INDEPENDENT OBIC ALGEBRAS WITH ANNIHILATING SUBSETS

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Abstract. This paper develops a unified algebraic framework that extends the structure of the classical obic algebra (O-algebra) into two broader systems: the generalized obic algebra (GO-algebra) and the independent obic algebra (IO-algebra). An IO-algebra is defined by adopting one key identity from O-algebras and one from GO-algebras, together with an additional axiom that establishes its own independent behavior. This construction makes IO-algebras structurally autonomous while maintaining algebraic coherence with both O- and GO-algebras. In addition, two special subsets—the zero-annihilating subset and the self-annihilating subset—are introduced and analyzed to describe the interactions among internal elements in these systems. Their algebraic roles as subalgebras and ideals are examined in the settings of O-, GO-, and IO-algebras. The proposed framework contributes to the development of new algebraic structures that broaden the conceptual foundation and structural hierarchy of obic-type algebras.

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1. Introduction

The study of algebraic systems derived from logical connectives has remained an active and expanding field since the pioneering works of Imai and Iséki [15,17], who established the foundations of BCK- and BCI-algebras as algebraic models of propositional logic. Subsequent research sought to refine and generalize these structures in various directions. Hu and Li [8] proposed BCH-algebras, and Jun, Roh, and Kim [18] introduced BH-algebras as further extensions. Dudek [6] developed the concept of BCC-algebras, providing a more general and unified framework encompassing both BCK- and BCI-algebras. In parallel, Neggers and Kim, along with their collaborators, introduced several related systems, including d-algebras [26], Q-algebras [25], and B-algebras [27], all aimed at capturing diverse logical and order-theoretic behaviors. Their ideas inspired subsequent extensions, including BN-algebras [21] and Bd-algebras [4], which further enriched the theoretical landscape of

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implication-type algebras. Further developments were made by Kim and Kim with the introduction of BE-algebras [22], followed by BM-, BG-, and GE-algebras [3,19,20], each extending the scope of logical operations and algebraic properties. A notable addition to this lineage is the QI-algebra proposed by Bandaru [2], which unifies the structural features of Q-algebras and implication-based systems, thereby expanding the conceptual reach of these algebraic frameworks. Meanwhile, other families such as PU-algebras [7], KU-algebras and their extensions [23], and UP-algebras [9], together with their independent and bialgebraic variants [10,24], have offered new perspectives on algebraic semantics for non-classical logics. Further theoretical expansions through pseudo-type generalizations—such as pseudo-BE [5], pseudo-eBE [29,30], and pseudo-CI-algebras [28]—have strengthened the algebraic foundations of logical reasoning and ideal theory. In addition, the introduction of JU-algebras and the concept of p-closure ideals by Ansari, Haider, and Koam [1] has extended these systems toward closure-based approaches and ideal-theoretic interpretations. Collectively, these developments illustrate a long-standing and systematic effort to generalize, connect, and unify diverse classes of implication-type algebras, providing a broad framework for studying the algebraic structures underlying non-classical and many-valued logics.

In the recent development of algebraic systems related to logical implications, obic algebras (O-algebras) were first introduced by E. Ilojide, a Nigerian mathematician, in 2019 [11]. This structure offered a novel perspective on implication-type operations by emphasizing the interaction between the binary operation and a distinguished zero element, thereby extending beyond the scope of classical implication algebras. Building upon this foundation, Ilojide proposed kreb algebras [12], which generalized the axiomatic behavior of O-algebras by introducing new forms of associativity and identity interaction. In subsequent works, he further expanded this line of inquiry through the definitions of lojid algebras [13] and polian algebras [14], each designed to reinterpret the role of the zero element and to explore additional symmetries and dualities in non-classical algebraic reasoning. Collectively, these systems demonstrate Ilojide's ongoing effort to establish a new family of logical algebras grounded in implication-like operations but exhibiting distinct internal consistency and generalized identity structures. Motivated by these pioneering contributions and the growing need for a more unified theoretical framework, the present study aims to develop two extended systems—namely, the generalized obic algebra (GO-algebra) and the independent obic algebra (IO-algebra)—as natural extensions of the original obic concept, intended to capture broader logical and structural relationships among these emerging algebraic forms.

The present paper aims to establish a unified algebraic framework that extends the structure of the classical O-algebra into two more comprehensive systems: the GO-algebra and the IO-algebra. The IO-algebra is formulated by combining one characteristic identity of O-algebras with a key property of

GO-algebras, together with an additional axiom that defines its own independent structure. This construction allows IO-algebras to operate as a structurally autonomous yet algebraically coherent system situated between O- and GO-algebras. In this study, we systematically investigate the fundamental properties, axiomatic relations, and equivalence conditions among these three systems and demonstrate that GO-algebras constitute the most general class encompassing both O- and IO-algebras. Moreover, we introduce and analyze two special subsets within these algebras, namely the zero-annihilating subset and the self-annihilating subset, which play crucial roles in describing the internal behavior of elements under the binary operation. Their algebraic characteristics are explored in detail, particularly regarding the conditions for forming subalgebras and ideals in O-, GO-, and IO-algebras. The results obtained contribute to a deeper understanding of the internal structure of obic-type systems and provide a foundation for future extensions of these logical-algebraic models. The paper is organized as follows: Section 2 recalls the preliminary notions and basic properties of O-, GO-, and IO-algebras; Section 3 presents the main properties and interrelations among these systems, including the study of the zero-annihilating and self-annihilating subsets and their roles as subalgebras and ideals; and Section 4 concludes the paper with remarks and directions for further research.

2. Preliminaries and the Formulation of GO- and IO-Algebras

Before proceeding to the formal development of our proposed systems, it is essential to recall the basic notion of O-algebra, which serves as the foundational structure for this study. An O-algebra was first introduced by E. Ilojide in 2019 as an algebraic system designed to model implication-type operations through a binary operation $\star$ and a distinguished element 0, representing the zero or annihilator of the system. This structure generalizes the idea of implication in non-classical logic by imposing specific interaction rules between elements and the zero element, thereby extending beyond the conventional framework of implication algebras. The axioms of an O-algebra encode the essential properties governing these interactions and form the basis for further generalizations, including the IO-algebra and the GO-algebra, which will be introduced later in this paper. For completeness, we restate below the definition of an O-algebra as given in [11].

Definition 2.1. [11] *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called an obic algebra (O-algebra), where X is a nonempty set, $\star$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(x \star 0 = x) \tag{O-1}$$

$$(\forall x, y, z \in X)((x \star (y \star z)) \star x = x \star (y \star (z \star x))) \tag{O-2}$$

$$(\forall x \in X)(x \star x = 0) \tag{O-3}$$

Before proceeding to the generalized structures, such as IO- and GO-algebras, it is essential to recall several fundamental properties that hold within the framework of O-algebras. These identities, originally established and further analyzed in [11,16], illustrate the algebraic flexibility and internal consistency of the obic operation $\star$ under the axioms defining an O-algebra. In particular, the identity (2.1) was first proved in [11], while the set of properties (2.2)–(2.20) were systematically derived and verified in [16]. These results provide deeper insight into how the zero element 0 interacts with the binary operation, capturing nontrivial symmetries and self-referential behaviors within the structure. Moreover, they serve as a theoretical foundation for the forthcoming generalizations to IO- and GO-algebras, where additional axioms extend and refine the underlying logical behavior of obic systems. The following proposition summarizes the principal identities established in this algebraic framework.

