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TRAIN ALGEBRAS OF DEGREE 2 AND EXPONENT 4 THAT ARE PRINCIPAL TRAIN OF RANK 4

WENDIN-GOUDI ACHILE ZANGRE[1], SIDMANAGDÉ EMILE OUEDRAOGO[2,*], PAUL BEREMWIDOUGOU[2], ANDRÉ CONSEIBO[2]

[1]Ecole Polytechnique de Ouagadougou, 18 BP 234, Ouagadougou, Burkina Faso

[2]Département de Mathématiques, Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso

[*]Corresponding author: andreconsebo@yahoo.fr

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Abstract. The structure of train algebras of degree 2 and exponent 4 have been studied in [8]. The objective of this paper is to describe the structure of the algebras of this class which are principal train of rank 4.

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1. Introduction

Principal train algebras were introduced by Etherington around 1939 in the dynamics of population genetic modeling using algebraic structures. Train algebras of degree 2 and exponent $n \geq 3$, i.e baric algebras satisfying the ω-polynomial identity $(x^n)^2 - \omega(x)^n x^n = 0$ have been partially studied in [3].The case $n = 4$ has been investigated in [8].The purpose of this paper is to study the structure of a subclass of algebras satisfying a polynomial identity $(x^4)^2 - \omega(x)^4 x^4$ that are principal train algebras of rank 4.

2. Preliminaries

Definition 2.1. *A commutative algebra A over a field K is said to be:*

i) *power-associative if $x^i x^j = x^{i+j}$ for any x in A and for all integer $i, j \geq 1$;*

ii) *Jordan if $(x^2 y)x - x^2(yx) = 0$, for all $x, y \in A$;*

iii) *baric if it exists $\omega : A \to K$ a nonzero morphism of algebras;*

It should be noted that every Jordan algebra is power-associative ([6]).

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Definition 2.2. *Let (A, ω) be a baric algebra over a field K. The algebra A is a principal train algebra of rank $n \geq 2$, if there are $\alpha_1, \cdots, \alpha_{n-1} \in K$ such that $x^n + \alpha_1\omega(x)x^{n-1} + \cdots + \alpha_{n-1}\omega(x)^{n-1}x = 0$, $\forall x \in A$, where n is such a smallest integer.*

In this paper, A is a finite-dimensional train $\mathbb{C}$-algebra of degree 2 and exponent 4, i.e a commutative not necessarily associative algebra satisfying the ω-polynomial identity $(x^4)^2 - \omega(x)^4x^4 = 0$, where $\mathbb{C}$ is the field of complex numbers, $\omega : A \to \mathbb{C}$ an nonzero morphism of algebras. The authors in [3], showed that A admits a nonzero idempotent. In this section, we give some results on the structure of train algebra of degree 2 and exponent 4.

Proposition 2.1. ([8, *Proposition 2.2*]) *The algebra A has a Peirce decomposition as follows: $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$, where $A_\alpha = \{y \in \ker(\omega)/ey = \alpha y\}$, with $\alpha \in \{0, \frac{1}{2}, \lambda = -\frac{1+i\sqrt{7}}{4}, \overline{\lambda} = \frac{-1+i\sqrt{7}}{4}\}$.*

The products of the Peirce components are given by the following theorem.

Theorem 2.1. ([8, *Theorem 2.5*])

i) $A_0^2 \subset A_0 \oplus A_{1/2}$;
ii) $A_{1/2}^2 \subset A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$;
iii) $A_\lambda^2 \subset A_{1/2}$;
iv) $A_{\overline{\lambda}}^2 \subset A_{1/2}$;
vi) $A_{1/2}A_0 \subset A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$;
vii) $A_0A_\lambda \subset A_{1/2}$;
viii) $A_0A_{\overline{\lambda}} \subset A_{1/2}$;
ix) $A_{1/2}A_\lambda \subset A_{1/2} \oplus A_0 \oplus A_{\overline{\lambda}}$;
x) $A_{1/2}A_{\overline{\lambda}} \subset A_{1/2} \oplus A_0 \oplus A_\lambda$;
xi) $A_\lambda A_{\overline{\lambda}} \subset A_{1/2}$.

In [8], the authors showed that the dimensions of Peirce components are independent of the choice of the idempotent in the Peirce decomposition of A and give the following definition.

Definition 2.3. *Let $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition of A. The type of A is defined by $(1 + r, s, t_1, t_2)$ where r, s, t_1, t_2 denote respectively the dimensions of the vector spaces $A_{1/2}, A_0, A_\lambda, A_{\overline{\lambda}}$ and $1 + r + s + t_1 + t_2$ is the dimension of A (see [8]).*

Proposition 2.2. *A train algebra of degree 2 and exponent 4 is a principal train algebra of rank 4 if and only if its train equation is one of the following forms:*

i) $x^4 - (1 + \alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$ (1), *where* $\alpha \in \{0, \lambda, \overline{\lambda}\}$;
ii) $x^4 - \frac{3+2\alpha}{2}\omega(x)x^3 + \frac{1+3\alpha}{2}\omega(x)^2x^2 - \frac{\alpha}{2}\omega(x)^3x = 0$ (2), *where* $\alpha \in \{0, \frac{1}{2}, \lambda, \overline{\lambda}\}$;
iii) $x^4 - (1 + 2\alpha)\omega(x)x^3 + (\alpha^2 + 2\alpha)\omega(x)^2x^2 - \alpha^2\omega(x)^3x = 0$ (3), *where* $\alpha \in \{\lambda, \overline{\lambda}\}$;

iv) $x^4 - \frac{1}{2}\omega(x)x^3 - \frac{1}{2}\omega(x)^3 x = 0$ (4).