Proposition 2.1. [11,16] *In an O-algebra $X = (X, \star, 0)$, the following assertions are valid:*

$$(\forall x, y \in X)(x \star y = (x \star (y \star x)) \star x) \tag{2.1}$$

$$(\forall x, y \in X)((x \star y) \star x = x \star (y \star (0 \star x))) \tag{2.2}$$

$$(\forall x, y, z, u \in X)((x \star (y \star (z \star (u \star y)))) \star x = x \star ((y \star (z \star u)) \star (y \star x))) \tag{2.3}$$

$$(\forall x, y \in X)(x \star (y \star (z \star x)) = x \star ((y \star z) \star (0 \star x))) \tag{2.4}$$

There is only one element 0 *that satisfies* (O-1) (2.5)

$$(\forall x, y \in X)(x \star (y \star (y \star x)) = 0) \tag{2.6}$$

$$(\forall x, y \in X)((x \star (x \star y)) \star y = 0) \tag{GO-1}$$

$$(\forall x, y, z \in X)((x \star y) \star x = x \star (y \star ((x \star (z \star (z \star x))) \star x))) \tag{2.7}$$

$$(\forall x, y \in X)(x \star (y \star ((y \star (x \star y)) \star y)) = 0) \tag{2.8}$$

$$(\forall x, y \in X)(x \star (y \star (0 \star (x \star y))) = 0) \tag{2.9}$$

$$(\forall x, y \in X)((x \star (0 \star (y \star x))) \star y = 0) \tag{2.10}$$

$$(\forall x, y \in X)(((y \star x) \star y) \star (0 \star x) = 0) \tag{2.11}$$

$$(\forall x, y \in X)((y \star (x \star (0 \star y))) \star (0 \star x) = 0) \tag{2.12}$$

$$(\forall x, y \in X)(y \star ((x \star (0 \star y)) \star x) = 0) \tag{2.13}$$

$$(\forall x, y \in X)(y \star (x \star ((0 \star y) \star (0 \star x))) = 0) \tag{2.14}$$

$$(\forall x, y, z \in X)(x \star (y \star ((z \star (z \star y)) \star x)) = 0) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \star ((y \star (y \star z)) \star (z \star x)) = 0) \tag{2.16}$$

$$(\forall x, y, z \in X)((x \star (x \star y)) \star (z \star (z \star y)) = 0) \tag{2.17}$$

$$(\forall x, y \in X)((0 \star ((x \star y) \star x)) \star (y \star 0) = 0) \tag{2.18}$$

$$(\forall x, y \in X)((0 \star (x \star (y \star (0 \star x)))) \star (y \star 0) = 0) \tag{2.19}$$

$$(\forall x, y \in X)((0 \star (y \star ((0 \star x) \star (0 \star y)))) \star (0 \star x) = 0) \tag{2.20}$$

Before introducing the new concept of a GO-algebra, let us recall that an O-algebra is characterized by its distinctive binary operation $\star$ and the constant element 0, which together model implication-type interactions under specific axioms. However, these axioms restrict the system to a narrow class of structures, limiting its flexibility in expressing wider algebraic behaviors. To overcome this limitation and to establish a more comprehensive framework that accommodates both the original and extended forms of obic operations, we now introduce the notion of a GO-algebra. This new system retains the essential properties of O-algebras while relaxing certain operational constraints, thereby enabling a broader class of algebraic systems to be included within the same conceptual scheme.

Definition 2.2. *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called a generalized obic algebra (GO-algebra), where X is a nonempty set, $\star$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(x \star 0 = x) \tag{O-1}$$

$$(\forall x, y \in X)((x \star (x \star y)) \star y = 0) \tag{GO-1}$$

$$(\forall x \in X)(x \star x = 0) \tag{O-3}$$

After establishing the axiomatic foundation of a GO-algebra, it is natural to explore the fundamental properties that follow from its definition. These properties not only clarify the internal structure of a GO-algebra but also reveal how it generalizes the behavior of the original O-algebra. In particular, the following results demonstrate several key relationships among the elements of the system and the fixed element 0, including conditions of uniqueness, equivalence, and structural invariance that characterize the internal consistency of GO-algebras.

Proposition 2.2. *In a GO-algebra $X = (X, \star, 0)$, the following assertions are valid:*

$$(\forall x, y \in X)(x \star (x \star y) = x \Leftrightarrow x \star y = 0) \tag{2.21}$$

There is only one element 0 that satisfies (O-1) (2.22)

$$(\forall x, y \in X)(x \star y = y) \Rightarrow X = \{0\} \tag{2.23}$$

$$(\forall x \in X)(x \star 0 = 0 \Leftrightarrow x = 0) \tag{2.24}$$

$$(\forall x, y \in X)(x \star (x \star (y \star y)) = (x \star (x \star y)) \star y) \tag{2.25}$$

$$(\forall x \in X)(0 \star x = 0 \Leftrightarrow 0 \star (0 \star x) = 0) \tag{2.26}$$

Proof. (2.21) Let $x, y \in X$ be such that $x \star (x \star y) = x$. Then

$$\begin{aligned} x \star y &= (x \star (x \star y)) \star y && \text{(by assumption)} \\ &= 0. && \text{(by (GO-1))} \end{aligned}$$

Conversely, let $x, y \in X$ be such that $x \star y = 0$. Then

$$\begin{aligned} x \star (x \star y) &= x \star 0 && \text{(by assumption)} \\ &= x. && \text{(by (O-1))} \end{aligned}$$

(2.22) If $x \star y = x$ for all $x \in X$, then

$$\begin{aligned} y &= y \star y && \text{(by assumption)} \\ &= 0. && \text{(by (O-3))} \end{aligned}$$

Therefore, 0 is the unique element that satisfies (O-1).

(2.23) If $x \star y = y$ for all $x, y \in X$, then for all $x \in X$,

$$\begin{aligned} x &= x \star x && \text{(by assumption)} \\ &= 0. && \text{(by (O-3))} \end{aligned}$$

Hence, $X = \{0\}$.

(2.24) Let $x \in X$ be such that $x \star 0 = 0$. Then

$$\begin{aligned} x &= x \star 0 && \text{(by (O-1))} \\ &= 0. && \text{(by assumption)} \end{aligned}$$

Conversely, it is immediately true by (O-3).

(2.25) Let $x, y \in X$. Then

$$\begin{aligned} x \star (x \star (y \star y)) &= x \star (x \star 0) && \text{(by (O-3))} \\ &= x \star x && \text{(by (O-1))} \\ &= 0 && \text{(by (O-3))} \\ &= (x \star (x \star y)) \star y. && \text{(by (GO-1))} \end{aligned}$$

(2.26) Let $x \in X$ be such that $0 \star x = 0$. Then

$$\begin{aligned} 0 &= 0 \star 0 && \text{(by (O-3))} \\ &= 0 \star (0 \star x). && \text{(by assumption)} \end{aligned}$$

Conversely, let $x \in X$ be such that $0 \star (0 \star x) = 0$. Then

$$\begin{aligned} 0 &= (0 \star (0 \star x)) \star x && \text{(by (GO-1))} \\ &= 0 \star x. && \text{(by assumption)} \end{aligned}$$

□

While the GO-algebra extends the structure of the classical O-algebra by relaxing its operational constraints, it still preserves a unified dependence among its axioms. In contrast, we now propose a distinct algebraic system in which one of the central identities of O-algebras and one of GO-algebras are both retained, but supplemented by a new axiom that introduces an independent operational behavior. This system, called an IO-algebra, is constructed to emphasize symmetry and reversibility between elements through the role of the zero element, thereby exhibiting both autonomy and coherence within the broader class of obic-type algebras.

Definition 2.3. *An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called an independent obic algebra (IO-algebra), where X is a nonempty set, $\star$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(x \star 0 = x) \tag{O-1}$$

$$(\forall x, y \in X)((x \star (x \star y)) \star y = 0) \tag{GO-1}$$

$$(\forall x, y \in X)(0 \star (x \star y) = y \star x) \tag{IO-1}$$

Having established the axiomatic structure of an IO-algebra, we now turn to examine the algebraic consequences that arise from its definition. Unlike the GO-algebra, whose behavior is primarily governed by a generalized extension of the obic operation, the IO-algebra reveals an additional level of symmetry and duality induced by the independent axiom (IO-1). This axiom enables bidirectional interactions between elements through the zero element, producing several unique identities that distinguish IO-algebras from both O- and GO-algebras. The following propositions summarize the fundamental properties and internal relationships that naturally follow from the axioms of IO-algebras.

Proposition 2.3. *In an IO-algebra $X = (X, \star, 0)$, the following assertions are valid:*

$$(\forall x \in X)(x \star x = 0) \tag{O-3}$$

$$(\forall x, y \in X)(x \star (y \star (y \star x)) = (x \star (y \star y)) \star x) \tag{2.27}$$

$$(\forall x, y, z \in X)(((0 \star ((x \star y) \star x)) \star y) \star (0 \star z) = z) \tag{2.28}$$

$$(\forall x \in X)(0 \star x = 0 \Leftrightarrow x = 0) \tag{2.29}$$

$$(\forall x \in X)(0 \star (0 \star x) = x) \tag{2.30}$$

Proof. (O-3) Let $x \in X$. Then

$$\begin{aligned} 0 &= (0 \star (0 \star x)) \star x && \text{(by (GO-1))} \\ &= (x \star 0) \star x && \text{(by (IO-1))} \\ &= x \star x. && \text{(by (O-1))} \end{aligned}$$

(2.27) Let $x, y \in X$. Then

$$
\begin{aligned}
(x \star (y \star y)) \star x &= (x \star 0) \star x && \text{(by (O-3))}\\
&= x \star x && \text{(by (O-1))}\\
&= 0 && \text{(by (O-3))}\\
&= 0 \star ((y \star (y \star x)) \star x) && \text{(by (GO-1))}\\
&= x \star (y \star (y \star x)). && \text{(by (IO-1))}
\end{aligned}
$$

(2.28) Let $x, y, z \in X$. Then

$$
\begin{aligned}
((0 \star ((x \star y) \star x)) \star y) \star (0 \star z) &= ((x \star (x \star y)) \star y) \star (0 \star z) && \text{(by (IO-1))}\\
&= 0 \star (0 \star z) && \text{(by (GO-1))}\\
&= z \star 0 && \text{(by (IO-1))}\\
&= z. && \text{(by (O-1))}
\end{aligned}
$$

(2.29) Let $x \in X$ be such that $0 \star x = 0$. Then

$$
\begin{aligned}
0 &= 0 \star 0 && \text{(by (O-3))}\\
&= 0 \star (0 \star x) && \text{(by assumption)}\\
&= x \star 0 && \text{(by (IO-1))}\\
&= x. && \text{(by (O-1))}
\end{aligned}
$$

Conversely, it is immediately true by (O-3).

(2.30) Let $x \in X$. Then

$$
\begin{aligned}
0 \star (0 \star x) &= x \star 0 && \text{(by (IO-1))}\\
&= x. && \text{(by (O-1))}
\end{aligned}
$$

□

3. Structural Properties and Subset Analysis of O-, GO-, and IO-Algebras

Having established the foundational definitions and axiomatic systems of O-, GO-, and IO-algebras in the previous section, we now examine their intrinsic structural relationships and algebraic behaviors. This section is devoted to investigating how these systems interact, how one may be embedded within another, and which conditions characterize their equivalence or independence. We first demonstrate that both O- and IO-algebras can be naturally viewed as particular cases of GO-algebras, though they are not necessarily connected. Following this, we introduce two special subsets in each algebraic structure, called the zero-annihilating subset and the self-annihilating subset, and study their fundamental properties, conditions for being subalgebras, and their behavior as ideals within O-, GO-, and IO-algebras. Overall, this section provides a comprehensive view of the internal organization and hierarchical relationships among these three algebraic systems.

Theorem 3.1. *Every O-algebra is a GO-algebra.*

Proof. Assume that X is an O-algebra. By Proposition 2.1, we have (GO-1). Since we have (O-1) and (O-3), we see that X is a GO-algebra. □

Example 3.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	1	2	3	0	5
1	1	0	2	2	1	0
2	2	0	0	0	5	4
3	3	2	1	0	3	0
4	4	1	5	3	0	2
5	5	0	4	0	2	0

Thus, $X = (X, \star, 0)$ forms a GO-algebra. However, since

$$(1 \star (2 \star 2)) \star 1 = 0 \neq 2 = 1 \star (2 \star (2 \star 1)),$$

the structure X fails to satisfy the defining condition for O-algebras, and therefore is not an O-algebra.

Theorem 3.2. *Every IO-algebra is a GO-algebra.*

Proof. Assume that X is an IO-algebra. By Proposition 2.3, we have (O-3). Since we have (O-1) and (GO-1), we see that X is a GO-algebra. □

Example 3.2. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	1	2	0	4	5	6
1	1	0	2	1	2	4	0
2	2	0	0	1	0	0	3
3	3	1	4	0	5	6	2
4	4	2	1	5	0	0	0
5	5	4	0	6	1	0	0
6	6	0	3	2	0	0	0

Thus, $X = (X, \star, 0)$ forms a GO-algebra. However, since

$$0 \star (2 \star 4) = 0 \neq 1 = 4 \star 2,$$

it follows that X fails to be an IO-algebra.