Proof. Let $A = \mathbb{C}e \oplus A_0 \oplus A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$, an algebra satisfying $(x^4)^2 = \omega(x)^4 x^4$ that is principal train of rank 4. Then the train equation of A is of the form $x^4 - (1+\alpha+\beta)\omega(x)x^3 + \alpha\omega(x)^2x^2 + \beta\omega(x)^3 x = 0$ where $\alpha, \beta \in \mathbb{C}$. According to Zariski's topology, the set $H = \{x \in A, \omega(x) = 1\}$ is dense in A, so identity $x^4 - (1+\alpha+\beta)\omega(x)x^3 + \alpha\omega(x)^2x^2 + \beta\omega(x)^3x = 0$ for all $x \in A$ is equivalent to $x^4 - (1+\alpha+\beta)x^3 + \alpha x^2 + \beta x = 0$ for all $x \in H$. Let $P(X) = X^4 - (1+\alpha+\beta)X^3 + \alpha X^2 + \beta X = X(X-1)(X-\alpha_1)(X-\alpha_2)$ be the train polynomial of A, where $\alpha_1, \alpha_2 \in \mathbb{C}$. Thus $\alpha = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$ and $\beta = -\alpha_1\alpha_2$. We consider the following four cases:

1^{st} **Case**: $\alpha_1 \neq \frac{1}{2}, \alpha_2 \neq \frac{1}{2}$

According to ([5], Theorem 5) and Proposition 2.1, A has a Peirce decomposition as follows $A = \mathbb{C}e \oplus A_{1/2} \oplus A_{\alpha_1} \oplus A_{\alpha_2}$ with $\alpha_1, \alpha_2 \in \{0, \lambda, \overline{\lambda}\}$.

- If $\alpha_1 = 0$ and $\alpha_2 = \lambda$ then $\alpha = \lambda$, $\beta = 0$, and A satisfies the train equation $x^4 - (1+\lambda)\omega(x)x^3 + \lambda\omega(x)^2x^2 = 0$;
- If $\alpha_1 = 0$ and $\alpha_2 = \overline{\lambda}$ then $\alpha = \overline{\lambda}$, $\beta = 0$, and A satisfies the train equation $x^4 - (1+\overline{\lambda})\omega(x)x^3 + \overline{\lambda}\omega(x)^2x^2 = 0$;
- If $\alpha_1 = \lambda$ and $\alpha_2 = \overline{\lambda}$ then $\alpha = 0$, $\beta = -\frac{1}{2}$, and A satisfies the train equation $x^4 - \frac{1}{2}\omega(x)x^3 - \frac{1}{2}\omega(x)^3x = 0$;

2^{nd} **Case**: $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \frac{1}{2}$

According to Theorem 1 from ([4]) and Proposition 2.1, A has the following decomposition $A = \mathbb{C}e \oplus A_{1/2} \oplus A_{\alpha_2}$ with $\alpha_2 \in \{0, \lambda, \overline{\lambda}\}$.

- If $\alpha_2 = 0$ then $\alpha = \frac{1}{2}$ and $\beta = 0$; thus A satisfies the train equation $x^4 - \frac{3}{2}\omega(x)x^3 + \frac{1}{2}\omega(x)^2x^2 = 0$;
- If $\alpha_2 = \lambda$ then $\alpha = \frac{1+3\lambda}{2}$ and $\beta = -\frac{\lambda}{2}$; thus A satisfies the train equation $x^4 - \frac{3+2\lambda}{2}\omega(x)x^3 + \frac{1+3\lambda}{2}\omega(x)^2x^2 - \frac{\lambda}{2}\omega(x)^3x = 0$;
- If $\alpha_2 = \overline{\lambda}$ then $\alpha = \frac{1+3\overline{\lambda}}{2}$ and $\beta = -\frac{\overline{\lambda}}{2}$; thus A satisfies the train equation $x^4 - \frac{3+2\overline{\lambda}}{2}\omega(x)x^3 + \frac{1+3\overline{\lambda}}{2}\omega(x)^2x^2 - \frac{\overline{\lambda}}{2}\omega(x)^3x = 0$.

3^{rd} **Case**: $\alpha_1 = \alpha_2 \neq \frac{1}{2}$

From Theorem 1 in ([4]) and Proposition 2.1, A has a nonzero idempotent and a Peirce decomposition $A = \mathbb{C}e \oplus A_{1/2} \oplus B$, where $B = \ker(\ell_e - \alpha_1 Id)^2$ and $\ell_e : \ker\omega \to \ker\omega, x \mapsto ex$. If $B = 0$ then A satisfies the train equation $x^2 = \omega(x)x$, which is a contradiction as A is of rank 4.