Example 3.3. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	6	5
1	1	0	3	2	5	5	2
2	2	3	0	1	0	0	1
3	3	2	1	0	0	0	0
4	4	6	0	0	0	0	1
5	5	6	0	0	0	0	1
6	6	2	1	0	1	1	0

Thus, $X = (X, \star, 0)$ forms an IO-algebra. However, notice that

$$(2 \star (3 \star 4)) \star 2 = (2 \star 0) \star 2 = 0 \neq 1 = 2 \star (3 \star 0) = 2 \star (3 \star (4 \star 2)),$$

so X fails the required identity and therefore is not an O-algebra.

Example 3.4. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	0	0
2	2	4	0	6	6	0	4	4
3	3	5	6	0	0	6	5	5
4	4	2	6	0	0	6	2	2
5	5	3	0	6	6	0	3	3
6	6	0	4	5	2	3	0	0
7	7	0	3	2	5	4	0	0

Thus, $X = (X, \star, 0)$ forms an O-algebra. However, observe that

$$0 \star (1 \star 2) = 0 \star 3 = 3 \neq 4 = 2 \star 1,$$

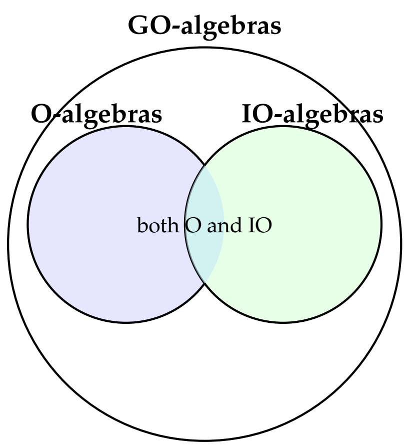
so the IO-condition fails. Consequently, X is not an IO-algebra.

Example 3.5. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	0	4	5	4	4
2	2	0	0	4	3	4	4
3	3	4	4	0	2	0	0
4	4	5	3	2	0	1	2
5	5	4	4	0	1	0	0
6	6	4	4	0	2	0	0

Consequently, $X = (X, \star, 0)$ satisfies the axioms of both an O-algebra and an IO-algebra.

Remark 3.1. *From Theorems 3.1 and 3.2, it follows that both O-algebras and IO-algebras are subclasses of GO-algebras. That is, every O-algebra satisfies all axioms of a GO-algebra, and every IO-algebra can likewise be viewed as a GO-algebra. Although O-algebras and IO-algebras are generally independent systems (i.e., neither is contained in the other), there exist algebraic structures that satisfy the axioms of both systems simultaneously. Hence, the relationship among these three algebraic systems can be represented schematically as follows.*



In algebraic systems, subsets that remain closed under the binary operation often provide insight into the algebra's internal structure. Such subsets, when equipped with the same operation, form subalgebras, and when they additionally satisfy certain absorption conditions with respect to the distinguished element, they are called ideals. These notions were first adapted to the framework of O-algebras by Ilojide in [11] and later extended to kreb algebras in [12]. Following these developments, we now reformulate the concepts of subalgebra and ideal within the broader settings of O-, GO-, and IO-algebras, which will serve as a foundation for studying more specialized subsets, such as zero-annihilating and self-annihilating subsets, in the next subsection.

Definition 3.1. [11, 12] *A nonempty subset S of an O-, GO-, or IO-algebra $X = (X, \star, 0)$ is called* *(i) a subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \star y \in S) \tag{3.1}$$

(ii) an ideal of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S \tag{3.2}$$

$$(\forall x, y \in X)((0 \star x) \star (0 \star y) \in S, y \in S \Rightarrow x \in S) \tag{3.3}$$

Theorem 3.3. *Let S be an ideal of an IO-algebra X. Then $0 \star x \in S$ for all $x \in S$.*

Proof. Let $x \in S$. Then

$$\begin{aligned} x &= x \star 0 && \text{(by (O-1))} \\ &= (0 \star (0 \star x)) \star 0 && \text{(by (2.30))} \\ &= (0 \star (0 \star x)) \star (0 \star 0). && \text{(by (O-3))} \end{aligned}$$

By (3.2) and (3.3), we have $0 \star x \in S$. □

Remark 3.2. *$\{0\}$ is a subalgebra of an O-, GO-, and IO-algebra.*

Remark 3.3. *$\{0\}$ is an ideal of an O- and IO-algebra.*

From Example 3.6, note that $(0 \star 3) \star (0 \star 0) \in \{0\}$ and $0 \in \{0\}$, but $3 \notin \{0\}$. This immediately shows that $\{0\}$ fails to be an ideal of the GO-algebra X.

Example 3.6. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	0	0	0	4	5	6
1	1	0	1	3	4	6	3
2	2	1	0	2	0	0	0
3	3	3	2	0	0	0	0
4	4	4	2	0	0	6	2
5	5	0	0	0	0	0	0
6	6	0	2	0	2	4	0

Then $X = (X, \star, 0)$ forms a GO-algebra. Consider $S = \{0, 1, 2, 3, 5\}$. This set is an ideal of X, but not a subalgebra: although $1, 5 \in S$, their operation $1 \star 5 = 6 \notin S$, so S fails to be closed under $\star$ and thus is not a subalgebra of X.

Example 3.7. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	0	0	3	0	5
1	1	0	1	4	4	5
2	2	1	0	3	2	0
3	3	4	3	0	0	0
4	4	0	2	0	0	0
5	5	5	0	0	0	0

Thus, $X = (X, \star, 0)$ forms a GO-algebra. Consider the subset $S = \{0, 1, 2, 4, 5\}$. This S is closed under the operations of X, so it is a subalgebra. However, observe that

$$(0 \star 3) \star (0 \star 5) = 0 \in S \text{ and } 5 \in S, \text{ but } 3 \notin S.$$

Hence, S fails the ideal condition, and therefore S is not an ideal of X.

Example 3.8. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	1	0	3	1	0
1	1	0	2	2	0	1
2	2	0	0	0	4	0
3	3	2	1	0	5	1
4	4	0	4	5	0	3
5	5	1	0	4	3	0

Thus, $X = (X, \star, 0)$ forms a GO-algebra. Consider the subset $S = \{0, 2, 5\}$. Consequently, S satisfies the axioms of both a subalgebra and an ideal of X.

Theorem 3.4. *Let X be an IO-algebra. Every ideal of X is a subalgebra of X.*

Proof. Let S be an ideal of X. Let $x, y \in S$. By Theorem 3.3, we have $0 \star x, 0 \star y \in S$. Then

$$\begin{aligned} 0 &= (y \star (y \star x)) \star x && \text{(by (GO-1))} \\ &= (0 \star ((y \star x) \star y)) \star x && \text{(by (IO-1))} \\ &= (0 \star ((y \star x) \star y)) \star (0 \star (0 \star x)). && \text{(by (2.30))} \end{aligned}$$

Since $0 \star x \in S$ and by (3.2) and (3.3), we have $(y \star x) \star y \in S$. Thus

$$\begin{aligned} (y \star x) \star y &= (0 \star (x \star y)) \star y && \text{(by (IO-1))} \\ &= (0 \star (x \star y)) \star (0 \star (0 \star y)). && \text{(by (2.30))} \end{aligned}$$

Since $0 \star y \in S$ and $(y \star x) \star y \in S$ and by (3.2) and (3.3), we have $x \star y \in S$. Hence, S is a subalgebra of X. □

Example 3.9. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	6	2	3	4	5	1
1	1	0	6	2	0	3	2
2	2	1	0	6	0	0	6
3	3	2	1	0	5	4	0
4	4	0	0	5	0	3	0
5	5	3	0	4	3	0	0
6	6	2	1	0	0	0	0

Then $X = (X, \star, 0)$ forms an IO-algebra. Consider $S = \{0, 1, 2, 4, 6\}$. This set is a subalgebra of X. However, note that

$$(0 \star 3) \star (0 \star 6) = 2 \in S \text{ and } 6 \in S, \text{ but } 3 \notin S.$$

Thus, S fails the ideal condition in X, and therefore S is not an ideal of X.

Theorem 3.5. *Let X be an O-algebra. Every ideal of X is a subalgebra of X.*

Proof. Let S be an ideal of X.

The properties we will use in the proof include:

(1) There is only one element 0 that satisfies (O-1) (2.5)

(2) $(\forall x \in X)(x \star 0 = x)$ (O-1)

(3) $(\forall x, y \in X)((x \star y) \star x = x \star (y \star (0 \star x)))$ (2.2)

(4) $(\forall x, y, z \in X)(x \star (y \star (z \star x)) = x \star ((y \star z) \star (0 \star x)))$ (2.4)

(5) $(\forall x, y \in X)(x \star (y \star (y \star x)) = 0)$ (2.6)

(6) $(\forall x, y, z \in X)(x \star (y \star ((z \star (z \star y)) \star x)) = 0)$ (2.15)

(7) $(\forall x, y \in X)((0 \star ((x \star y) \star x)) \star (y \star 0) = 0)$ (2.18)

(8) $(\forall x, y \in X)((0 \star (x \star (y \star (0 \star x)))) \star (y \star 0) = 0)$ (2.19)

The conditions of an ideal include:

(9) $0 \in S$ (3.2): ideal condition

(10) $(\forall x, y \in X)((0 \star x) \star (0 \star y) \in S \wedge y \in S \Rightarrow x \in S)$ (3.3): ideal condition

Goal:

(11) $(\forall x, y \in X)(x \in S \wedge y \in S \Rightarrow x \star y \in S)$ (3.1): goal

Suppose that S is not a subalgebra of X.

The procedure for proving a contradiction is as follows:

(12) $c_1 \in S$	(11): the negation of the goal
(13) $c_2 \in S$	(11): the negation of the goal
(14) $c_1 \star c_2 \notin S$	(11): the negation of the goal
(15) $x \star ((y \star z) \star (0 \star x)) = x \star (y \star (z \star x))$	(4): flipping
(16) $x \star y \neq x \vee y = 0$	(1): logical disjunction
(17) $x \star y \neq x \vee 0 = y$	(16): flipping
(18) $((0 * x) * (0 * y)) \notin S \vee y \notin S \vee x \in S$	(10): logical disjunction
(19) $x \star (y \star (z \star x)) \neq x \vee 0 = (y \star z) \star (0 \star x)$	(17): $y := (y \star z) \star (0 \star x)$ and (15)
(20) $x \star (y \star (z \star x)) \neq x \vee (y \star z) \star (0 \star x) = 0$	(19): flipping
(21) $x \star (y \star ((z \star (z \star y)) \star x)) = 0$	(6)
(22) $(x \star y) \star 0 \neq x \star y \vee (y \star x) \star (0 \star (x \star y)) = 0$	(20): $z := x, x := x \star y$ and (5): $y \star (x \star (x \star y)) = 0$
(23) $x \star y \neq x \star y \vee (y \star x) \star (0 \star (x \star y)) = 0$	(22): (O-1)
(24) $(x \star y) \star (0 \star (y \star x)) = 0$	(23): $x := y, y := x$
(25) $(0 \star (x \star (y \star (0 \star x)))) \star y = 0$	(8): (O-1)
(26) $0 \notin S \vee x \notin S \vee (y \star ((0 \star x) \star (0 \star y))) \in S$	(18): $y := x, x := y \star ((0 \star x) \star (0 \star y))$ and (2.20)
(27) $0 \notin S \vee x \notin S \vee (y \star (0 \star (x \star y))) \in S$	(15): $x := y, y := 0, z := x$ and (26)
(28) $x \notin S \vee (y \star (0 \star (x \star y))) \in S$	(27) and (9)
(29) $(x \star (0 \star (c_2 \star x))) \in S$	(28): $x := c_2, y := x$ and (13)
(30) $x \notin S \vee (y \star (0 \star (x \star y))) \in S$	(28)
(31) $(x \star (0 \star (c_1 \star x))) \in S$	(30): $y := x, x = c_1$ and (12)
(32) $(c_2 \star (0 \star x)) \notin S \vee x \in S$	(29): $x := 0 \star x$ and (18)
(33) $(c_2 \star (0 \star (c_1 \star c_2))) \notin S$	(32): $x := c_1 \star c_2$ and (14)
(34) A contradictory proposition	Contradiction: (31): $x := c_2$ and (33)

Hence, S is a subalgebra of X. $\square$

Example 3.10. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6	7
0	0	7	2	4	3	5	6	1
1	1	0	7	2	0	7	7	2
2	2	1	0	3	4	0	0	7
3	3	5	4	0	5	4	4	0
4	4	0	3	5	0	3	3	5
5	5	1	0	3	4	0	0	7
6	6	1	0	3	4	0	0	7
7	7	2	1	0	2	1	1	0

Thus, $X = (X, \star, 0)$ forms an O-algebra. Now take $S = \{0, 2, 3, 4, 5, 6\}$. This set is closed under $\star$ and contains 0, so S is a subalgebra of X. However, observe that

$$(0 \star 7) \star (0 \star 4) = 2 \in S \text{ and } 4 \in S, \text{ but } 7 \notin S.$$

Hence, S fails the absorption property required of ideals, and therefore S is not an ideal of X.

In algebraic systems that involve a distinguished zero element, certain subsets can be characterized by the way they interact with this element under the binary operation. Motivated by this observation, we now introduce two special subsets of the O-, GO-, and IO-algebras: the zero-annihilating subset and the self-annihilating subset. These subsets capture essential annihilation properties of elements with respect to the zero element and provide insight into the internal consistency of the algebraic structure. In particular, we investigate the conditions under which these subsets form subalgebras or ideals within each of the three systems, thereby linking annihilating behavior to structural stability in obic-type algebras.

Definition 3.2. *Let $X = (X, \star, 0)$ be an algebra of type $(2, 0)$, which may be an* O-, GO-, *or* IO-*algebra. We define the following special subsets of X:*

$$\mathbb{A}_0 = \{x \in X \mid 0 \star x = 0\}$$

$$\mathbb{A}_s = \{x \in X \mid x \star (0 \star x) = 0\}$$

The subset $\mathbb{A}_0$ is called the zero-annihilating subset of X, and the subset $\mathbb{A}_s$ is called the self-annihilating subset of X.