We then have the following possibilities:

- If $\alpha_1 = \alpha_2 = 0$, A satisfies $x^4 - \omega(x)x^3 = 0$;
- If $\alpha_1 = \alpha_2 = \lambda$, then $\alpha = \lambda^2 + 2\lambda$, $\beta = -\lambda^2$, and A satisfies $x^4 - (1+2\lambda)\omega(x)x^3 + (\lambda^2 + 2\lambda)\omega(x)^2x^2 - \lambda^2\omega(x)^3x = 0$;

- If $\alpha_1 = \alpha_2 = \overline{\lambda}$, then $\alpha = \overline{\lambda}^2 + 2\overline{\lambda}$, $\beta = -\overline{\lambda}^2$, and A satisfies $x^4 - (1+2\overline{\lambda})\omega(x)x^3 + (\overline{\lambda}^2 + 2\overline{\lambda})\omega(x)^2x^2 - \overline{\lambda}^2\omega(x)^3x = 0$.

4^{th} **Case**: $\alpha_1 = \alpha_2 = \frac{1}{2}$

In this case, $\alpha = \frac{5}{4}$, $\beta = -\frac{1}{4}$, and A satisfies the train equation $x^4 - 2\omega(x)x^3 + \frac{5}{4}\omega(x)^2x^2 - \frac{1}{4}\omega(x)^3x = 0$. □

3. Power associativity

In [2], authors showed that the train equation of power-associative principal train algebra of rank 4 is of form $x^4 - \omega(x)x^3 = 0$, $x^4 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$ or $x^4 - 3\omega(x)x^3 + 3\omega(x)^2x^2 - \omega(x)^3x = 0$. We show that in our case, the equation is necessarily $x^4 - \omega(x)x^3 = 0$.

Corollary 3.1. *Any train algebra of degree 2 and exponent 4 is power-associative principal train algebra of rank 4 if and only if its train equation is* $x^4 - \omega(x)x^3 = 0$.

Proof. Indeed, let A be a train algebra of degree 2 and exponent 4 which is power-associative. Then, A satisfies the train equation $x^8 - \omega(x)^4x^4 = 0$. By direct calculation, it can be shown that if A satisfies one of the other two equations then it satisfies a train equation of rank 3. □

In [1], the authors show that such algebras are Jordan algebras because its satisfy the equation $(x^2)^2 - \omega(x)x^3 = 0$.

However, there exist train algebras of rank 4 and train equation $x^4 - \omega(x)x^3 = 0$ which are not power-associative as shown in the following example.

Example 3.1. [1] *Let* $(A =< e, a, b, c >, \omega)$ *where* $\omega : A \to \mathbb{C}$ *is a morphism of algebras such that* $\omega(e) = 1$, $\omega(a) = \omega(b) = \omega(c) = 0$, *and* A *a commutative* $\mathbb{C}$*-algebra with multiplication table given by:* $e^2 = e, ea = \frac{1}{2}a, eb = c, b^2 = c$, *non mentioned products being zero. By a simple calculation, we can observe that* A *satisfies* $x^4 - \omega(x)x^3 = 0$ *but does not satisfy the identities* $(x^2)^2 - x^4 = 0$ *and* $(x^4)^2 - \omega(x)^4x^4 = 0$.

In the rest of the paper our study will focus on the class of principal train algebras of equation (1).

4. Study of equation (1) for $\alpha = 0$

In this section, A is a train algebra of degree 2 and exponent 4 that is a principal train algebra of rank 4 with train equation $x^4 - \omega(x)x^3 = 0$. A partial linearization of identity $x^4 - \omega(x)x^3 = 0$ gives $x^3y + x(x^2y) + 2x(x(xy)) - \omega(y)x^3 - \omega(x)(x^2y + 2x(xy)) = 0$. Setting $x = e$ and $y \in \ker\omega$, we have $(2\ell_e^3 - \ell_e^2)(y) = 0$ where $\ell_e := L_e/\ker\omega$ and $L_e : A \to A, x \mapsto ex$. Therefore $2X^3 - X^2$ is the minimal annihilator polynomial of ℓ_e and A admits a following Peirce decomposition $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0'$ with $A_{1/2} = \{x \in \ker\omega, 2ex = x\}$ and $A_0' = \{x \in \ker\omega, e(ex) = 0\}$

Using Proposition 2.1 and Theorem 2.1, we have $A_0' = A_0 = \{x \in \ker \omega, ex = 0\}$ and the following result.

Theorem 4.1. *We have* $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0$ *and*

i) $A_0^2 \subset A_0$;

ii) $A_{1/2}^2 \subset A_0$;

iii) $A_{1/2}A_0 \subset A_{1/2}$.