Throughout this section, $\mathbb{A}_0$ and $\mathbb{A}_s$ will be used to denote these subsets in the corresponding algebraic system. We shall later examine the conditions under which $\mathbb{A}_0$ and $\mathbb{A}_s$ form subalgebras or ideals within O-, GO-, and IO-algebras.

Remark 3.4. *Since (O-3) is true in an O-, GO-, and IO-algebra $X = (X, \star, 0)$, we have $0 \star 0 = 0$. Hence, $0 \in \mathbb{A}_0$.*

Remark 3.5. *Since (O-3) is true in an O-, GO-, and IO-algebra $X = (X, \star, 0)$, we have $0 \star (0 \star 0) = 0$. Hence, $0 \in \mathbb{A}_s$.*

3.1. **Zero- and Self-Annihilating Subsets of O-Algebras.** In O-algebras, subsets defined through their interaction with the zero element 0 reveal essential aspects of the system's internal structure. In particular, zero- and self-annihilating subsets describe how elements behave under annihilation and provide a foundation for analyzing subalgebras and ideals within O-, IO-, and GO-algebras.

Theorem 3.6. *Let X be an O-algebra. Then $\mathbb{A}_0 = \{0\}$.*

Proof. Assume that $\mathbb{A}_0 \neq \{0\}$. Then there exists $y \in \mathbb{A}_0$ such that $y \neq 0$. Thus,

$$
\begin{aligned}
y &= y \star 0 && \text{(by (O-1))} \\
&= y \star (0 \star 0) && \text{(by (O-3))} \\
&= y \star (0 \star (0 \star y)) && \text{(by } \mathbb{A}_0\text{'s property)} \\
&= (y \star (0 \star 0)) \star y && \text{(by (O-2))} \\
&= (y \star 0) \star y && \text{(by (O-3))} \\
&= y \star y && \text{(by (O-1))} \\
&= 0. && \text{(by (O-3))}
\end{aligned}
$$

which is a contradiction. Hence, $\mathbb{A}_0 = \{0\}$. □

With Remark 3.2 and Theorem 3.6, we obtain the corollary.

Corollary 3.1. $\mathbb{A}_0$ *is a subalgebra of an O-algebra.*

With Remark 3.3 and Theorem 3.6, we obtain the corollary.

Corollary 3.2. $\mathbb{A}_0$ *is an ideal of an O-algebra.*

Theorem 3.7. $\mathbb{A}_s$ *is an ideal of an O-algebra* X*.*

Proof. Let X be an O-algebra.

The properties we will use in the proof include:

(1) There is only one element 0 that satisfies (O-1) (2.5)

(2) $(\forall x \in X)(x \star 0 = x)$ (O-1)

(3) $(\forall x \in X)(x \star x = 0)$ (O-3)

(4) $(\forall x, y \in X)((x \star y) \star x = x \star (y \star (0 \star x)))$ (2.2)

(5) $(\forall x, y, z \in X)(x \star (y \star (z \star x)) = x \star ((y \star z) \star (0 \star x)))$ (2.4)

(6) $(\forall x, y \in X)(x \star (y \star (y \star x)) = 0)$ (2.6)

(7) $(\forall x, y \in X)((x \star (x \star y)) \star y = 0)$ (GO-1)

(8) $(\forall x, y \in X)((x \star (0 \star (y \star x))) \star y = 0)$ (2.10)

The conditions of $\mathbb{A}_s$ include:

(9) $x \in \mathbb{A}_s \Leftrightarrow x \star (0 \star x) = 0$ By $\mathbb{A}_s$'s property

Goal:

(10) $0 \in \mathbb{A}_s$ (3.2): ideal condition

(11) $(\forall x, y \in X)((0 \star x) \star (0 \star y) \in \mathbb{A}_s \wedge y \in \mathbb{A}_s \Rightarrow x \in \mathbb{A}_s)$ (3.3): ideal condition

Suppose that $\mathbb{A}_s$ is not an ideal of X.

The procedure for proving a contradiction is as follows:

(12) $(0 \star c_1) \star (0 \star c_2) \in \mathbb{A}_s$ (11): the negation of the goal

(13) $c_2 \in \mathbb{A}_s$ (11): the negation of the goal

(14) $c_1 \notin \mathbb{A}_s$ (11): the negation of the goal

(15) $x \notin \mathbb{A}_s \vee x \star (0 \star x) = 0$ (9): the negation of the $\mathbb{A}_s$

(16) $x \in \mathbb{A}_s \vee x \star (0 \star x) \neq 0$ (9): the negation of the $\mathbb{A}_s$

(17) $x \star y \neq x \ \vee \ y = 0$ (1): logical disjunction

(18) $x \star y \neq x \ \vee \ 0 = y$ (17): flipping

(19) $x \star y = x \ \vee \ y \neq 0$ (1): logical disjunction

(20) $x \star y = x \ \vee \ 0 \neq y$ (19): flipping

(21) $x \star ((y \star z) \star (0 \star x)) = x \star (y \star (z \star x))$ (5): symmetric form

(22) $x \star (y \star (z \star x)) \neq x \vee 0 = (y \star z) \star (0 \star x)$ (18): $x := x, y := y \star (z \star x)$ and (21)

(23) $x \star (y \star (z \star x)) \neq x \vee (y \star z) \star (0 \star x) = 0$ (22): flipping

(24) $0 \star x = x \star ((y \star (y \star x)) \star (0 \star x))$ (4): $y := y \star (y \star x)$ and (6)

(25) $0 \star x = x \star (y \star ((y \star x) \star x))$ (24): $x := x$, $y := y$, $z := y \star x$ and (21)

(26) $x \star (y \star ((y \star x) \star x)) = 0 \star x$ (25): symmetric form

(27) $0 \neq x \star (x \star y) \vee 0 = y$ (18): $x := x \star (x \star y)$, $y := y$ and (7)

(28) $x \star (x \star y) \neq 0 \vee 0 = y$ (27): flipping

(29) $0 \neq x \star (0 \star (y \star x)) \vee 0 = y$ (18): $x := x \star (0 \star (y \star x))$, $y := y$ and (8)

(30) $x \star (0 \star (y \star x)) \neq 0 \vee 0 = y$ (29): symmetric form

(31) $((0 \star c_1) \star (0 \star c_2)) \star (0 \star ((0 \star c_1) \star (0 \star c_2))) = 0$ From (15) and (12): since $(0 \star c_1) \star (0 \star c_2) \in \mathbb{A}_s$ holds

(32) $c_2 \star (0 \star c_2) = 0$ From (13) and (15): $c_2 \in \mathbb{A}_s$ holds

(33) $c_1 \star (0 \star c_1) \neq 0$ From (14) and (15): $c_1 \notin \mathbb{A}_s$ holds

(34) $x \star (c_1 \star (0 \star c_1)) \neq x$ (18): $c_1 \star (0 \star c_1) \neq 0$ implies product changes x

(35) $x \star (0 \star (c_1 \star (0 \star c_1))) \neq x$ (18): $0 \star (c_1 \star (0 \star c_1)) \neq 0$

(36) $0 \star c_2 = c_2 \star ((0 \star c_2) \star (0 \star c_2))$ (4): $x := c_2, y := (0 \star c_2)$ and (32)

(37) $0 \star c_2 = c_2 \star 0$ (36): (O-3)

(38) $0 \star c_2 = c_2$ (37): (O-1)

(39) $((0 \star c_1) \star c_2) \star (0 \star ((0 \star c_1) \star (0 \star c_2))) = 0$ (38): $0 \star c_2 = c_2$ into (31)

(40) $((0 \star c_1) \star c_2) \star (0 \star ((0 \star c_1) \star c_2)) = 0$ (38): $0 \star c_2 = c_2$ into (39)

(41) $c_2 \star ((x \star y) \star c_2) = c_2 \star (x \star (y \star c_2))$ (21): $x := c_2, y := x, z := y$ and (38)

(42) $(x \star y) \star 0 \neq x \star y \vee (y \star x) \star (0 \star (x \star y)) = 0$ (23): $x := x \star y, y := y, z := x$ and (6)

(43) $x \star y \neq x \star y \vee (y \star x) \star (0 \star (x \star y)) = 0$ (42): (O-1)

(44) $(x \star y) \star (0 \star (y \star x)) = 0$ Zero-symmetry established and (43): $x := y, y := x$

(45) $0 \neq (0 \star c_1) \star c_1$ (35): $(0 \star c_1) \star c_1 := x$ and (44)

(46) $(0 \star c_1) \star c_1 \neq 0$ (45): flipping

(47) $(0 \star (x \star y)) \star (0 \star 0) \neq 0 \vee 0 = y \star x$ (30): $x := 0 \star (x \star y), y := y \star x$ and (44)

(48) $(0 \star (x \star y)) \star 0 \neq 0 \vee 0 = y \star x$ (47): (O-3)

(49) $0 \star (x \star y) \neq 0 \vee 0 = y \star x$ (48): (O-1)

(50) $0 \star (x \star y) \neq 0 \vee y \star x = 0$ (49): flipping

(51) $x \star (x \star ((0 \star c_1) \star c_1)) \neq 0$ (28): $y := (0 \star c_1) \star c_1$ and (46)

(52) $0 \star ((x \star ((0 \star c_1) \star c_1)) \star x) \neq 0$ (50): $x := x \star ((0 \star c_1) \star c_1), y := x$ and (51)

(53) $0 \star (x \star (((0 \star c_1) \star c_1) \star (0 \star x))) \neq 0$ (4): $x := x, y := (0 \star c_1) \star c_1$

(54) $0 \star (x \star ((0 \star c_1) \star (c_1 \star x))) \neq 0$ (21): $x := x, y := (0 \star c_1)$ and $z := c_1$

(55) $0 \neq x \star ((0 \star c_1) \star (c_1 \star x))$ (54) and (20): $y := x \star ((0 \star c_1) \star (c_1 \star x))$

(56) $x \star ((0 \star c_1) \star (c_1 \star x)) \neq 0$ (55): symmetric form

(57) $(x \star ((x \star c_1) \star c_1)) \star ((0 \star c_1) \star (0 \star c_1)) \neq 0$ (56): $x := x \star ((x \star c_1) \star c_1)$ and (26): $c_1 \star x := 0 \star c_1$

(58) $(x \star ((x \star c_1) \star c_1)) \star 0 \neq 0$ (57): (O-3)

(59) $x \star ((x \star c_1) \star c_1) \neq 0$ (58): (O-1)

(60) $0 \star (((x \star c_1) \star c_1) \star x) \neq 0$ (50): $x := (x \star c_1) \star c_1, y := x$ and (59)

(61) $0 \neq ((x \star c_1) \star c_1) \star x$ (20): $x := 0, y := ((x \star c_1) \star c_1) \star x$

(62) $((x \star c_1) \star c_1) \star x \neq 0$ (61): symmetric form

(63) $x \star (((y \star c_1) \star c_1) \star y) \neq x$ (18): $x := x, y := ((y \star c_1) \star c_1) \star y$ and (62)

(64) $x \star ((0 \star c_1) \star (y \star (0 \star (c_1 \star y)))) \neq x$ (63): $y := (y \star (0 \star (c_1 \star y)))$ and (8): $(y \star c_1) \star c_1 := 0 \star c_1$

(65) $((0 \star c_1) \star c_2) \star 0 = ((0 \star c_1) \star c_2) \star ((0 \star c_1) \star (c_2 \star ((0 \star c_1) \star c_2)))$ (21): $x := (0 \star c_1) \star c_2, y := 0 \star c_1, z := c_2$ and (40).

(66) $(0 \star c_1) \star c_2 = ((0 \star c_1) \star c_2) \star ((0 \star c_1) \star (c_2 \star ((0 \star c_1) \star c_2)))$ (65): (O-1)

(67) $(0 \star c_1) \star c_2 = ((0 \star c_1) \star c_2) \star ((0 \star c_1) \star (c_2 \star (0 \star (c_1 \star c_2))))$ (41): $c_2 \star ((0 \star c_1) \star c_2) := c_2 \star (0 \star (c_1 \star c_2))$

(68) $((0 \star c_1) \star c_2) \star ((0 \star c_1) \star (c_2 \star (0 \star (c_1 \star c_2)))) = (0 \star c_1) \star c_2$ (67): flipping

(69) A contradictory proposition Contradiction: (68) and (64)

Hence, $\mathbb{A}_s$ is an ideal of X. □

3.2. **Zero- and Self-Annihilating Subsets of GO-Algebras.** Within GO-algebras, the concepts of zero- and self-annihilation extend those defined in O-algebras to a more general setting. These subsets capture how the generalized obic operation interacts with the zero element 0, revealing broader structural behaviors that may not appear in the classical case. Their analysis provides key insights into the internal consistency and flexibility of GO-algebras.

Theorem 3.8. *$\mathbb{A}_0$ is an ideal of a GO-algebra X.*

Proof. Let X be a GO-algebra. By Remark 3.4, we have $0 \in \mathbb{A}_0$. Let $x, y \in X$ be such that $(0 \star x) \star (0 \star y), y \in \mathbb{A}_0$. Then

$$
\begin{aligned}
0 &= 0 \star ((0 \star x) \star (0 \star y)) && \text{(by } \mathbb{A}_0\text{'s property)} \\
&= 0 \star ((0 \star x) \star 0) && \text{(by } \mathbb{A}_0\text{'s property)} \\
&= 0 \star (0 \star x). && \text{(by (O-1))}
\end{aligned}
$$

By (2.26), we have $0 \star x = 0$. Thus, $x \in \mathbb{A}_0$. Hence, $\mathbb{A}_0$ is an ideal of X. □

Example 3.11. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	0	0	3	3	0
1	1	0	3	2	4	1
2	2	3	0	0	5	3
3	3	2	0	0	0	0
4	4	4	5	0	0	0
5	5	1	0	0	0	0

Then $X = (X, \star, 0)$ forms a GO-algebra. Consider $\mathbb{A}_0 = \{0, 1, 2, 5\}$. However, note that

$$1 \in \mathbb{A}_0 \text{ and } 2 \in \mathbb{A}_0, \text{ but } 1 \star 2 = 3 \notin \mathbb{A}_0.$$

Thus, $\mathbb{A}_0$ fails the subalgebra condition in X, and therefore $\mathbb{A}_0$ is not a subalgebra of X.