Proof. Indeed, if we consider Proposition 2.1 and Theorem 2.1, we have $A_0' = A_0 = \{x \in \ker \omega, ex = 0\}$ and therefore $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0$. We have also that $A_0^2 \subset A_0 \oplus A_{1/2}$, $A_{1/2}^2 \subset A_0$ and $A_{1/2}A_0 \subset A_{1/2}$. Let $x = e + \beta x_0$ in A with $x_0 \in A_0$. The equality $0 = x^4 - \omega(x)x^3 = -\frac{1}{2}\beta^2 e x_0^2 + \beta^3(ex_0^3 + x_0(ex_0^2) - x_0^3) + \beta^4 x_0^4$ implies that $ex_0^2 = 0$ and $A_0^2 \subset A_0$. □

Lemma 4.1. *Let* $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0$ *be a Peirce decomposition of* A*, we have the following identities:*

i) $2x_{1/2}(x_0x_{1/2}) - x_0x_{1/2}^2 = 0$;

ii) $x_{1/2}(x_{1/2}(x_0x_{1/2})) = 0$;

iii) $x_{1/2}^3 = 0$;

iv) $(x_{1/2}^2)^2 = 0$;

v) $(x_0^2)^2 = 0$;

vi) $x_0^3 = 0$;

vii) $x_0(x_0x_{1/2}) = 0$;

viii) $x_{1/2}x_0^2 = 0$;

ix) $x_0^2x_{1/2}^2 = 0$;

x) $(x_0x_{1/2})^2 = 0$;

xi) $x_0(x_{1/2}(x_0x_{1/2})) = 0$;

xii) $x_{\frac{1}{2}}^2(x_0x_{\frac{1}{2}}) = 0$;

xiii) $x_0^2(x_{\frac{1}{2}}x_0) = 0$.

Proof. The proof is done by setting $x = e + \beta x_0 + \mu x_{1/2}$ respectively in identities $(x^4)^2 - \omega(x)^4x^4 = 0$ and $x^4 - \omega(x)x^3 = 0$. The identification of coefficients of $\mu^i\beta^j$, for $1 \leq i + j \leq 4$ in each identity. □

A principal train algebra of rank 4 and equation $x^4 - \omega(x)x^3 = 0$ that satisfies the identity $(x^4)^2 - \omega(x)^4x^4 = 0$ is Jordan algebra, therefore it is power-associative as the following result indicates.

Proposition 4.1. *Let* A *be a train algebra of degree* 2 *and exponent* 4 *which is principal train of rank* 4 *and equation* $x^4 - \omega(x)x^3 = 0$*, then* A *is Jordan.*

Proof. Let $A = \mathbb{C}e \oplus A_0 \oplus A_{1/2}$ a Peirce decomposition of A. Let $x = e + x_0 + x_{1/2}$ be an element of A of weight 1, i.e such that $\omega(x) = 1$. Using identities of Lemma 4.1, we have $x^2 = e + x_{1/2} + x_{1/2}^2 + x_0^2 + 2x_0x_{1/2}$,

$x^3 = e + x_{1/2} + x_{1/2}^2 + 2x_0x_{1/2} + 2x_0x_{1/2}^2$ and $(x^2)^2 = e + x_{1/2} + x_{1/2}^2 + 2x_0x_{1/2} + 2x_0x_{1/2}^2$, so $(x^2)^2 = x^3$. According to Zariski's topology, the set of elements of A of weight 1 is dense in A, thus $(x^2)^2 - \omega(x)x^3 = 0$, $\forall x \in A$. In [1], the authors show that a such algebra is Jordan algebra. □

Proposition 4.2. *Let $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0$ be a Peirce decomposition of A. If $A_0^2 = 0$, then, for any $x \in A$, we have $x^3 - \omega(x)x^2 = 0$.*

Proof. Let $x = e + x_{1/2} + x_0$ in A. Suppose that $A_0^2 = 0$, according to Lemma 4.1, we have $x_{1/2}(x_0x_{1/2}) = 0$, $x_{1/2}^3 = 0$ and $x_0(x_0x_{1/2}) = 0$. Thus, $x^2 = e + x_{1/2} + x_{1/2}^2 + 2x_0x_{1/2}$, $x^2 - x = x_{1/2}^2 + 2x_0x_{1/2} - x_0$ and $x^3 - x^2 = 0$. □

Remark 4.1. *According to Proposition 4.2, for a train algebra of degree 2 and exponent 4 that principal train with train equation $x^4 - \omega(x)x^3 = 0$, the condition $A_0^2 = 0$ cannot be satisfied.*

We give the classification in dimension at most four.