Example 3.12. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	1	0	5	0	1	0
1	1	0	1	0	3	0	4
2	2	1	0	5	2	1	6
3	3	0	5	0	0	0	0
4	4	3	2	1	0	0	0
5	5	1	0	0	0	0	0
6	6	4	6	0	1	1	0

Thus, $X = (X, \star, 0)$ forms a GO-algebra. Consider the subset $\mathbb{A}_s = \{0, 1, 3\}$. We have

$$0 \in \mathbb{A}_s \text{ and } 3 \in \mathbb{A}_s, \text{ but } 0 \star 3 = 5 \notin \mathbb{A}_s,$$

so $\mathbb{A}_s$ fails to be closed under $\star$ and therefore is not a subalgebra of X.

Example 3.13. *Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6
0	0	1	0	0	4	0	4
1	1	0	1	3	3	4	5
2	2	1	0	2	6	0	4
3	3	0	2	0	0	0	0
4	4	3	6	1	0	0	0
5	5	4	0	0	1	0	0
6	6	5	0	0	0	1	0

Then $X = (X, \star, 0)$ is a GO-algebra. Consider the subset $\mathbb{A}_s = \{0, 1, 4, 6\}$. We note that

$$(0 \star 3) \star (0 \star 4) = 0 \star 4 = 4 \in \mathbb{A}_s \textit{ and } 4 \in \mathbb{A}_s, \textit{ but } 3 \notin \mathbb{A}_s,$$

so $\mathbb{A}_s$ is not an ideal of X.

3.3. **Zero- and Self-Annihilating Subsets of IO-Algebras.** In IO-algebras, the notions of zero- and self-annihilation are extended under the independent obic framework, where the operation $\star$ interacts with the zero element 0 in a more flexible manner. These subsets reveal how independence between obic and generalized identities affects internal algebraic behavior, providing a distinct perspective compared with those in O- and GO-algebras.

Theorem 3.9. *Let X be an IO-algebra. Then $\mathbb{A}_0 = \{0\}$.*

Proof. Let $y \in \mathbb{A}_0$. Then

$$\begin{aligned} y &= y \star 0 && \text{(by (O-1))} \\ &= 0 \star (0 \star y) && \text{(by (IO-1))} \\ &= 0 \star 0 && \text{(by } \mathbb{A}_0\text{'s property)} \\ &= 0. && \text{(by (O-3))} \end{aligned}$$

Hence, $\mathbb{A}_0 = \{0\}$. □

By Remark 3.2 and Theorem 3.9, we obtain the corollary.

Corollary 3.3. *$\mathbb{A}_0$ is a subalgebra of an IO-algebra.*

By Remark 3.3 and Theorem 3.9, we obtain the corollary.

Corollary 3.4. *$\mathbb{A}_0$ is an ideal of an IO-algebra.*

Example 3.14. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6	7
0	0	1	2	4	3	6	5	7
1	1	0	3	2	0	7	2	2
2	2	4	0	3	4	0	1	1
3	3	2	4	0	2	0	0	0
4	4	0	3	2	0	0	0	0
5	5	7	0	0	0	0	1	1
6	6	2	1	0	0	1	0	0
7	7	2	1	0	0	1	0	0

Thus, $X = (X, \star, 0)$ forms an IO-algebra. Consider $\mathbb{A}_s = \{0, 1, 2, 7\}$. Although

$$1 \in \mathbb{A}_s \text{ and } 2 \in \mathbb{A}_s, \text{ but } 1 \star 2 = 3 \notin \mathbb{A}_s.$$

Hence, $\mathbb{A}_s$ fails to be closed under $\star$ and therefore is not a subalgebra of X.

Example 3.15. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\star$	0	1	2	3	4	5	6	7
0	0	1	3	2	6	5	4	7
1	1	0	7	0	5	4	0	2
2	2	7	0	4	1	0	0	3
3	3	0	6	0	0	0	2	2
4	4	5	1	0	0	6	2	0
5	5	6	0	0	4	0	6	0
6	6	0	0	3	3	4	0	0
7	7	3	2	3	0	0	0	0

Then $X = (X, \star, 0)$ is an IO-algebra. Consider the subset $\mathbb{A}_s = \{0, 1, 5, 7\}$. We note that

$$(0 \star 4) \star (0 \star 7) = 6 \star 7 = 0 \in \mathbb{A}_s \text{ and } 7 \in \mathbb{A}_s, \text{ but } 4 \notin \mathbb{A}_s,$$

so $\mathbb{A}_s$ is not an ideal of X.

The following table summarizes the structural behaviors of $\mathbb{A}_0$ and $\mathbb{A}_s$ within the three algebraic systems discussed above. It highlights their respective roles as subalgebras or ideals and clarifies the distinctions in their closure properties across O-, GO-, and IO-algebras.

Table 1. Summary of the structural behaviors of $\mathbb{A}_0$ and $\mathbb{A}_s$ within O-, GO-, and IO-algebras.

Algebraic System	$\mathbb{A}_0$	$\mathbb{A}_s$
O-algebra	$\{0\}$	Ideal (also a subalgebra)
GO-algebra	Ideal	Cannot be generally guaranteed to form a subalgebra or an ideal
IO-algebra	$\{0\}$	Cannot be generally guaranteed to form a subalgebra or an ideal

4. Conclusion

In this paper, we have introduced a unified framework extending the classical obic algebra (O-algebra) into two broader algebraic systems, namely the generalized obic algebra (GO-algebra) and the independent obic algebra (IO-algebra). By combining essential identities from O- and GO-algebras with an additional independent axiom, the IO-algebra was shown to form a structurally autonomous yet algebraically consistent system situated between the two. Through comparative analysis, it was established that GO-algebras constitute the most general class encompassing both O- and IO-algebras, thereby clarifying their hierarchical and structural relationships. Furthermore, two special subsets—the zero-annihilating subset and the self-annihilating subset—were defined and examined, providing deeper insight into the internal interactions among elements in these algebraic systems. Their conditions for forming subalgebras and ideals were characterized, revealing new insights into the substructural behavior of obic-type algebras. Overall, the results obtained not only strengthen the theoretical foundation of implication-based algebraic systems but also open avenues for further research on generalizations, homomorphic properties, and potential applications in non-classical logic and information theory.

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