Remark 4.2. *If A is train algebra of degree 2 and exponent 4 which is principal train of equation $x^4 - \omega(x)x^3 = 0$, the type of A is $(1 + r, s, 0, 0)$.*

Proposition 4.3. *Every algebra of this class is of dimension at least 3.*

Proof. According to Proposition 4.2, $A_0 \neq 0$ and $s \geq 1$, therefore the dimension of A is $1 + r + s \geq 2$. Suppose that A is two-dimensional, thus $r = 0$. We can set $A =< e, v >$ where $e^2 = e$ and $A_0 =< v >$. Then, $ev = 0, v^2 = av$ where $a \in \mathbb{C}$. We have $0 = v^3 = a^2v$, thus $a = 0$ and $v^2 = 0$ which contradicts the Proposition 4.2, A is of dimension at least 3. □

Theorem 4.2. *Up to isomorphism, there exists only one three-dimensional train algebra of degree 2 and exponent 4 that is principal train of equation $x^4 - \omega(x)x^3 = 0$: $A =< e, e_1, e_2 >$ with multiplication table $e^2 = e, e_1^2 = e_2$, the other products being zero.*

Proof. Assume that A is three-dimensional. Since the dimension of A_0 is at least 1, then the type of A is $(1, 2, 0, 0)$ or $(2, 1, 0, 0)$. If type of A is $(2, 1, 0, 0)$, setting $A =< e, u, v >$ with $e^2 = e$, $A_0 =< v >$ and $A_{1/2} =< u >$. We have $v^2 = \beta v$ with $\beta \in \mathbb{C}$. Then $0 = v^3 = \beta^2 v$, so $\beta = 0$ and $v^2 = 0$ which is impossible according to Proposition 4.2 Therefore, the type of A is $(1, 2, 0, 0)$. Setting $A =< e, v_1, v_2 >$ where $e^2 = e$ and $A_0 =< v_1, v_2 >$. We have $ev_1 = ev_2 = 0$, $v_1^2 = a_1v_1 + a_2v_2$, $v_2^2 = b_1v_1 + b_2v_2$, $v_1v_2 = c_1v_1 + c_2v_2$ where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{C}$. We have $0 = v_1^3 = a_1^2v_1 + a_1a_2v_2 + a_2c_1v_1 + a_2c_2v_2$, $0 = v_2^3 = b_2b_1v_1 + b_2^2v_2 + b_1c_1v_1 + b_1c_2v_2$, $0 = v_1^2v_2 + 2v_1(v_1v_2) = (3c_1a_1 + 2c_1c_2 + a_2b_1)v_1 + (c_2a_1 +$

$2c_2^2 + a_2b_2 + 2a_2c_1)v_2$, $0 = v_2^2v_1 + 2v_2(v_1v_2) = (b_1a_1 + c_1b_2 + 2c_1^2 + 2c_2b_1)v_1 + (3b_2c_2 + a_2b_1 + 2c_1c_2)v_2$,

$$\text{so}\begin{cases} a_1^2 + a_2c_1 & = 0 \quad (1)\\ a_1a_2 + a_2c_2 & = 0 \quad (2)\\ b_2b_1 + b_1c_1 & = 0 \quad (3)\\ b_2^2 + b_1c_2 = 0 & = 0 \quad (4)\\ 3c_1a_1 + 2c_1c_2 + a_2b_1 & = 0 \quad (5)\\ c_2a_1 + 2c_2^2 + a_2b_2 + 2a_2c_1 & = 0 \quad (6)\\ a_1b_1 + b_2c_1 + 2c_1^2 + a_2b_2 + 2c_2b_1 & = 0 \quad (7)\\ 3b_2c_2 + a_2b_1 + 2c_1c_2 & = 0 \quad (8) \end{cases}$$

From (2), we have $a_2 = 0$ or $a_1 = -c_2$.

- Suppose that $a_2 = 0$, then (1), (4), (5), (6), (7) and (8) imply that $a_1 = 0$, $c_2 = 0$, $b_2 = 0$, $c_1 = 0$. Therefore $v_1^2 = 0$, $v_2^2 = b_1v_1$, $v_1v_2 = 0$. by a change of variable, the multiplication table of A becomes: $e^2 = e$, $v_2^2 = v_1$, non mentioned products being zero.
- If $a_2 \neq 0$, $v_1^2 \neq 0$. Then (v_1, v_1^2) is a basis of A_0. We can set $v_2 = v_1^2$. Thus $v_1v_2 = v_1^3 = 0$, $v_2^2 = (v_1^2)^2 = 0$ and the multiplication table of A becomes: $e^2 = e$, $v_1^2 = v_2$, non mentioned products being zero.

□

Theorem 4.3. *Up to isomorphism, we have six four-dimensional train algebras of degree 2 and exponent 4 that are principal train of equation $x^4 - \omega(x)x^3 = 0$. Their multiplication tables are as follows, where the products not mentioned are zero:*

$T_1(a) : e^2 = e, ee_1 = \frac{1}{2}e_1, e_2^2 = e_3, e_1^2 = ae_3$, $a \in \mathbb{C}^\star$, $T_1(a) \simeq T_1(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2$;

$T_2 : e^2 = e, ee_1 = \frac{1}{2}e_1, e_2^2 = e_3$;

$T_3 : e^2 = e, e_1^2 = e_2$;

$T_4(a) : e^2 = e, e_1^2 = e_2, e_3^2 = ae_2$, $a \in \mathbb{C}^\star$, $T_4(a) \simeq T_4(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2$;

$T_5 : e^2 = e, e_1^2 = e_2, e_1e_3 = e_2$;

$T_6(a) : e^2 = e, e_1^2 = e_2, e_3^2 = ae_2, e_1e_3 = e_2$, $a \in \mathbb{C}^\star$, $T_6(a) \simeq T_6(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2$.

Proof. Assume that A is four-dimensional. Since the dimension of A_0 is at least 1, then the type of A is $(3,1,0,0)$, $(2,2,0,0)$ or $(1,3,0,0)$. If the type of A is $(3,1,0,0)$, then $A_0^2 = 0$, that is impossible. Let us examine the two remaining cases.

i) Case 1: type of A is $(2,2,0,0)$. Then, $A =< e, u, v_1, v_2 >$ with $A_{1/2} =< u >$ and $A_0 =< v_1, v_2 >$. Since $A_0^2 \neq 0$, it exists a basis (v_1, v_2) of A_0 such that $v_1^2 \neq 0$. We can set $v_2 = v_1^2$ and we have $eu = \frac{1}{2}u, ev_1 = ev_2 = 0$, $u^2 = \gamma v_1 + \eta v_2$, $uv_1 = \mu u$, $uv_2 = uv_1^2 = 0$, $v_2^2 = (v_1^2)^2 = 0$, $v_1v_2 = v_1^3 = 0$, $\gamma, \mu, \eta \in \mathbb{C}$. We have, $0 = v_1(uv_1) = \mu^2u$, so $\mu = 0$. Since $0 = (u^2)^2 = \gamma^2v_2$, then $\gamma = 0$. Therefore, the multiplication table of A is one of the following:

$e^2 = e, eu = \frac{1}{2}u, v_1^2 = v_2$, the others products being zero;

$e^2 = e, eu = \frac{1}{2}u, v_1^2 = v_2$, $u^2 = \eta v_2$, the others products being zero;

ii) Case 2: type of A is $(1,3,0,0)$. Then $A =< e, v_1, v_2, v_3 >$ with $A_{1/2} = 0$ and $A_0 =< v_1, v_2, v_3 >$. Since $A_0^2 \neq 0$, It exists a basis such that $v_1^2 \neq 0$. Setting $v_2 = v_1^2$. We have $ev_1 = ev_2 = ev_3 = 0$, $v_1^2 = v_2$, $v_2^2 = (v_1^2)^2 = 0$, $v_1v_2 = v_1^3 = 0$. Setting $v_3^2 = a_1v_1 + a_2v_2 + a_3v_3$, $v_1v_3 = b_1v_1 + b_2v_2 + b_3v_3$, $v_2v_3 = c_1v_1 + c_2v_2 + c_3v_3$, $a_i, b_i, c_i \in \mathbb{C}$ for $i \in \{1,2,3\}$. Using the equalities $v_3^3 = 0$, $(v_3^2)^2 = 0$, $v_1^2v_3 + 2v_1(v_1v_3) = 0$, $v_1^2v_3^2 + 2(v_1v_3)^2 = 0$, $v_2(v_1v_3) = 0$, $v_2(v_2v_3) = 0$, $v_1(v_2v_3) = 0$,$v_3^2(v_1v_3) = 0$, $v_3^2(v_2v_3) = 0$, $v_3^2v_1 + 2v_3(v_3v_1) = 0$, $v_3^2v_2 + 2v_3(v_3v_2) = 0$, $(v_2v_3)^2 = 0$, $2(v_1v_3)^2 + v_2v_3^2 = 0$, we obtain $a_1 = a_3 = b_1 = b_3 = c_1 = c_2 = c_3 = 0$ and $v_3^2 = a_2v_2$, $v_1v_3 = b_2v_2$, $v_2v_3 = 0$.

□

5. Study of equation (1) for $\alpha \in \{\lambda, \overline{\lambda}\}$

In this section, A is a train of degree 2 and exponent 4 that is principal train algebra of rank 4 with train equation $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$, where $\alpha \in \{\lambda, \overline{\lambda}\}$. A partial linearization of identity (1) gives $x^3y + x(x^2y) + 2x(x(xy)) - (1+\alpha)\omega(y)x^3 - (1+\alpha)\omega(x)(x^2y + 2x(xy)) + 2\alpha\omega(x)^2xy + 2\alpha\omega(xy)x^2 = 0$. Setting $x = e$ and $y \in \ker\omega$ in this last identity, we have $[2\ell_e^3 - (1+2\alpha)\ell_e^2 + \alpha\ell_e](y) = 0$ where $\ell_e : \ker\omega \to \ker\omega, x \mapsto ex$. Therefore $2X^3 - (2\alpha+1)X^2 + \alpha X$ is the minimal annihilator polynomial of ℓ_e and A admits a following Peirce decomposition $A = \mathbb{C}e \oplus A_0 \oplus A_{1/2} \oplus A_\alpha$ where $A_\mu = \{x \in \ker\omega, ex = \mu x\}$ with $\mu \in \{0, 1/2, \alpha\}$. The assertion $i)$ of the following result is obtain by direct calculation. Using Proposition 2.1 and Theorem 2.1, we obtain assertions $ii)$ to $vi)$.

Corollary 5.1. *We have $A = \mathbb{C}e \oplus A_0 \oplus A_{1/2} \oplus A_\alpha$ and*

i) $A_0^2 = 0$;
ii) $A_{1/2}^2 \subset A_0 \oplus A_\alpha$;
iii) $A_\alpha^2 \subset A_{1/2}$;
iv) $A_0A_{1/2} \subset A_{1/2} \oplus A_\alpha$;
v) $A_0A_\alpha \subset A_{1/2}$;
vi) $A_{1/2}A_\alpha \subset A_0 \oplus A_{1/2}$.

Lemma 5.1. *Let $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0 \oplus A_\alpha$ be a Peirce decomposition of a train algebra of degree 2 and exponent 4 which is train of rank 4 and equation $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$. If $A_\alpha = 0$, then, for all $x_0 \in A_0$ and $x_{1/2} \in A_{1/2}$, we have the following identities:*

i) $x_0^2 = 0$;
ii) $x_{1/2}^3 = 0$;
iii) $x_0(x_0x_{1/2}) = 0$;
iv) $x_{1/2}(x_0x_{1/2}) = 0$;

v) $(x_0x_{1/2})^2 = 0$.

Proof. The proof is done by setting $x = e + \beta x_0 + \mu x_{1/2}, \beta, \mu \in \mathbb{C}$ respectively in identities $(x^4)^2 - \omega(x)^4x^4 = 0$ (see)and $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$. The identification of coefficients of $\mu^i\beta^j$, for $1 \leq i + j \leq 4$ in each identity. □

Lemma 5.2. *Let $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0 \oplus A_\alpha$ be a Peirce decomposition of a train algebra of degree 2 and exponent 4 which is train of rank 4 and equation $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$. If $A_0 = 0$, then, for all $x_\alpha \in A_\alpha$ and $x_{1/2} \in A_{1/2}$, we have the following identities:*

i) $x_\alpha^2 = 0$;

ii) $x_{1/2}^3 = 0$;

iii) $(x_{1/2}^2)^2 = 0$;

iv) $x_{1/2}(x_\alpha x_{1/2}) = 0$;

v) $x_\alpha(x_\alpha x_{1/2}) = 0$.

Proof. These identities follow by the same argument as in the previous lemma. □

Proposition 5.1. *Let $A = \mathbb{C}e \oplus A_{1/2} \oplus A_0 \oplus A_\alpha$ be a Peirce decomposition of a train algebra of degree 2 and exponent 4 which is principal train of rank 4 and equation $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$.*

i) *If $A_0 = 0$, then A satisfies the identity $x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2x = 0$.*

ii) *If $A_\alpha = 0$, A satisfies the equation $x^3 - \omega(x)x^2 = 0$.*

Proof. By direct calculation, using Lemma 5.2, we establish the assertion $i)$ and Lemma 5.1 allows us to obtain $ii)$. □

We give now the classification in dimension at most four.

Remark 5.1. *If A is train algebra of degree 2 and exponent 4 which is principal train of equation $x^4 - (1+\alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$, the type of A is $(1+r, s, t, 0)$ or $(1+r, s, 0, t)$ where r, s, t are respectively the dimensions of $A_{1/2}$, A_0 and A_α. We can verify that the results obtained in the case of the type $(1+r, s, t, 0)$ are similar to those in the case $(1+r, s, 0, t)$. Assuming that the type of A is $(1+r, s, t, 0)$.*

Proposition 5.2. *Every algebra in this class is of dimension at least 3.*

Proof. Since $A_\alpha \neq 0$ and $A_0 \neq 0$, then $t \geq 1$ and $s \geq 1$, so $1 + r + s + t \geq 3$. □

Theorem 5.1. *There exists, up to isomorphism, only one 3-dimensional algebra: $e^2 = e, ee_2 = \alpha e_2$, the other products being zero.*

Proof. Suppose that A is three-dimensional, then $r + s + t = 2$ and the type of A is $(1, 1, 1, 0)$. Setting $A =< e, e_1, e_2)$ with $A_{1/2} = 0$, $A_0 =< e_1 >$ and $A_\alpha =< e_2 >$, then $e^2 = e, ee_2 = \alpha e_2, ee_1 = e_1^2 = e_2^2 = e_1e_2 = 0$. □

Theorem 5.2. *Up to isomorphism, the four-dimensional train algebras of this class are, where non mentioned products being zero:*

$Alg_1 : e^2 = e, ee_2 = \alpha e_2, ee_3 = \alpha e_3;$

$Alg_2 : e^2 = e, ee_3 = \alpha e_3;$

$Alg_3 : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = e_3, e_1e_3 = e_2;$

$Alg_4 : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = e_3, e_1e_2 = e_3;$

$Alg_5 : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3;$

$Alg_6 : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1e_2 = e_3;$

$Alg_7 : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1e_3 = e_2;$

$Alg_8(a) : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = ae_2, a \in \mathbb{C}^\star$, $Alg_8(a) \simeq Alg_8(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2;$

$Alg_9(a) : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = ae_2, e_1e_3 = e_2, a \in \mathbb{C}^\star$, $Alg_9(a) \simeq Alg_9(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2;$

$Alg_{10}(a) : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = ae_2, e_1e_2 = e_3, a \in \mathbb{C}^\star$, $Alg_10(a) \simeq Alg_10(a') \Leftrightarrow a'a^{-1} \in (\mathbb{C}^\star)^2;$

$Alg_{11} : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3;$

$Alg_{12} : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3, e_1e_3 = e_2;$

$Alg_{13} : e^2 = e, ee_1 = \frac{1}{2}e_1, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3, e_1e_2 = e_3.$

Proof. Suppose A is four-dimensional, then $r + s + t = 3$ with $t \geq$ and $r + s \geq 1$. The type of A is $(1, 1, 2, 0)$, $(1, 2, 1, 0)$ or $(2, 1, 1, 0)$.

i) Type $A = (1, 1, 2, 0)$, i.e $A =< e, e_1, e_2, e_3 >$ where $A_{1/2} = 0$, $A_0 =< e_1 >$ and $A_\alpha =< e_2, e_3 >$. Then, $e^2 = e, ee_1 = 0, ee_2 = \alpha e_2, ee_3 = \alpha e_3, e_1^2 = e_2^2 = e_3^2 = e_1e_2 = e_1e_3 = e_2e_3 = 0.$

ii) Type $A = (1, 2, 1, 0)$ i.e $A =< e, e_1, e_2, e_3 >$ where $A_{1/2} = 0$, $A_0 =< e_1, e_2 >$ and $A_\alpha =< e_3 >$. Then $e^2 = e, ee_1 = 0, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_2^2 = e_3^2 = e_1e_2 = e_1e_3 = e_2e_3 = 0.$

iii) Type $A = (2, 1, 1, 0)$, i.e $A =< e, e_1, e_2, e_3 >$ where $A_{1/2} =< e_1 >$, $A_0 =< e_2 >$ and $A_\alpha =< e_3 >$. Then $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2 + a_2e_3, e_2^2 = 0, e_3^2 = b_1e_1, e_1e_2 = c_1e_1 + c_2e_3, e_1e_3 = d_1e_1 + d_2e_2, e_2e_3 = b_2e_1.$

1^{st}case: $A_\alpha^2 \neq 0$, we can suppose $b_1 = 1$. Since $e_3^4 = 0$, then $d_1 = 0$ and $d_2b_2 = 0$. We have by using the fact that $e_1^4 = 0$, $a_2c_1d_2 = 0$, $d_2(a_1c_1 + c_2d_2) = 0$, $a_1(a_1c_1 + c_2d_2) = 0$.

- If $d_2 \neq 0$ and $a_2 = 0$, then $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2, e_2^2 = 0, e_3^2 = e_1, e_1e_2 = c_1e_1 + c_2e_3, e_1e_3 = d_2e_2, e_2e_3 = 0$. We show that A not satisfies the equation $x^4 - (1 + \alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$.
- If $d_2 = 0$, then $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2 + a_2e_3, e_2^2 = 0, e_3^2 = e_1, e_1e_2 = c_1e_1 + c_2e_3, e_1e_3 = 0, e_2e_3 = b_2e_1$. We show that A not satisfies the equation $x^4 - (1 + \alpha)\omega(x)x^3 + \alpha\omega(x)^2x^2 = 0$.

2^{nd} case: $A_\alpha^2 = 0$, then $b_1 = 0$, $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2 + a_2e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = c_1e_1 + c_2e_3, e_1e_3 = d_1e_1 + d_2e_2, e_2e_3 = 0$.

- Suppose $a_1 = 0$ and $a_2 \neq 0$. we can set $a_2 = 1$, then $d_1 = c_1d_2 = c_2d_2 = 0$. We have the following multiplications tables:
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = e_2, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = e_3, e_1e_3 = 0, e_2e_3 = 0$;

- If $a_1 = a_2 = 0$, then $c_1 = d_1 = c_2d_2 = 0$, thus we have the following multiplications tables:
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = 0, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = 0, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = 0, e_2^2 = 0, e_3^2 = 0, e_1e_2 = e_3, e_1e_3 = 0, e_2e_3 = 0$
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = 0, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = e_2, e_2e_3 = 0$
- Suppose that $a_1 \neq 0$ and $a_2 = 0$. Then we have $c_1 = d_1 = c_2d_2 = 0$ and therefore, the following multiplication tables:
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = 0, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = e_2, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = a_1e_2, e_2^2 = 0, e_3^2 = 0, e_1e_2 = e_3, e_1e_3 = 0, e_2e_3 = 0$.
- Suppose that $a_1a_2 \neq 0$. We can set $a_1 = a_2 = 1$ and therefore $c_1 = d_1 = c_2d_2 = 0$. Thus we have the multiplication tables:
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = 0, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = 0, e_1e_3 = e_2, e_2e_3 = 0$;
  $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = 0, ee_3 = \alpha e_3, e_1^2 = e_2 + e_3, e_2^2 = 0, e_3^2 = 0, e_1e_2 = e_3, e_1e_3 = 0, e_2e_3 = 0$.

□

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